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The mechanism of the diffractive nonreciprocity of counterpropagating waves in a ring gas laser

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Abstract. The mechanism of the appearance of the diffraction nonreciprocity of counterpropagating waves in a ring gas laser is examined. Formulas were obtained for the difference between their frequencies and intensities, the dependences of which on the detuning agree with those obtained earlier and with experiment. Explicit dependences of the differences between the frequencies and intensities of the counterpropagating waves on the cross sections of the aperture and of the Gaussian beams incident on the latter were found.

1. Introduction

The aim of the present study is the discovery of the mechanism of the diffraction nonreciprocity [1, 2] of counterpropagating waves in a single-mode ring gas laser. It follows from the experiments that there is a relationship between the splitting of the lasing frequencies of the counterpropagating waves and the inequality of their intensities [3 – 5]. Thus the frequency nonreciprocity of the cavity is in essence of energy nature and is associated with the nonreciprocal energy transfer between counterpropagating waves. The relationship between the amplitude and frequency characteristics of the lasing field is described by nonlinear dynamic equations, by virtue of which the saturation effects (the nonlinearity of the active medium caused by the 'self-interaction' of the field) should play an active role in the establishment of the amplitude – frequency nonreciprocity. The experimentally observed direct dependence of the splitting of the counterpropagating-wave frequency on the average continuous-lasing intensity indicates directly the nonlinearity of the amplitude – frequency nonreciprocity.

A theory of the diffraction nonreciprocity of the counterpropagating waves of a ring laser has been developed [1, 2]. The diffraction splitting of the transverse structure of counterpropagating waves was used as the cornerstone of the theory. Such splitting leads to a nonreciprocity of the nonlinear polarisability of the medium for counterpropagating waves, which in fact induces the amplitude – frequency splitting of the continuous single-mode lasing field. Analytical studies, carried out for the lowest transverse mode TEM_{00q} yielded the following expression for the splitting of counterpropagating-wave frequencies in a ring gas laser:

$$\omega_1 - \omega_2 = -\frac{c}{L}\frac{N_d I\left[(\beta'' - \theta'')^2 + (\beta' - \theta')^2\right]}{\beta'' - \theta''} = -\frac{c}{L}\frac{N_d I|\beta - \theta|^2}{\beta'' - \theta''}\,, \qquad (1)$$

where $\beta = \beta' + i\beta''$; $\theta = \theta' + i\theta''$ are complex self- and cross-saturation coefficients (here and in Sections 6 and 7 the primes are used to denote the real and imaginary parts of complex parameters, whereas elsewhere in the text the primes imply differentiation with respect to a spatial coordinate); I is the average oscillator intensity; L is the cavity perimeter; c is the velocity of light in a vacuum; N_d is a multiplier determined by the transverse-field inhomogeneity effects. The complex computational method, based on the expansion of the diffraction distortions in the transverse modes of a passive cavity, did not allow the determination of the analytical dependence of the coefficient N_d on the aperture size, but numerical calculations yielded a qualitative agreement with experimental data [3 – 5]. The dependence of the splitting $\Delta\omega$ on the difference $|\beta - \theta|^2$ is a characteristic feature.

The problem in which the amplitude – phase continuous lasing equations are obtained from the condition that the fields at the 'entry' to and 'exit' from a ring cavity with a nonlinear active medium are identical in a round trip has been investigated for the laser field [6]. According to the results obtained [6], the diffraction splitting of the transverse structure of counterpropagating waves leads merely to the inequality of their intensities but not to frequency splitting. This paradoxical conclusion, essentially decoupling the frequency nonreciprocity from the energy nonreciprocity, is in our view associated with the incorrect construction of the round-trip matrix of the active cavity in the above study [6] and hence the consequent incorrect continuous lasing equations. Despite this, the above study [6] is interesting because it makes more acute the question of the mechanism of the conversion of the spatial nonreciprocity into the amplitude – frequency nonreciprocity.

It was shown in the present study that the nature of the appearance of the spatial nonreciprocity and its relationship with the energy nonreciprocity of the dynamic system play a key role in this mechanism. Precisely this kind of relationship arises in the stopping down of the field both by a slit and by an induced aperture, i.e. by a transversely inhomogeneous nonlinear active medium. A specific feature of diffraction is that it functions simultaneously also as a factor establishing the state of the optical beam (its spatial structure) and an irreversible converter (dissipator) of the energy, the operation of which depends on the transverse field distribution. As a result of such mutual influence, the spatial structure becomes incorporated in the energy – frequency field characteristics

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and hence in the continuous lasing equations. Since both the diffraction and the nonlinearity are attributes of the laser, the nonreciprocity caused by them may be regarded as 'immanent symmetry breaking' of the active cavity.

