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Some models of propagation of extremely short electromagnetic pulses in a nonlinear medium

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Abstract. Some cases of model media considered in this paper allow analytical solutions to nonlinear wave equations to be found and the time dependence of the electric field strength to be determined in the explicit form for arbitrarily short electromagnetic pulses. Our analysis does not employ any assumptions concerning a harmonic carrier wave or the variation rate of the field in such pulses. The class of models considered includes two-level resonance and quasi-resonance systems. Nonresonance media are analysed in terms of models of anharmonic oscillators — the Duffing and Lorentz models. In most cases, only particular solutions describing the stationary propagation of a video pulse (a unipolar transient of the electric field or a pulse including a small number of oscillations of the electric field around zero) can be found. These solutions correspond to sufficiently strong electromagnetic fields when the dispersion inherent in the medium is suppressed by nonlinear processes.

1. Introduction

The last two decades have seen advances in nonlinear optics and laser physics to the range of femtosecond pulses of optical (to be more accurate, electromagnetic) radiation [1 – 12]. One of the methods of producing such pulses is the compression of an initial pulse with various time-domain compressors [1, 2, 5, 7], including fibre-grating compressors. This method permitted pulses with a duration of 6 fs to be obtained [2]. The possibility of compressing a pulse to a duration of 1 fs in experiments on the scattering of free relativistic electrons in the field of a high-power short radiation pulse was also discussed [8]. Methods for producing attosecond electromagnetic pulses were considered in Ref. [9].

Another method of producing femtosecond pulses is to generate such pulses directly in laser systems [6, 10]. Specifically, Sartania et al. [6] generated 20 fs pulses of energy 1.5 mJ and a repetition rate of 1 kHz (5 fs pulses with an energy of 0.5 mJ were obtained with a fibre – prism compressor). The authors of Ref. [10] employed a Ti: sapphire laser to generate 6.5 fs pulses with a mean power of 200 mW and a repetition rate of 86 MHz. Parametric wave mixing, self-action of laser pulses (self-focusing and self-modulation), and coherent transient processes in the field of femtosecond pulses were considered in Ref. [11]. The review by Andreev et al. [12] was devoted to the generation of high-power short laser pulses.

In the context of rapid progress in the generation of femtosecond, and even shorter, pulses of electromagnetic radiation, it is of interest to analyse theoretical models describing the propagation of very short pulses in nonlinear dispersive media. Naturally the Maxwell equations supplemented with equations governing the evolution of the polarisation or currents arising in a medium subject to electromagnetic radiation [13 – 25] or with the Schrödinger equation for electrons interacting with an incident electromagnetic field [26, 27] provide a background for all the theories considered. Since it seldom occurs that exact analytical results can be obtained in such problems, various approximations simplifying the consideration and allowing analytical expressions to be derived are often employed.

An important and, at the same time, simple approximation corresponds to the propagation of electromagnetic waves in one of many possible directions. The model of unidirectional waves lowers the order of the wave equation without imposing limitations on pulse duration. Obviously, in certain cases this approximation is a priori inapplicable (e.g., for waves in periodic or scattering media).

Another broad class of approximations is associated with the properties of a medium. Specifically, gaseous media (molecular gases and metal vapours) and impurities in glasses are characterised by discrete absorption spectra. If the frequency of monochromatic radiation coincides with the frequency of an atomic or molecular transition or is close to such a frequency, then the interaction has a resonant character. A radiation pulse is not a monochromatic wave, but it can be represented as a quasi-monochromatic wave:

$$E(\boldsymbol{r},\, t) = \mathscr{E}(\boldsymbol{r},\, t)\exp(-\mathrm{i}\omega_0 t + \mathrm{i}\boldsymbol{k}_0\boldsymbol{r}) + \mathrm{c.c}\,, \qquad (1)$$

where a plane scalar wave with a wave vector $\boldsymbol{k}_0$ corresponding to the frequency ω_0 of the carrier monochromatic wave is considered for simplicity. This representation is adequate if the pulse envelope $\mathscr{E}(\boldsymbol{r}, t)$ is a function slowly varying in space and time, i.e., the inequalities

$$\left|\frac{\partial\mathscr{E}}{\partial t}\right| \ll \omega_0|\mathscr{E}| \quad \text{and} \quad |\nabla\mathscr{E}| \ll |\boldsymbol{k}_0||\mathscr{E}|\,.$$

are satisfied.

Such a representation of a pulse of electromagnetic radiation is called the slowly varying envelope approximation. The resonance condition defined for a monochromatic wave can be extended to the case of a radiation pulse if the spectral width of the envelope $\mathscr{E}(\boldsymbol{r},\, t)$ is much smaller

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than the frequency of the carrier wave and the difference in the frequencies of transitions between the energy levels adjacent to the resonance state (here, we consider dipole-allowed transitions). Energy levels other than those coupled by the resonance condition are often neglected when the above conditions are specified, and a resonance medium is represented as an ensemble of two-level atoms. In a more general situation, an ensemble of N-level atoms can be employed as a model of a resonance medium.

If the duration of an electromagnetic pulse is less than all the relaxation times in the medium under study, then the propagation of such a pulse is accompanied only by stimulated absorption and re-emission. With $N = 2$ these processes may give rise to self-induced transparency when certain conditions imposed on the pulse amplitude are satisfied [28]. Electromagnetic pulses whose duration is shorter than the polarisation and population relaxation times of resonance levels are called ultrashort pulses. Although this term sometimes has a broader meaning, for definiteness we will take ultrashort pulses to mean sufficiently short pulses that can be represented as quasi-monochromatic waves.

In the slowly varying envelope approximation, the Maxwell equations or the D'Alembert wave equation can be reduced to first-order equations for $\mathscr{E}(\boldsymbol{r}, t)$ and the resonance medium can be described by the Bloch equations. If an electromagnetic pulse is represented as a superposition of quasi-monochromatic waves with different carrier frequencies corresponding to different pairs of resonance transitions, then we should employ a set of reduced wave equations and a set of Bloch equations generalised to the case of a multilevel medium. Examples of such situations are considered in Refs. [19, 29 – 32].

Quasi-harmonic signals are employed widely in theoretical studies devoted to nonlinear coherent phenomena and in nonlinear fibre optics. However, this is not the only example of a solitary electromagnetic wave. Since when the phenomena of coherent interaction of electromagnetic radiation with resonance systems were discovered, much attention has been focused on the theoretical consideration of the propagation of short light pulses under conditions when the slowly varying envelope approximation becomes inapplicable. As is well known [13, 33, 34] the complete set of Maxwell – Bloch equations has a solution that describes the propagation of a pulse of electromagnetic radiation without a high-frequency pulse carrier. Such pulses were later called video pulses [36 – 39]. Numerical simulations of the propagation of video pulses [40] have demonstrated that such pulses are unstable with respect to collisions with each other. Goldstein [41] has shown that the complete set of Maxwell – Bloch equations does not possess the Painlevé property, which indicates the nonsoliton character of video pulses.

Furthermore the complete set of Maxwell – Bloch equations has solutions corresponding to solitary waves that contain a few cycles of electric and magnetic fields. Such waves are referred to as extremely short electromagnetic pulses. If the number of field cycles in a pulse is large, then such a pulse can be approximated with a quasi-harmonic wave (or a superposition of quasi-harmonic waves if there are several carrier frequencies). In this case, such pulses can be called ultrashort pulses. The duration of ultrashort pulses in such a situation is assumed to be much shorter than the time of irreversible polarisation relaxation.

We should start analyzing the propagation of video pulses by considering the applicability conditions of the two-level approximation for this problem. In the case of a quasi-monochromatic wave we reduce our analysis to a resonance transition if the frequency of the carrier wave is close to the frequency of an atomic transition and the spectral half-width of the pulse is substantially less than this frequency. The spectrum of a video pulse reaches its maximum at zero frequency, and only the spectral half-width $\Delta\omega_{\rm p}$ of a video pulse can be employed to define a criterion that would allow the consideration to be reduced to a single transition. Such a criterion will be introduced as the condition that the spectral half-width of a video pulse should be less than or of the order of the frequency of transition from the ground state to the neighbouring (on the energy scale) excited state, whereas the other excited states should be separated from the ground state by frequency intervals exceeding several half-widths $\Delta\omega_{\rm p}$.

Spectra with such a structure can be found among ionic or atomic spectra. (For example, in the case of a potassium atom, the frequency of transitions from the ground $4S$ state to the $4P$ state is approximately two times lower than the frequency of the $4S - 5P$ transitions. One arrives at the same relation for the frequencies of transitions from the ground $^4I_{15/2}$ state to the lower $^4I_{13/2}$ excited state and the frequencies of $^4I_{15/2} \rightarrow {}^4I_{9/2}$ and $^4I_{15/2} \rightarrow {}^4F_{9/2}$ transitions in Er^{3+} ions.) In such situations, real atoms or ions can be approximated with two-level atoms (in the above-specified sense). To include the degeneracy of energy levels of two-level atoms, we would then have to generalise the Bloch equations. However, at the first stage, we can restrict our analysis to the minimum degeneracy degree.

Let ω_a be the frequency of transitions from the ground state to the lowest excited state. This frequency can be considered to be a natural time scale of the problem. When the pulse duration $t_{\rm p}$ satisfies the inequality $t_{\rm p}\omega_a \gg 1$, the propagation of ultrashort pulses can be described in the slowly varying envelope approximation. Applying this approximation we assume that the relevant conditions are satisfied for the variation rate of the envelope of ultrashort pulses. Conversely, provided that $t_{\rm p}\omega_a \leqslant 1$, this approximation becomes inapplicable, but we can at least employ the approximation of unidirectional wave propagation.

The model of a two-level medium provides us with yet another parameter that defines the time scale — the Rabi frequency $\omega_{\rm R}$. In addition, the ratio $\varepsilon = \omega_{\rm R}/\omega_a$ may be small. In this case, we can try to find the solution to the Bloch equation as a power series in ε. Then, dividing this series, we can find the polarisation with an accuracy up to some order of smallness in ε and, thus, derive an approximate wave equation for the electric field of a pulse without invoking the slowly varying envelope approximation. The condition $\varepsilon \approx 1$ implies that the electric field strength in the pulse is comparable with the strength of the atomic field. Therefore, the parameter ε controls the applicability of the notion of the strong field. With $\omega_{\rm R} \ll \omega_a$, the condition $\Delta\omega_{\rm p} \ll \omega_a$ is called the quasi-resonance condition.

We should note that the Bloch equations have been mentioned above in connection with the use of a resonance medium as an example of equations governing the behaviour of the polarisation of a medium. In other cases, equations related to other models of a nonlinear medium can be employed, including a model of an anharmonic oscillator, a model of an electron plasma in a metal, conductivity electrons in semiconductors, excitons in molecular crystals, or spin waves in magnetic dielectrics.

In the following sections, we will consider electromagnetic solitary waves whose durations are so small that the slowly varying envelope approximation is inapplicable. Some simple models employed in nonlinear optics will be considered as examples of nonlinear media where such pulses propagate. The choice of a specific model of a nonlinear medium makes it possible to derive equations describing approximately the evolution of ultrashort pulses. Our analysis employs neither the assumption that light pulses have slowly varying envelopes nor the quasi-harmonic approximation (which assumes that a harmonic wave can be selected as a carrier). Most of the equations presented below cannot be solved exactly with the use of analytical methods. However, we can find some particular solutions to these equations. The main part of this paper will be devoted to the consideration of such solutions.

2. Approximation of unidirectional waves

The set of Maxwell equations in an isotropic dielectric can be reduced to a single wave equation for the electric field strength $\boldsymbol{E} = E\boldsymbol{l}$. For a plane wave with a constant polarisation vector $\boldsymbol{l}$, we derive the wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}\,, \tag{2}$$

where the polarisation of the medium P is determined by the model chosen for the description of the nonlinear medium.

Following Refs [14, 42] we introduce an auxiliary function $B(z,t)$ satisfying the equation $\partial B/\partial t = c\partial E/\partial z$. Then, equation (2) can be represented in the equivalent form:

$$\frac{\partial B}{\partial t} - c\frac{\partial E}{\partial z} = 0\,, \qquad \frac{\partial E}{\partial t} - c\frac{\partial B}{\partial z} = -4\pi\frac{\partial P}{\partial t}\,.$$

Performing the relevant transformations, we arrive at

$$\left(\frac{\partial B}{\partial t} + c\frac{\partial B}{\partial z}\right) - \left(\frac{\partial E}{\partial t} + c\frac{\partial E}{\partial z}\right) = 4\pi\frac{\partial P}{\partial t}\,, \tag{3.1}$$

$$\left(\frac{\partial B}{\partial t} - c\frac{\partial B}{\partial z}\right) + \left(\frac{\partial E}{\partial t} - c\frac{\partial E}{\partial z}\right) = -4\pi\frac{\partial P}{\partial t}\,. \tag{3.2}$$