Like the earlier investigation [1, 2, 6], this study deals with a single-mode ring gas laser having a Gaussian aperture and a square-shaped active medium which does not distort the spatial structure of the TEM_{00q} mode (Fig. 1). The continuous (cw) lasing is considered in the dynamic aspect as a fixed point of a discrete dynamic operator $\hat{M}$ determined by the ABCD matrix for the conversion of a Gaussian beam in round trip through the cavity. In this interpretation, the change in the field per pass (the cavity constant) behaves as the Floquet index corresponding to the fixed point of the conversion $\hat{M}$. By considering a ring laser as a discrete dynamic system, it is possible to discover the fundamental relationship between the diffraction nonreciprocity and the energy irreversibility caused by dissipation.

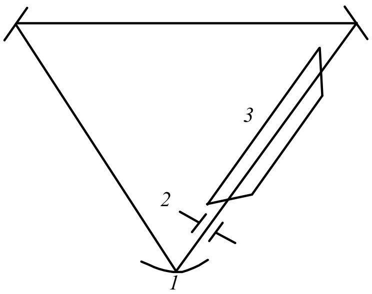

Figure 1. Schematic illustration of a ring laser with a spherical mirror (1), a Gaussian aperture (2), and a square-shaped active medium (3).

2. Gaussian beam in an active medium

Within the framework of the classical theory, an electric field is described by the electrical induction vector $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$, where $\boldsymbol{E}$ is the field strength and P is the polarisation of the medium. Under quasi-neutrality ($\nabla\boldsymbol{D} = 0$) conditions, the vectors E and P obey the wave equation

$$\Delta\boldsymbol{E} + 4\pi\nabla(\nabla\boldsymbol{P}) - \frac{1}{c^2}\frac{\partial^2(\boldsymbol{E} + 4\pi\boldsymbol{P})}{\partial t^2} = 0 \,. \tag{2}$$

For single-mode cw lasing,

$$\begin{aligned} \boldsymbol{E} &= \vec{\mathscr{E}}(x,\,y,\,z)\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\,, \\ \boldsymbol{P} &= \vec{\mathscr{P}}(x,\,y,\,z)\exp(-\mathrm{i}\omega t) + \mathrm{c.c.}\,. \end{aligned} \tag{3}$$

The polarisation of an inhomogeneously broadened gaseous medium can be represented in the following form neglecting the transverse spatial dispersion [2]:

$$\vec{\mathscr{P}} = \varkappa\vec{\mathscr{E}}(x,\,y,\,z) + \mathrm{c.c.}\,, \tag{4}$$

where $\varkappa = K\chi/2\pi k$ is the polarisability of a nonlinear medium in the active transition; $k = \omega/c$; K is the gain per unit length of the medium; $\chi \sim 1$ is the dimensionless polarisability. For a beam with a diameter w and neglecting quantities of the order of $(kw)^{-4}$, Eqn (2) is converted into the scalar Helmholtz equation in the steady-state case:

$$\left[\varDelta + k^2(1 + 2\chi K k^{-1})\right]E = 0 \,, \tag{5}$$

which assumes the following from in the Eikonal representation $E = E_0\exp[\mathrm{i}\Psi(x,\,y,\,z)]$:

$$\mathrm{i}\Delta\Psi - |\nabla\Psi|^2 + k^2(1 + 2\chi K k^{-1}) = 0 \,. \tag{6}$$

The diffraction aspects are described by the diffusion term $\mathrm{i}\Delta\psi$. According to formula (6), its relative contribution to the Eikonal is $\Delta\Psi/k^2 \sim (1/kw)^2$. Having confined the treatment to the lowest cavity mode, we represent the field E in the form [7]

$$\begin{aligned} E &= E(z)\exp[\mathrm{i}\Psi(x,\,y,\,z;\,z_0)]\,, \\ \Psi(x,\,y,\,z;\,z_0) &= \frac{x^2Q_x(z) + y^2Q_y(z)}{2} + \int_{z_0}^{z} f(z)\,\mathrm{d}z\,, \end{aligned} \tag{7}$$

where z_0 is the position of the reference plane on the z axis; $f(z)$ is a complex function defined below. The complex functions Q_x and Q_y are large (in terms of the modulus) quantities and $Q_x,\,Q_y \to \mathrm{const}$ when $k \to \infty$.