Equations determining the characteristics of this set are written as $\xi = t + z/c$ and $\eta = t - z/c$. Thus the characteristic form of Eqns (3) is

$$\frac{\partial B}{\partial \eta} + \frac{\partial E}{\partial \eta} = -2\pi\frac{\partial P}{\partial t}\,, \qquad \frac{\partial B}{\partial \xi} - \frac{\partial E}{\partial \xi} = 2\pi\frac{\partial P}{\partial t}\,. \tag{4}$$

Now, suppose that E, B, and P are the waves propagating predominantly in one direction, which is characterised, for example, by the parameter $\eta = t - z/c$. If the medium under consideration consisted of atoms or molecules responding linearly to the applied field, then all the variables E, B, and P would depend only on η. However, a reflected wave exists in the general case. Let us employ an auxiliary replacement $P \to \varepsilon P$ and expand E, B, and P as power series in ε assuming that the parameter ε is small:

$$E = E^{(0)}(\eta) + \varepsilon E^{(1)}(\eta,\,\xi) + \varepsilon^2 E^{(2)}(\eta,\,\xi) + \ldots\,,$$

$$B = B^{(0)}(\eta) + \varepsilon B^{(1)}(\eta,\,\xi) + \varepsilon^2 B^{(2)}(\eta,\,\xi) + \ldots\,,$$

$$P = P^{(0)}(\eta) + \varepsilon P^{(1)}(\eta,\,\xi) + \varepsilon^2 P^{(2)}(\eta,\,\xi) + \ldots\,.$$

Substituting these expansions into the first equation in the set (4), we derive in the first order in ε

$$\frac{\partial}{\partial \eta}\left(B^{(0)} + \varepsilon B^{(1)}\right) + \frac{\partial}{\partial \eta}\left(E^{(0)} + \varepsilon E^{(1)}\right) = -\varepsilon\frac{\partial P^{(0)}}{\partial \eta}\,,$$

or

$$B + E = -\varepsilon P^{(0)}\,.$$

In deriving the equations presented above, we assumed that $P^{(0)}$ is independent of ξ and the fields vanish simultaneously with the polarisation of the medium. To be more specific we assumed that the fields and the polarisation both vanish for $t \to \pm\infty$.

Substituting the expression derived above into the second equation in the set (4) and keeping the terms up to the first order in ε on the right-hand side of this equation, we find that

$$\frac{\partial E}{\partial \xi} = -\frac{\varepsilon}{2}\frac{\partial P^{(0)}}{\partial \eta}\,.$$

Using the initial variables z and t, we can rewrite this equation as

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t}\,, \tag{5}$$

which corresponds to the wave equation written in the approximation of a unidirectional wave.

3. Resonance media

To be able to apply Eqn (2) or (5) to describe the propagation of short pulses of electromagnetic radiation, we should specify how the polarisation of the nonlinear medium should be calculated. For many years, the approximation of a resonance medium has been the most popular model of a nonlinear dispersive medium. Moreover, such a nonlinear medium can be represented as an ensemble of two-level atoms. Among recent studies we should mention Refs [24, 25], which were devoted to the propagation of pulses with a duration of a few cycles of the carrier in a medium of two-level atoms. In such a situation, the rules that can be employed to calculate the polarisation of a medium are especially simple. These rules are formulated as the Bloch equations for a vector whose components are related in a certain way with the elements of the density matrix of a two-level atom (see the details in Ref. [14]).

3.1 The stationary solution to the Maxwell–Bloch equations

In the scalar approach free of the slowly varying envelope approximation, the Maxwell–Bloch equations are written as [13, 14, 43]

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi n_{\rm at} d}{c^2}\frac{\partial^2 r_1}{\partial t^2}\,, \tag{6.1}$$

$$\frac{\partial r_1}{\partial t} = -\omega_a r_2\,, \qquad \frac{\partial r_2}{\partial t} = \omega_a r_1 + \frac{2d}{\hbar}E r_3\,, \qquad \frac{\partial r_3}{\partial t} = -2dEr_2, \tag{6.2}$$

where $n_{\rm at}$ is the density of resonance atoms and d is the dipole moment of the atomic transition. These equations ignore the inhomogeneous broadening of the resonance absorption line (all the atoms have equal transition frequencies ω_a). Introducing new variables $\tau = \omega_a t$, $\xi = \omega_a z/c$, and $q(\tau,\xi) = 2dE/\hbar\omega_a$, we can rewrite these equations as

$$\frac{\partial^2 q}{\partial \xi^2} - \frac{\partial^2 q}{\partial \tau^2} = \alpha\frac{\partial^2 r_1}{\partial \tau^2}\,, \tag{7.1}$$

$$\frac{\partial r_1}{\partial \tau} = -r_2 , \quad \frac{\partial r_2}{\partial \tau} = r_1 + q r_3 , \quad \frac{\partial r_3}{\partial \tau} = -q r_2 , \qquad (7.2)$$

where $\alpha = 8\pi n_{\rm at} d^2/\hbar\omega_a$ is the dimensionless parameter that can be expressed in terms of the time $t_{\rm c}^{-1} = 4\pi n_{\rm at} d^2 \hbar^{-1}$ characteristic of a two-level system [44, 45]; $\alpha = 2/t_{\rm c}\omega_a$.

To derive equations governing the stationary propagation of a pulse of electromagnetic radiation, we should assume that the components of the Bloch vector and the normalised pulse envelope depend only on one variable $t \pm z/V$, or the dimensional variable $\varsigma = \omega_a(t \pm z/V)$, where V is the velocity of pulse propagation. This finding implies that stationary waves propagate only in one direction. In this case, the set (7) can be reduced to a set of ordinary equations. Writing the boundary conditions for $|\varsigma| \to \infty$ as $dq/d\varsigma = q = 0$, $r_1 = r_2 = 0$, and $r_3 = -1$, we find the following solution to the set of equations considered:

$$q(\varsigma) = \frac{2}{\theta}\,\mathrm{sech}\left[\frac{\omega_a}{\theta}\left(t \pm \frac{z}{V}\right)\right] ,$$

where $\theta^2 = (c^2 - V^2)T/[(1+\alpha)V^2 - c^2] > 0$. As can be seen from this expression, the duration of a stationary pulse can be defined as $t_{\rm p} = \theta/\omega_a$. Then the expression for the electric field strength can be written as

$$E(t, z) = E_0\,\mathrm{sech}\left[\frac{dE_0}{\hbar}\left(t \pm \frac{z}{V}\right)\right] , \qquad (8)$$

where $E_0 = \hbar t_{\rm p}^{-1} d^{-1}$ is the field amplitude. This expression for a stationary pulse coincides with the formula derived by Bullough and Ahmad [13] and presented by Bullough et al. [14]. Since expression (8) does not involve a carrier wave, such an electromagnetic pulse is an example of a video pulse.