We substitute formula (7) in formula (6) and divide the terms into those which do and do not depend on the transverse coordinates. Having set the separation constant equal to zero (since we are considering the TEM_{00q} mode), we obtain

$$\mathrm{i}(Q_x + Q_y) + \mathrm{i}\,f' - f^2 + k^2[1 + 2\chi(z)Kk^{-1})] = 0\,, \tag{8}$$

$$\begin{aligned} &x^2\left(fQ'_x + Q_x^2 - \mathrm{i}\frac{Q''_x}{2}\right) + y^2\left(fQ'_y + Q_y^2 - \mathrm{i}\frac{Q''_y}{2}\right) \\ &\quad + \frac{1}{4}\left(x^2Q'_x + y^2Q'_y\right)^2 - 2\chi(x,\,y,\,z)Kk = 0\,, \end{aligned} \tag{9}$$

where $\chi(z)$, and $\chi(x,\,y,\,z)$ are the corresponding components of the polarisability

$$\chi = \chi(z) + \chi(x,\,y,\,z)\,. \tag{10}$$

We shall seek the solution to formula (8) in the form of an asymptotic expansion in terms of $1/k$:

$$f(z) = kf_0 + f_1 + O(1/k)\,. \tag{11}$$

On substituting formula (11) in formula (8), we find $f_0 = (-1)^{j+1}$;

$$\begin{aligned} f_1 &= K_j\chi_j(z) + \frac{\mathrm{i}}{2k_j}(Q_x + Q_y) \\ &= K_j\chi_j(z) + \frac{\mathrm{i}}{2}(\xi_{x_j} + \xi_{y_j})\,, \quad j = 1,\,2\,, \end{aligned} \tag{12}$$

where

$$K_j = f_{0j}K = (-1)^{j+1}K\,;\ \xi_{pj} = Q_p/k_j\,;\ k_j = f_{0j}k = (-1)^{j+1}k\,; \tag{13}$$

the index j is used to number counterpropagating waves. Taking into account formulas (12) and (13), the expression for the field (7) assumes the form of an astigmatic Gaussian beam of the TEM_{00q} mode of a ring cavity:

$$\begin{aligned} E &= \sum_{j=1}^{2} E_{0j}(z_{0j})\frac{\exp[\mathrm{i}k_j(\xi_{xj}x^2 + \xi_{yj}y^2)/2}{(m_{xj}m_{yj})^{1/2}} \\ &\times \exp\left[\mathrm{i}\int_{z_{0j}}^{z} K_j\chi_j\,\mathrm{d}z + \mathrm{i}k_j(z - z_{0j})\right], \quad m_{pj} = \int_{z_{0j}}^{z}\xi_{pj}\,\mathrm{d}z\,; \end{aligned} \tag{14}$$

the complex parameters ξ_x and ξ_y determine the curvatures $(S_x,\,S_y)$ and the diameters $(w_x,\,w_y)$ of the beams incident on the aperture; $\xi_{pj} \equiv S_{pj} + 2\mathrm{i}/(k_j w_{pj}^2)$. The transition from the 'forward' to the 'reverse' wave corresponds to the transition to the complex-conjugate variable ξ ($\xi \to \xi^*$), which determines the spatial structure of the beam.

On substituting formulas (12) and (13) in formula (8), we obtain the equation for the phase variables ξ_x and ξ_y, which assumes the following form in the region $x,\ y \geqslant w$ in which we are interested:

$$x^2\{\xi'_{xj} + \xi_{xj}^2 + \mathrm{O}[(kw)^{-2}]\} + y^2\{\xi'_{yj} + \xi_{yj}^2 + \mathrm{O}[(kw)^{-2}]\} \\ -2\chi_j(x,\ y,\ z)K_j k_j^{-1} = 0\ . \tag{15}$$

This is the parabolic approximation equation modified for the case of a medium with a very weak polarisability $[K/k \sim (kw)^{-2}]$. According to formula (14), the saturation effects have a Gaussian structure with an aperture determined by the beam diameter w.

The polarisabilities of the medium assume the following form for counterpropagating waves:

$$\chi_j = \chi_j^{(1)} + \sum_{n=1}^{2} \beta_{jn}|E_n|^2$$

and hence lead to the distortion of the Gaussian field structure [7]. The nonlinear medium is approximately quadratic only for the paraxial part of the beam when $x \ll w_x,\ y \ll w$:

$$\begin{aligned} \chi_j &= \chi_j^{(1)} + \sum_{n=1}^{2} \chi_{jn}\left(1 - \frac{2x^2}{w_{xn}^2} - \frac{2y^2}{w_{yn}^2}\right)\ , \\ \chi_{jn} &= \beta_{jn}|E(z_{0n})|^2 \exp\left[-\int_{z_{0n}}^{z} \mathrm{Re}(\xi_{xn} + \xi_{yn})\,\mathrm{d}z \right. \\ &\qquad \left. -2\int_{z_{0n}}^{z} K_n \mathrm{Im}\chi_n^{(1)}(z)\,\mathrm{d}z\right]\ . \end{aligned} \tag{16}$$