The velocity of propagation of the stationary video pulse (8) can be found from the above-defined duration and the amplitude of this pulse. Similar to the McCall – Hahn theory of self-induced transparency (SIT) [28], these quantities are related to each other in our case in such a way that a pulse may invert a two-level system and switch it back to the initial state within the pulse duration. After some algebra, we find that

$$\frac{1}{V^2} = \frac{1}{c^2}\left(1 + \frac{\alpha\theta^2}{1+\theta^2}\right) = \frac{1}{c^2}\left[1 + \frac{4(t_{\rm p}/t_{\rm c})^2}{1 + (t_{\rm p}\omega_a)^2}\right] ,$$

or

$$\frac{1}{V^2} = \frac{1}{c^2}\left[1 + \frac{8\pi n_{\rm at} d^2 \hbar\omega_a}{(dE_0)^2 + (\hbar\omega_a)^2}\right] , \qquad (9)$$

which coincides with the expression presented by Bullough et al. [14].

Apart from solitary waves, the Maxwell – Bloch Eqns (6) or (7) allow for the existence of another class of stationary solutions — cnoidal waves. Solutions of this class describe periodic extended waves. Since Eqns (6) hold true so long as the duration of a solitary wave is less or much less than the relaxation times in the atomic subsystem, cnoidal waves simply represent a mathematical example that falls beyond the limits of the physical meaning of the starting equations. Nevertheless it seems appropriate to mention also this class of solutions to the Maxwell – Bloch equations in order to illustrate the properties of the model considered.

3.2 A video pulse of polarised radiation in a resonance medium

Consider a light pulse propagating in a resonance medium consisting of two-level atoms with quantum transitions between levels that are degenerate in projections of angular momenta j_a and j_b [46, 47]. We will study the case of $j_a = 1 \to j_b = 0$ transitions. It is convenient to introduce the following notations for the elements of the density matrix $\hat\rho$ governing transitions between the states $|a, m\rangle = |j_a = 1, m = \pm 1\rangle$ and $|b\rangle = |j_b = 0, m = 0\rangle$:

$$\rho_{12} = \langle a, -1|\hat\rho|a, +1\rangle , \quad \rho_{13} = \langle a, -1|\hat\rho|b\rangle ,$$

$$\rho_{23} = \langle a, +1|\hat\rho|b\rangle , \quad \rho_{11} = \langle a, -1|\hat\rho|a, -1\rangle ,$$

$$\rho_{22} = \langle a, +1|\hat\rho|a, +1\rangle , \quad \rho_{33} = \langle b|\hat\rho|b\rangle ,$$

$$\rho_{kl} = \rho_{lk}^* , \quad l, k = 1, 2, 3 .$$

The generalised set of Maxwell – Bloch equations can be written as

$$\frac{\partial E^{(+1)}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E^{(+1)}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\langle d_{13}\rho_{31} + d_{31}\rho_{13}\rangle , \qquad (10.1)$$

$$\frac{\partial E^{(-1)}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E^{(-1)}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\langle d_{23}\rho_{32} + d_{32}\rho_{23}\rangle , \qquad (10.2)$$

$$i\hbar\frac{\partial\rho_{13}}{\partial t} = -\hbar\omega_a\rho_{13} + d_{13}(\rho_{33} - \rho_{11})E^{(+1)} - d_{23}\rho_{12}E^{(-1)} , \qquad (11.1)$$

$$i\hbar\frac{\partial\rho_{23}}{\partial t} = -\hbar\omega_a\rho_{23} + d_{23}(\rho_{33} - \rho_{22})E^{(-1)} - d_{13}\rho_{21}E^{(+1)} , \qquad (11.2)$$

$$i\hbar\frac{\partial\rho_{12}}{\partial t} = d_{13}\rho_{32}E^{(+1)} - d_{32}\rho_{13}E^{(-1)} , \qquad (11.3)$$

$$i\hbar\frac{\partial}{\partial t}(\rho_{11} - \rho_{33}) = 2(d_{13}\rho_{31} - d_{31}\rho_{13})E^{(+1)} + (d_{23}\rho_{32} - d_{32}\rho_{23})E^{(-1)} , \qquad (11.4)$$

$$i\hbar\frac{\partial}{\partial t}(\rho_{22} - \rho_{33}) = (d_{13}\rho_{31} - d_{31}\rho_{13})E^{(+1)} + 2(d_{23}\rho_{32} - d_{32}\rho_{23})E^{(-1)} . \qquad (11.5)$$

Here, $E^{(q)}$ is the spherical q component of the vector of the electric field strength in the light wave ($q = \pm 1$), d_{kl} are the matrix elements of the dipole moment operator of the atomic transition $j_a = 1 \to j_b = 0$ ($d_{13} = d_{23} = d_{31}^* = d_{32}^*$), and the angle brackets indicate summation over all the atoms with the frequency ω_a.

It would be convenient to use real variables and to introduce the dimensionless electric field strength:

$$\rho_{13} = r_1 + i r_2 , \quad \rho_{23} = s_1 + i s_2 ,$$

$$\rho_{12} = p_1 + i p_2 , \quad \rho_{33} - \rho_{11} = n_1 ,$$

$$\rho_{33} - \rho_{22} = n_2 , \quad \frac{dE^{(+1)}}{\hbar\omega_a} = q_1 , \quad \frac{dE^{(-1)}}{\hbar\omega_a} = q_2 . \qquad (12)$$

Employing real variables, we can rewrite the considered set of equations as

$$\frac{\partial r_1}{\partial \tau} = -r_2 - q_2 p_2\,, \quad \frac{\partial r_2}{\partial \tau} = r_1 - q_1 n_1 + q_2 p_1\,,$$

$$\frac{\partial s_1}{\partial \tau} = -s_2 + q_1 p_2\,, \quad \frac{\partial s_2}{\partial \tau} = s_1 - q_2 n_2 + q_1 p_1\,,$$

$$\frac{\partial p_1}{\partial \tau} = -q_1 s_2 - q_2 r_2\,, \quad \frac{\partial p_2}{\partial \tau} = -q_1 s_1 + q_2 r_1\,, \tag{13.1}$$

$$\frac{\partial n_1}{\partial \tau} = 4 q_1 r_2 + 2 q_2 s_2\,, \quad \frac{\partial n_2}{\partial \tau} = 2 q_1 r_2 + 4 q_2 s_2\,,$$

$$\frac{\partial^2 q_1}{\partial \xi^2} - \frac{\partial^2 q_1}{\partial \tau^2} = \alpha \frac{\partial^2 r_1}{\partial \tau^2}\,, \quad \frac{\partial^2 q_2}{\partial \xi^2} - \frac{\partial^2 q_2}{\partial \tau^2} = \alpha \frac{\partial^2 s_1}{\partial \tau^2}\,, \tag{13.2}$$

where $\tau = \omega_a t$; $\xi = \omega_a z c^{-1}$, and the parameter α appeared above in equation (7.1).