When $\chi_{nj} \geqslant (w/r)^2$ (r is the radius of the gas-discharge tube), the effects due to the field-induced transverse inhomogeneity of the medium predominate over effects associated with the inhomogeneity of a linear medium [8]. In this case where $w \ll r$, this inequality holds even for fairly small pump powers. Subsequently, in the calculation of the transverse inhomogeneity of the medium, we shall indicate only the nonlinear terms.

On substituting formula (16) in formula (15), we obtain the equation for the parameters ξ_{pj} which determine the transverse structure of each of the counterpropagating beams ($j = 1,\ 2$) in the paraxial region:

$$\begin{aligned} &\xi'_{pj} + \xi_{pj}^2 + \sigma_{pj}(z) = 0\,, \qquad \sigma_{pj}(z) = \sum_{n=1,2} \frac{4\chi_{jn}K_j}{k_j w_{pn}^2(z)}\ , \\ &p = x,\ y\ . \end{aligned} \tag{17}$$

Formally, Eqn (17) has the form of the Riccati equation, but each coefficient σ_{pj} depends on ξ_{p1} and ξ_{p2}. The parameter σ_{pj} describes a distributed optical system, $\mathrm{Re}\,\sigma_{pj}$ playing the role of a distributed ideal lens [9] and $\mathrm{Im}\,\sigma_{pj}$ the role of a distributed Gaussian aperture. The condition that the perturbation of the beam of such a system is small is expressed by the inequality

$$\left|\int_{(l)} \sigma_{pj}\,\mathrm{d}z\right| \ll |\xi_{pj}|\ , \tag{18}$$

where l is the length of the active medium.

3. Construction of a ray matrix for a lens-like active medium

Substitution of $\xi = \partial(\ln m)/\partial z$ transforms Eqn (17) into (the indices p and j are omitted)

$$m'' + \sigma m = 0\,, \qquad m(z_0) = 1\ . \tag{19}$$

Suppose that m_1 and m_2 are two linearly independent solutions of this equation satisfying the conditions

$$\begin{pmatrix} m_1(z_0) \\ m'_1(z_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} m_2(z_0) \\ m'_2(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{20}$$

The solutions for m and ξ then assume the form

$$\begin{aligned} m &= m_1 + m_2\xi_0 = a + b\xi_0\ , \\ \xi &= \frac{m'_1 + m'_2\xi_0}{m_1 + m_2\xi_0} = \frac{c + d\xi_0}{a + b\xi_0}\ . \end{aligned} \tag{21}$$

Here, $\xi_0 = \xi(z)$ is the value of ξ in the reference plane $z = z_0$. Formally, the solutions (21) are the usual ABCD solution for a Gaussian beam [7].

We shall seek the functions m_1 and m_2 by the method of successive approximations in terms of σ. For $\sigma = 0$, we have $m_1 = 1$ and $m_2 = z - z_0$. On substituting these solutions in Eqn (19), we find

$$\begin{aligned} m'_1 &= -\int_{z_0}^{z} \sigma \mathrm{d}z\,, \qquad m'_2 = 1 - \int_{z_0}^{z} \sigma(z - z_0)\,\mathrm{d}z'\ , \\ m_1 &= 1 - \int_{z_0}^{z} \mathrm{d}z' \int_{z_0}^{z'} \sigma\,\mathrm{d}z'' \\ &= 1 - (z - z_0)\int_{z_0}^{z} \sigma \mathrm{d}z' + \int_{z_0}^{z} \sigma(z' - z_0)\,\mathrm{d}z'\ , \\ m_2 &= (z - z_0) - \int_{z_0}^{z} \mathrm{d}z' \int_{z_0}^{z'} \sigma(z'' - z_0)\,\mathrm{d}z'' = (z - z_0) \\ &\quad -(z - z_0)\int_{z_0}^{z} \sigma(z' - z_0)\,\mathrm{d}z' + \int_{z_0}^{z} \sigma(z' - z_0)^2\,\mathrm{d}z'\ , \end{aligned} \tag{22}$$

where the double integrals are converted into single integrals with the aid of integration by parts. When condition (18) holds, it is possible to confine the treatment to the approximation (22), linear in terms of the intensity, in which the beam parameters, entering into σ [see formulas (16) and (17)], are defined for a passive system ($\sigma = 0$). The unimodularity of the ABCD matrix is then retained in terms of the same order of precision: $\Delta = AD - CB = m_1 m'_2 - m'_2 m_2 = 1 + \mathrm{O}(\sigma^2)$. The complex character of matrix (22) indicates irreversible losses due to light scattering by transverse inhomogeneities of the medium.

We may draw attention to the fact that the matrices for the passage of the counterpropagating waves ($j = 1,\ 2$) through the entire medium have the structure of the forward and reverse matrices, for which

$$\begin{aligned} &a_{pj}(\sigma_{pj}) = d_{p3-j}(\sigma_{pj})\,, \qquad d_{pj}(\sigma_{pj}) = a_{p3-j}(\sigma_{pj})\ , \\ &b_{pj}(\sigma_{pj}) = -b_{p3-j}(\sigma_{pj})\,, \qquad c_{pj}(\sigma_{pj}) = -c_{p3-j}(\sigma_{pj})\ . \end{aligned} \tag{23}$$

Indeed, assuming that the input (reference) plane of one wave coincides in formulas (22) with the output plane of the counterpropagating wave, we obtain

$$a_{p2}(\sigma_{p2}) = 1 - (z_1 - z_2)\int_{z_2}^{z_1} \sigma_{p2}\,\mathrm{d}z + \int_{z_2}^{z_1} (z - z_2)\sigma_{p2}\,\mathrm{d}z$$

$$= 1 - \int_{z_1}^{z_2} (z - z_1)\sigma_{p2}\,\mathrm{d}z = d_{p1}(\sigma_{p2})$$

etc. At the same time, the difference between the coefficients σ_{p1} and σ_{p2}, caused both by the difference between the self- and cross-saturation coefficients β_{nn} and β_{jn}, $j \neq n$, although $\beta_{nm} = \beta_{ij}$, $\beta_{jn} = \beta_{nj}$) and by the spatial nonreciprocity ($w_{p1} \neq w_{p2}$), leads to failure by mutual transformations to reflect the passage of counterpropagating waves through a nonlinear active medium. As we shall see later, this fact actually induces the amplitude – frequency nonreciprocity of counerpropagating waves in the laser.

4. Boundary conditions on mirrors and apertures

In the simplest situation, abrupt changes in the Gaussian beam on mirrors and apertures are determined by the boundary conditions

$$\xi_{pj} \to \xi_{pj} + 2\rho_{pj}\,, \qquad \frac{1}{\rho} = \frac{k_j}{k} R_p \cos\psi_{pj} = (-1)^{j+1} R_p \cos\psi_{pj}\,,$$

$$\xi_{pj} \to \xi_{pj} + 2\mathrm{i}N_{pj}\,, \qquad \frac{1}{N_{pj}} = k_j a_p^2\,, \tag{24}$$

where R_p are the main radii of curvature of the mirror; ψ_{pj} is the angle of incidence of the jth wave on the mirror ($\cos\psi_{pj} < 0$); a_p is the linear size of the aperture. The boundary conditions indicated are described by the matrices

$$\hat{R} = \begin{pmatrix} 1 & 0 \\ 2\rho & 1 \end{pmatrix}, \qquad \hat{N} = \begin{pmatrix} 1 & 0 \\ 2\mathrm{i}N & 1 \end{pmatrix}. \tag{25}$$

For counterpropagating waves, the reflection and stopping-down matrices [expressions (25)] are mutually reciprocal:

$$\hat{R}_2 = \hat{R}_1^{-1}\,, \qquad \hat{N}_2 = \hat{N}_1^{-1}\,. \tag{26}$$

The complex character of the stopping down matrix $\hat{N}$ reflects the fact that the narrowing of the beam is accompanied by irreversible energy losses with a decrement $\sim w^2/a^2$, where a is the maximum aperture size.

5. Discrete field dynamics in a ring laser

According to formula (14), the state of a Gaussian beam along each transverse coordinate x, y is determined by the complex phase variable ξ, which depends on the longitudinal coordinate z as a parameter. Since the points z and $z + L$ are identical in a ring laser with a perimeter L, the stationary field has a period L [$E(x, y, z + L) = E(x, y, z)$], which is equivalent to the requirement of the L-periodicity of the transverse structure and of the field on the cavity axis:

$$\xi(z + L) = \xi(z)\,, \qquad E(0, 0, z + L) = E(0, 0, z)\,. \tag{27}$$

Suppose that the transformation $\hat{M}$ maps the point $\xi(z)$ onto the point $\xi(z + L)$: $\xi(z + L) = \hat{M}\xi(z)$. The periodicity condition (27) then defines $\xi(z + L)$ as a fixed mapping point $\hat{M}$:

$$\xi = \hat{M}\xi\,, \tag{28}$$

for which the parameter

$$\mu = \exp\left(-\frac{1}{2}\int_0^L \xi\,\mathrm{d}z\right) = \exp(\Lambda L) \tag{29}$$

plays the role of a multiplier, whereas Λ is the Floquet index of the fixed point [10].