Eqns (13) describe the propagation of ultrashort (including extremely short) pulses of polarised electromagnetic radiation in a resonance medium. It is unlikely that the complete solution to this set of equations can be found in the analytical form. However, following the conventional approach to the investigation of nonlinear waves, we can find stationary solutions.

To do this, we assume again that the sought-for solution to the set of Eqns (13) is described by functions depending on the variable $\varsigma = \omega_a(t \pm z/V)$. Omitting the details of the derivation of the set of equations for normalised electric field strengths q_1 and q_2, presented in Ref. [48], we will write this set of equations in the final form:

$$\frac{d^2 q_1}{dy^2} + \left(q_1^2 + q_2^2\right) q_1 = \frac{1}{2}\left(q_0^2 n_{10} - 1\right) q_1\,, \tag{14.1}$$

$$\frac{d^2 q_2}{dy^2} + \left(q_1^2 + q_2^2\right) q_2 = \frac{1}{2}\left(q_0^2 n_{20} - 1\right) q_2\,, \tag{14.2}$$

where $y = \sqrt{2}\varsigma$; and $q_0^2 = \alpha V^2 (c^2 - V^2)^{-1}$.

Let us introduce parameters $a_i^2 = (q_0^2 n_{i0} - 1)/2$, $i = 1, 2$. If the populations of excited levels are equal to each other and all the atoms are in the ground state, we should set $n_{10} = n_{20} = 1$ and $a_1 = a_2 = \Omega$, which corresponds to the case of an initially nonpolarised resonance medium. The solution to the set (13) can be written in the following simple form:

$$q_1(\varsigma) = \sqrt{2\Omega} e^{(+)} \operatorname{sech}\left(\sqrt{2\Omega}\varsigma\right)\,,$$

$$q_1(\varsigma) = \sqrt{2\Omega} e^{(-)} \operatorname{sech}\left(\sqrt{2\Omega}\varsigma\right)\,,$$

where $e^{(\pm)}$ are the components of the unit vector determining the polarisation states of the field in the pulse of electromagnetic radiation. The solution presented above depends on the propagation velocity. However, it would be more convenient to employ the parameter q_0^2. The real variables of the density matrix of the medium are related to these solutions by the formulas:

$$q_1 = q_0^2 r_1\,, \quad q_2 = q_0^2 s_1\,,$$

$$\frac{dq_1}{d\varsigma} = -q_0^2 r_2\,, \quad \frac{dq_2}{d\varsigma} = -q_0^2 s_2\,,$$

$$p_1 = q_0^{-2} q_1 q_2\,, \quad n_1 + q_0^{-2}\left(2q_1^2 + q_2^2\right) = n_{10}\,,$$

$$n_2 + q_0^{-2}\left(q_1^2 + 2q_2^2\right) = n_{20}\,.$$

Now, switching back to the initial physical variables, we can represent the electric field strength in the pulse of electromagnetic radiation as

$$E^{(\pm)}(t, z) = e^{(\pm)} E_0 \operatorname{sech}\left[\frac{dE_0}{\hbar}\left(t \pm \frac{z}{V} - t_0\right)\right]. \tag{15}$$

Eqn (15) is a simple extension of the results presented in Ref. [13] to the case of a vector (polarised) ultrashort pulse and a specific model of a resonance medium. Similar to the expressions derived in Ref. [13], the duration t_p of a stationary ultrashort pulse in Eqn (15) is expressed in terms of the peak amplitude of the pulse, $t_p = \hbar (dE_0)^{-1}$, whereas the velocity of pulse propagation can be determined from the definitions of parameters q_0 and Ω:

$$\frac{1}{V^2} = \frac{1}{c^2}\left[1 + \frac{8\pi n_{at} |d|^2 \hbar\omega_a}{(\hbar\omega_a)^2 + 2(dE_0)^2}\right].$$

Along with solution (15), which describes a polarised video pulse, we can also formally derive periodic solutions corresponding to nonlinear anharmonic waves, including cnoidal waves.

The populations of excited states corresponding to different projections of the angular momentum are usually equal to each other. However, we can break this symmetry, for example by irradiating a resonance medium with weak circularly polarised light. Then, in the case of cw radiation, the steady-state population difference between energy levels coupled by different transitions depends on the intensity and the polarisation type of incident radiation. For a high-power short pulse passing through a medium prepared in such a way, the medium becomes polarised; i.e., the populations of excited levels differ from each other: $n_{10} \neq n_{20}$. For such a medium, we should set $a_1 \neq a_2$ in Eqn (14). If, following Refs. [49, 50], we set $q_1 = g/f$ and $q_2 = h/f$, then equations (14) can be rewritten in the bilinear form:

$$D^2(g \cdot f) = a_1^2 g f\,, \quad D^2(h \cdot f) = a_2^2 h f\,,$$
$$D^2(f \cdot f) = g^2 + h^2\,, \tag{16}$$

where $D(a \cdot b) = (da/dy)b - a(db/dy)$ are the Hirota operators [51, 52].

These bilinear equations can be solved in the following way. The functions g, h, and f are represented as polynomials, e.g.,

$$g = \varepsilon g_1 + \varepsilon^3 g_3\,, \quad h = \varepsilon h_1 + \varepsilon^3 h_3\,, \quad f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4\,.$$

Substituting these expansions into Eqns (16) and equating the coefficients appearing with the same degrees of ε, we arrive at a set of chained linear equations with variable coefficients:

$$D^2(g_1 \cdot 1) = a_1^2 g_1\,,$$
$$D^2(h_1 \cdot 1) = a_2^2 h_1\,,$$
$$2D^2(f_2 \cdot 1) = g_1^2 + h_1^2\,,$$
$$D^2(g_3 \cdot 1) = a_1^2 g_3 + a_1^2 g_1 f_2 - D^2(g_1 \cdot f_2)\,,$$
$$D^2(h_3 \cdot 1) = a_2^2 h_3 + a_2^2 h_1 f_2 - D^2(h_1 \cdot f_2)\,,$$
$$2D^2(f_4 \cdot 1) = 2(g_1 g_3 + h_1 h_3) - D^2(f_2 \cdot f_2)\,,$$
$$D^2(g_3 \cdot f_2) + D^2(g_1 \cdot f_4) = a_1^2(g_1 f_4 + g_3 f_2)\,,$$
$$D^2(h_3 \cdot f_2) + D^2(h_1 \cdot f_4) = a_2^2(h_1 f_4 + h_3 f_2)\,,$$

$$2D^2(f_2 \cdot f_4) = g_3^2 + h_3^2 , \quad D^2(g_3 \cdot f_4) = a_1^2 g_3 f_4 ,$$
$$D^2(h_3 \cdot f_4) = a_2^2 h_3 f_4 , \quad D^2(f_4 \cdot f_4) = 0 .$$

Solving these equations sequentially, we derive the solution to the set (16):

$$g = 2\sqrt{2} a_1 \exp(\theta_1)\left[1 + \exp\left(2\theta_2 + a_{12}\right)\right] ,$$

$$h = 2\sqrt{2} a_2 \exp(\theta_2)\left[1 - \exp\left(2\theta_1 + a_{12}\right)\right] ,$$

$$f = 1 + \exp(2\theta_1) + \exp(2\theta_2) + \exp(2\theta_1 + 2\theta_2 + a_{12}) ,$$

where $\exp a_{12} = (a_1 - a_2)/(a_1 + a_2)$, $\theta_{1,2} = a_{1,2}(y - y_{1,2})$; $y_{1,2}$ are the integration constants, and the other integration constants are chosen in such a way as to obtain the solution in the form of a solitary wave. Now, the solution to the starting set of equations (14) can be written as

$$q_1(y) = \frac{2\sqrt{2}\exp(\theta_1)\left[1+\exp\left(2\theta_2 + a_{12}\right)\right]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\exp(2\theta_1 + 2\theta_2 + a_{12})} , \quad (17)$$

$$q_2(y) = \frac{2\sqrt{2}\exp(\theta_2)\left[1 - \exp\left(2\theta_2 + a_{12}\right)\right]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\exp(2\theta_1 + 2\theta_2 + a_{12})} . \quad (18)$$

Setting $n_{10} = n_{20}$, i.e., considering the case of a nonpolarised medium, we reduce the solution derived above to $q_1(y) = q_2(y) = a_1 \mathrm{sech}[a_1(y - y_1)]$. Rewriting this solution in terms of the initial physical variables, we arrive at the expression for the electric field in a circularly polarised video pulse determined by formula (15).

Thus we analysed the propagation of an extremely short stationary pulse of an electromagnetic field in a resonance medium with a degenerate upper energy level. This analysis has shown that the parameters of such a pulse depend on the state of the medium. The solution known for the scalar case [13, 14] can be easily generalised to the case of a vector video pulse if the condition $n_{10} = n_{20} = 1$ is satisfied for the population differences between the ground and excited states with different projections of the angular momentum. Thus we obtain a circularly polarised half-cycle pulse corresponding to an atomic transition. The velocity of such a video pulse is independent of polarisation and coincides with the velocity obtained by Bullough and Ahmad [13].

The new solution to the complete set of Maxwell – Bloch equations exists if a previously prepared resonance medium has an asymmetric population distribution in excited states with different projections of the angular momentum, $n_{10} \neq n_{20}$. One of the spherical components of the vector of the electric field strength behaving in a way similar to the electric field in the scalar case corresponds to a unipolar transient of the electric field. The second component corresponds to a sign-alternating solitary wave. Fig. 1 presents time dependences of the spherical components of the vector of the electric field strength in a light pulse for two different values of the parameters n_{10} and n_{20}. As $q_0^2 n_{01}$ or $q_0^2 n_{02}$ tends to unity from the right, the oscillating component vanishes, and the video pulse as a whole becomes circularly polarised.

A similar consideration can be performed for pulses of polarised radiation propagating in media with $j_a = 0 \to j_b = 1$ and $j_a = 1 \to j_b = 1$ transitions.

Based on solution (15), we can provide a more detailed analysis of the applicability of the model of two-level atoms. Let us find the spectral half-width of the video pulse. If the duration of the video pulse is defined as $\tau_p = (2\hbar d E_0)^{-1}$,

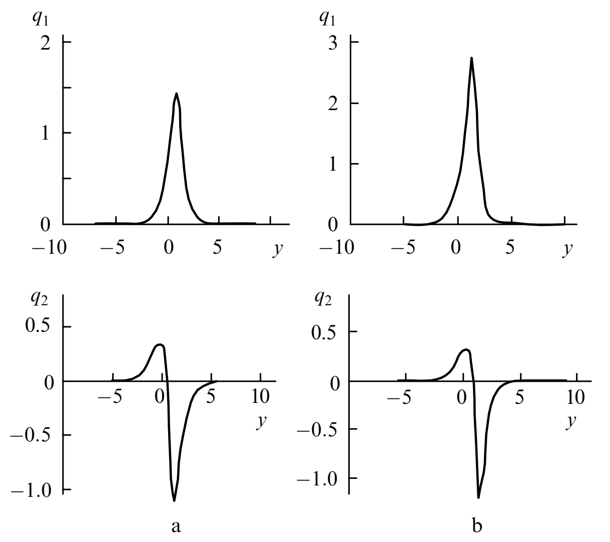

Figure 1. Dependence of spherical components of a video pulse on the dimensionless time $y = \sqrt{2}\omega_a(t \pm z/V)$ for (a) $q_0^2 n_{01} = 4$ and $q_0^2 n_{02} = 7$ and (b) $q_0^2 n_{01} = 9$ and $q_0^2 n_{02} = 8$.

then the Fourier transform of this pulse is given by

$$E(\omega) = \int_{-\infty}^{+\infty} \left|E^{(q)}(t, z)\right| \exp(\mathrm{i}\omega t)\mathrm{d}t = \frac{\pi}{2} E_0 \tau_p \,\mathrm{sech}\left(\frac{\pi}{4}\omega\tau_p\right) .$$

Consequently, the spectral half-width $\Delta\omega_p$ of the video pulse is determined from the relation $\Delta\omega_p \tau_p = (4/\pi)\ln(2 + \sqrt{3}) \approx 1.677$.