The steady-state condition (27) for each counterpropagating beam can be written in the form

$$(\Lambda_{xj} + \Lambda_{yj})L + \mathrm{i}\int_0^L K_j\chi_j(z)\,\mathrm{d}z + \mathrm{i}kL = 2\pi q\mathrm{i} + \varepsilon\,, \tag{30}$$

where ε represents the phenomenological losses on the cavity components (absorption in mirrors and other components). The complex equation (30) describes the amplitude – phase balance necessary for the establishment of a steady state in the operation of the laser. The real part of Eqn (30) ensures the steady-state nature of the laser output intensity, whereas the imaginary part ensures the steady-state nature of the laser-field frequency. In essence, Eqn (30) is the expression of the local dynamic steady-state condition in the 'spatial' representation. We may note that the Floquet indices which enter into it are not local quantities, but characterise the operation of the cavity as a whole.

For a cavity which does not distort the Gaussian structure of the TEM_{00q} mode, the point map $\hat{M}$ is determined by the unimodular ABCD transformation of the Gaussian beam [7]:

$$\xi = \frac{C + D\xi}{A + B\xi}\,, \qquad m = A + B\xi\,, \tag{31}$$

whence we have the following expressions for the fixed point [formula (28)], its multiplier, and the Floquet index [formula (29)]:

$$B\xi = \frac{D - A}{2} + \mathrm{i}\big(1 - G^2\big)^{1/2}\,, \qquad m = G + \mathrm{i}\big(1 - G^2\big)^{1/2}\,,$$

$$\Lambda L = \frac{1}{2\mathrm{i}}\arccos G\,, \qquad G = \frac{A + D}{2}\,. \tag{32}$$

The quantity m is the eigenvalue of the ABCD matrix of the cavity, which thus plays the role of the 'monodromy operator' of the point map $\hat{M}$.

A limit cycle, a piecewise-continuous closed curve (Fig. 2), corresponds to the fixed point (31) in the phase plane of the beam ($\mathrm{Re}\,\xi$, $\mathrm{Im}\,\xi$). Its continuous sections reflect changes in the field between the cavity components and are determined by the matrices (21) and (22) (for the empty sec-

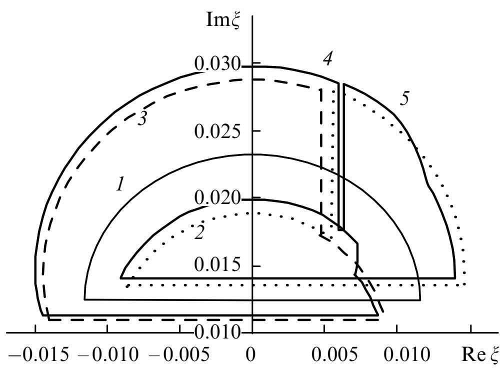

Figure 2. Phase portrait of the counterpropagating waves in limit cycles for a passive cavity without an aperture (1) and with an aperture (2, 3) and also for an active cavity with an aperture (4, 5).

tions, $\sigma = 0$). The discontinuities in the phase path correspond to abrupt changes in the field on the mirrors (horizontal jumps) and apertures (vertical jumps). The limit cycle in Fig. 2 represents a steady-state field in a ring cavity with one mirror and one Gaussian aperture [formula (25)], between which the active medium is placed. According to formula (14), the phase variables ξ_j for counterpropagating waves are complex conjugates: $\mathrm{Im}\,\xi_1 = -\mathrm{Im}\,\xi_2$. However, it is more convenient to plot phase paths on the phase plane $(\mathrm{Re}\,\xi, |\mathrm{Im}\,\xi|) = (\mathrm{Re}\,\xi_1, \mathrm{Im}\,\xi_1) = (\mathrm{Re}\,\xi_2, -\mathrm{Im}\,\xi_2)$. Under these conditions, the limit cycle for stable cavities is always located above the abscissa ($\mathrm{Re}\,\xi$) and the distance from it characterises the degree of stability of the cavity. The condition that the nonlinear perturbations in the beam structure defined by formula (18) are small requires that the point on the phase plane determined by the left-hand side of formula (18) should be much closer to the origin of coordinates ($\xi = 0$) than to the phase path of the beam.