In contrast to the spectrum of a quasi-monochromatic wave, $E(\omega)$ reaches its maximum at the point $\omega = 0$. However, the amplitudes of Fourier components constituting the wave packet (15) exponentially decrease with increase in ω. Thus, the model of a two-level atom is applicable in the case when $\hbar\Delta\omega_p$ is less than, for example, one tenth of the energy gap between the excited and ground states.

The application of the two-level model implies that we ignore cascade transitions. The role of transitions of this type was considered by Kaplan [19].

An additional restriction on the duration of a video pulse described by both formula (15) and expressions (17) and (18) is associated with the photoionisation limit for the amplitude of the electric field: $E_0 \leqslant E_{at} \sim 10^9$ V cm^{-1}. Since the duration τ_p is related to E_0, we can infer that $\tau_p \geqslant \tau_{at}$, where $\tau_{at} = 2\hbar(dE_{at})^{-1}$. Suppose that $d = 1$ D. Then we have $\tau_{at} \approx 70$ fs. Applying the relation between τ_p and $\Delta\omega_p$ derived above, we arrive at the photoionisation limit for the spectral half-width of a video pulse: $\Delta\omega_p \leqslant \Delta\omega_{ph} \approx 2.25 \times 10^{13}$ s^{-1}.

Consequently, stationary video pulses may propagate in resonance media, where the energy of excited states exceeds the energy of the ground state by at least $10\hbar\Delta\omega_{ph}$ (which corresponds to approximately 2×10^{-20} J). When the duration of a video pulse is less than this limit ($\tau_{at} \approx 70$ fs), then the amplitude of the stationary pulse may become so high that the perturbative treatment of the interaction of atoms with the electromagnetic field may become inadequate. Kaplan and Shkolnikov [20], for example, proposed to consider an atomic system classically if the strengths of the electric and atomic fields are comparable with each other. In such a situation, it would be interesting to investigate the interaction of a video pulse of polarised radiation with a nonlinear oscillator with two (or more) degrees of freedom.

3.3 The propagation of pulses of unidirectional waves

The stationary video pulses considered above are particular examples demonstrating the possibility of propagation of extremely short pulses. To find a broader class of solitary waves of this type by using analytical methods, we have to make additional assumptions. As mentioned above, the condition of the existence of stationary video pulses implies the implicit choice of one of possible directions of pulse propagation. Thus it would be natural to adopt from the very beginning the approximation of unidirectional waves in the basic complete set of Maxwell – Bloch equations without restricting ourselves to the requirement that the solutions should be stationary.

The set of equations governing the propagation of short electromagnetic pulses in this approximation can be derived from the complete set (6) by replacing the wave Eqn (6.1) by the reduced wave equation in accordance with the rule described in Section 2. Thus, Eqn (5) in the case under consideration can be rewritten as

$$\frac{\partial E}{\partial z}+\frac{1}{c}\frac{\partial E}{\partial t}=-\left(\frac{2\pi d}{c}\right)\left\langle\frac{\partial r_1}{\partial t}\right\rangle .$$

The Bloch Eqns (6.2) remain unchanged. Using new normalised variables $\tau=\omega_a(t-z/c)$, $\zeta=(4\pi n_{\rm at}d^2/c\hbar)z$, and $q=(2d/\hbar\omega_a)E$, we can represent the reduced Maxwell – Bloch (RMB) equations in the following form:

$$\frac{\partial q}{\partial \zeta}=-\frac{\partial}{\partial \tau}\langle r_1\rangle , \qquad \frac{\partial r_1}{\partial \tau}=-r_2 ,$$
$$\frac{\partial r_2}{\partial \tau}=r_1+qr_3 , \qquad \frac{\partial r_3}{\partial \tau}=-qr_2 . \tag{19}$$

Here, the angle brackets indicate summation over all the two-level atoms and division of the resulting sum by the concentration of these atoms $n_{\rm at}$.

It is well known [14, 15, 53] that the set of RMB equations can be represented as the condition of compatibility for a pair of linear matrix equations, which provides a background for the application of the method of the inverse scattering problem (ISP) [54, 55] to the solution of the set (19). Assuming that all the atoms of a resonance medium are in the ground state before the onset of the light pulse and after the propagation of the light pulse through the medium, we can find exact solutions to the RMB equations describing the propagation of video pulses and ultrashort pulses with a carrier. All the details of such a solution of the set (19) by the ISP method can be found in Ref. [53]. Below we present only the results of this analysis.

Generally, an N-soliton solution to the RMB equation is written as

$$q(\zeta,\,\tau)^2=4\frac{{\rm d}^2}{{\rm d}\tau^2}\ln\det\left(1+\hat H^*\hat H\right) , \tag{20}$$

where the matrix $\hat H$ is defined by its matrix elements

$$H_{nm}=\frac{(C_nC_m)^{1/2}\exp[{\rm i}\tau(\lambda_n-\lambda_m^*)]}{\lambda_n-\lambda_m^*} ,$$
$$C_n(\zeta)=C_n(0)\exp\left(\left\langle\frac{2{\rm i}\lambda_n\omega_a}{4\lambda_n^2-\omega_n^2}\right\rangle\zeta\right) .$$

The complex numbers λ_n and C_n, where $n,\ m=1,\,2,\ldots,N$, are determined by the initial conditions for the electric field in the pulse at the input of the medium (for $\zeta=0$).

Following Ref. [53] we can reduce expression (20) to a more symmetric and convenient form. First, we define matrix $\hat J$ such that

$$(\hat H\times\hat J\,)_{nm}=\frac{-{\rm i}\exp[{\rm i}(\lambda_n-\lambda_m)\tau-\alpha_n-\alpha_m]}{\lambda_n+\lambda_m} , \tag{21.1}$$

$$(\hat J\times\hat H^{-1})_{nm}=\frac{-{\rm i}\exp[{\rm i}(\lambda_n+\lambda_m)\tau+\alpha_n+\alpha_m+2(\beta_n+\beta_m)]}{\lambda_n+\lambda_m} , \tag{21.2}$$

where the parameters α_n and β_n can be determined from the following expressions:

$${\rm i}C(\zeta)=\exp[-2\alpha_n(\beta\zeta)] ,$$
$$\prod_j(\lambda_j+\lambda_n)\left[\prod_{j\neq n}(\lambda_j-\lambda_n)\right]^{-1}=-{\rm i}\exp(2\beta_n) .$$