In the absence of an aperture, the limit cycles $|\mathrm{Im}\,\xi| = f(\mathrm{Re}\,\xi)$ corresponding to counterpropagating waves coincide and only the directions of the round trip through them are different (curve 1 in Fig. 2). The fixed points ξ_j [equation (31)] for counterpropagating fields are then complex-conjugate quantities for each value of z.

In the presence of an aperture, the phase curves are split. This corresponds to the splitting of the spatial structure of the counterpropagating waves at each point in the cavity. The relative splitting is $|\Delta\xi_{pj}/\xi_{pj}| \sim w_{pj}^2/a_{pj}^2$. However, the amplitude – phase balance [formula (30)], which determines the steady-state lasing parameters, includes not the fixed points ξ_j themselves but their Floquet indices, determined (by virtue of unimodularity) solely by the trace of the ABCD matrix of the cavity — the 'monodromy operator'. As in a cavity with an active medium linear in terms of the field, in a passive cavity the 'monodromy operators' of counterpropagating waves are mutually reciprocal ($\hat{M}_2 = \hat{M}_1^{-1}$) and hence the multipliers in the Floquet index are equal. This leads to the amplitude – frequency reciprocity of the ring cavity at the lasing threshold [2].

6. Conditions for the appearance of the amplitude – frequency nonreciprocity in the laser

By subtracting the balance equations (30) for counterpropagating waves from one another and taking into account the reciprocity of the laser at the lasing threshold ($I = 0$), we obtain a complex equation for the balance of nonreciprocities:

$$(\beta - \theta)(I_1 W_1 - I_2 W_2) + (k_1 - k_2)L = 0\ , \tag{33}$$

where

$$\beta = \beta_{jj}\,;\ \ \theta = \theta_{ji}\,;\ \ W_j = W_{0j} + W_{\mathrm{g}j}\,;\ \ j,\ i = 1, h,\ 2\ (j \neq i)\ ;$$

$$W_{0j} = K\int_0^l \frac{w_x(l_j)w_y(l_j)}{w_x(z)w_y(z)}\exp\left(Kz\,\mathrm{Im}\chi^{(1)}\right)\mathrm{d}z\ ; \tag{34}$$

$$W_{\mathrm{g}j} = W'_{\mathrm{g}j} + \mathrm{i}W''_{\mathrm{g}j} = \frac{1}{2}\sum_{p=x,y}\frac{\partial G_{pj}/\partial(\beta_{pj}I_j)}{(1-G^2)^{1/2}}\bigg|_{I=0}\ ;$$

G_{pn} is half the trace of the corresponding active-cavity matrix.

We shall examine the conditions under which the frequencies of the counterpropagating waves are identical. For $k_1 = k_2$, we obtain from Eqn (33) $(\beta - \theta)(I_1 W_2 - I_2 W_1) = 0$, which holds either for $\beta = \theta$ or for $I_1/I_2 = W_2/W$. Since the intensities I_1 and I_2 are real quantities, the last equation requires that $W_1''W_2' - W_1'W_2'' = 0$. These conditions ensure the amplitude – frequency reciprocity of a single-mode laser, so that failure of these conditions leads to the splitting of the counterpropagating-beam frequencies of a single mode:

$$\begin{aligned}|\Delta kL| &\sim |\beta - \theta||W_1''W_2' - W_1'W_2''|I \\ &\sim |\beta - \theta||W''(W_1' - W_2')|I\ ,\end{aligned} \tag{35}$$

where $W'' \equiv (W_1'' + W_2'')/2$; I is the average lasing intensity. According to expression (35), for the reciprocal linear gain the necessary and sufficient conditions ensuring the splitting of the counterpropagating-wave frequencies of a single mode are a difference between the self- and cross-saturations [$(\beta - \theta)I \neq 0$], the presence of diffraction losses ($W'' = W_{\mathrm{g}}'' \sim w^2/a^2 \neq 0$), and different changes in the sizes of the counterpropagating-beam spots along the amplification direction ($W_1' - W_2' \neq 0$). The difference $W_1' = W_2'$ is determined both by the asymmetry of the disposition of the medium relative to the counterpropagating-beam waists and by the difference between the sizes of the waists caused by diffraction. If the medium is located asymmetrically relative to the waist of a beam unperturbed by the aperture (Fig. 1), then $|\Delta kL| \sim |\beta - \theta|(w^2a^{-2})I$; otherwise $|\Delta kL| \sim |\beta - \theta|(w^2a^{-2})^2 I$.

7. The differences between the frequencies and intensities of counterpropagating waves

Having added together the real parts of Eqns (30) for counterpropagating waves and having separated the real and imaginary parts in the difference equation (33) from one another, we obtain

$$\begin{aligned}&(\beta'' - \theta'')F_1 + (\beta' - \theta')F_2 = 0\ , \\ &(\beta'' - \theta'')F_2 - (\beta' - \theta')F_1 = (k_1 - k_2)L\ ,\end{aligned} \tag{36}$$

$$2\alpha = (\beta'' + \theta'')(I_1W_1' + I_2W_2') + (\beta' + \theta')(I_1W_1'' + I_2W_2'')\,, \tag{37}$$

where

$$F_1 \equiv I_1W_1' - I_2W_2'\,; \qquad F_2 \equiv I_1W_1'' - I_2W_2''\ ;$$

α is the excess of the nonlinear gain per pass through the cavity relative to the lasing threshold. The first equation of the system (36) establishes the equality of the nonreciprocity of the relative dissipation in the medium (the first term) and on the aperture (the second term). This is due to the nonreciprocal deformation of the spots of the beams incident on the aperture, arising as a consequence of the nonreciprocal effect of the induced gas lens ($\mathrm{Re}\,\sigma_1 \neq \mathrm{Re}\,\sigma_2$). We emphasise that the decrement of the field attenuation on the aperture is not a local characteristic determined exclusively by the aperture parameters. It depends on the state of the incident beam (its diameter), which is formed by the entire cavity and particularly by the nonlinear medium.

The quantities entering into the second (phase) equation of the system (36) are essentially nonlocal because they characterise the nonreciprocity of the phase velocities of counterpropagating waves in the cavity. The second term describes the nonreciprocal curvature of the wavefront under the influence of the distributed gas lens. The first term has a diffraction nature, the source of which is the curvature of the rays and their interference, arising as a result of the stopping down of the field by the nonlinear medium and the slit. After introducing

$$I = \frac{I_1 + I_2}{2}, \quad \Delta I = \frac{I_1 - I_2}{2}, \tag{38}$$

we find from Eqns (36) and (37) expressions for the amplitude–frequency splitting of the counterpropagating waves and for the average laser-radiation intensity:

$$\Delta I = -\frac{\alpha}{2}\frac{1}{A}\left[(\beta'' - \theta'')(W_1' - W_2') + (\beta' - \theta')(W_1'' - W_2'')\right],$$

$$\omega_1 - \omega_2 = \frac{c}{L}\frac{1}{A}\alpha\left[(\beta' - \theta')^2 + (\beta'' - \theta'')^2\right] \times (W_1''W_2' - W_1'W_2''), \tag{39}$$

$$I = \frac{\alpha}{2}\frac{1}{A}\left[(\beta'' - \theta'')(W_1' + W_2') + (\beta' - \theta')(W_1'' + W_2'')\right],$$

where

$$A = (\beta''^2 - \theta''^2)W_1'W_2' + (\beta'^2 - \theta'^2)W_1''W_2'' + (\beta'\beta'' - \theta'\theta'')(W_1''W_2' + W_1'W_2'').$$

When the cavity is tuned precisely to the centre of the gain line ($\beta' = \theta' = 0$), we obtain

$$I_1 - I_2 = \frac{\alpha(W_2' - W_1')}{2(\beta'' + \theta'')W_1'W_2'},$$

$$\omega_1 - \omega_2 = \frac{c}{L}\frac{\alpha(\beta'' - \theta'')(W_1''W_2' - W_1'W_2'')}{(\beta'' + \theta'')W_1'W_2'},$$

$$I = \frac{\alpha(W_2' + W_1')}{2(\beta'' + \theta'')W_1'W_2'}.$$

As a consequence of the incorrect calculation of the pass matrix in a nonlinear active medium, it was postulated in Ref. [6] that $F_2 = 0$, which leads to $F_1 = 0$ by virtue of formula (36) and hence to full energy–frequency reciprocity of the ring cavity.

Formula (39) for the difference between the frequencies is in structural agreement with formula (1), provided that one employs the average intensity I and it is postulated that

$$N_{\mathrm{d}} = -\frac{2(\beta'' - \theta'')(W_1''W_2' - W_1'W_2'')}{(\beta' - \theta')(W_1' + W_2') + (\beta'' - \theta'')(W_1'' + W_2'')}, \tag{40}$$

For this reason, the conclusions reached in Refs [1, 2] remain valid.

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