Recall that $q(\zeta,\tau)$ is a real quantity. Consequently, the numbers λ_m and c_n are either purely imaginary or form anti-Hermitian pairs, $\lambda_m^*=-\lambda_n$ and $C_m^*=-C_n$. Here, we do not discuss the proof of this statement, which is based on the properties of the spectral problem of the ISP method $\hat H\times\hat J=\hat J\times\hat H^*$ holds true, and Eqns (21) yield

$$\left(\hat J\times\hat H^{-1}+\hat J\times\hat H^*\right)_{nm}=M_{nm}\exp(\beta_n+\beta_m) ,$$

where

$$M_{nm}=\frac{\cosh(\vartheta_n+\vartheta_m)}{2{\rm i}(\lambda_n+\lambda_m)} ,$$
$$\vartheta_n=\frac14\left(k_n\tau-\left\langle\frac{4\omega_ak_n}{k_n^2+\omega_a^2}\right\rangle\zeta+\delta_n\right) , \qquad k_n=4{\rm i}\lambda_n .$$

Using these expressions, we can rewrite formula (20) in a more elegant form:

$$q(\zeta,\tau)^2=4\frac{{\rm d}^2}{{\rm d}\tau^2}\ln\det\left(\hat M\right) . \tag{22}$$

Since all the numbers λ_m can be grouped into L_1 purely imaginary numbers and L_2 anti-Hermitian pairs (with $N=L_1+2L_2$), N-soliton solutions to the RMB equations consist of L_1 fundamental solitons (1-solitons) and L_2 breathers (or bions, i.e., coupled soliton – antisoliton pairs). Breathers are defined as stable solitary waves that display intrinsic oscillations. In the theory of self-induced transparency, such solitary waves are called 0π pulses, as opposed to 2π pulses, which correspond to 1-solitons [14, 43]. Breathers, similar to 1-solitons, are stable with respect to collisions with each other and with other solitons.

The solution to the RMB equations corresponding to a 1-soliton is written as

$$q\omega_a(\zeta,\tau)=k_1{\rm sech}\left[\frac12k_1\left(t-\frac{z}{V_1}\right)\right] , \tag{23}$$

where $V_1=c[1+\langle\alpha'\omega_a(k_1^2+4\omega_a^2)^{-1}\rangle]^{-1}$ is the group velocity, and $\alpha'=4\pi n_{\rm at}d^2/\hbar\omega_a$. This solution is an RMB version of the stationary solution to the complete set of Maxwell – Bloch Eqns (8.2). An electromagnetic pulse of this type has no carrier wave and represents a unipolar transient of electromagnetic radiation. Such a wave can be considered to be a video pulse according to the definition given above.

One of the two-soliton solutions to the RMB Eqns (19) is written as

$$q\omega_a(t, z) = \left(\frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}\right) \times \frac{k_1 \operatorname{sech} \vartheta_1 + k_2 \operatorname{sech} \vartheta_2}{1 - B_{12}(\tanh \vartheta_1 - \tanh \vartheta_2 - \operatorname{sech} \vartheta_1 \operatorname{sech} \vartheta_2)}, \quad (24)$$

where

$$B_{12} = \frac{2k_1 k_2}{k_1^2 + k_2^2}, \quad \vartheta_n = \frac{k_n}{2}\left(t - \frac{z}{c}\left\langle 1 + \frac{4\alpha'\omega_a}{k_n^2 + 4\omega_a^2}\right\rangle\right).$$

This solution describes the collision of two video pulses (Fig. 2) in the same sense as in the collision of two McCall – Hahn 2π pulses in the theory of self-induced transparency. However, a 2π pulse is a soliton whose envelope varies slowly in time, whereas in the case under consideration, we deal with real strengths of the electric field. Moreover, the two-soliton solution to the set of Eqns (19) can be employed to generalise a concept of a 2π pulse. Suppose that λ_1 and λ_2 or k_1 and k_2 form a pair of anti-Hermitian complex numbers (i.e., $k_1 = -k_2^*$). In this case, the two-soliton solution to RMB equations corresponds to a breather representing a real solitary wave with intrinsic oscillations (Fig. 3). The envelope of this wave is an analogue of a McCall – Hahn 0π pulse.

Indeed, suppose that $k_1 = -k_2^* = k_0 + 2\mathrm{i}\Omega$ and $\delta_1 = -\delta_2^* = \delta' + \mathrm{i}\delta''$. Then, expression (24) gives the exact solution to Eqns (19) [43]:

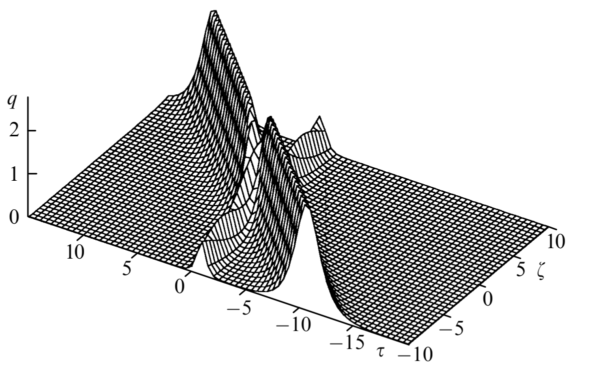

Figure 2. Collision of two video pulses in accordance with Eqn (24) for normalised pulse durations $\tau_{p1,2} = 1$ and 0.5, respectively.

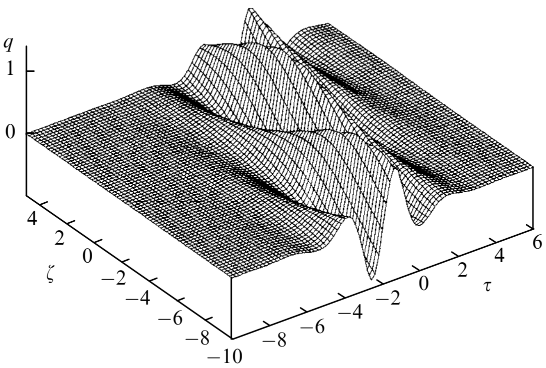

Figure 3. A video pulse in the form of a breather with a few cycles of the electric field strength, which is similar to the McCall – Hahn 0π pulse.

$$q\omega_a(t, z) = 2k_0 \operatorname{sech} \vartheta_{\mathrm{real}}\left(\frac{\cos \vartheta_{\mathrm{im}} - \gamma \sin \vartheta_{\mathrm{im}} \tanh \vartheta_{\mathrm{real}}}{1 + \gamma^2 \sin^2 \vartheta_{\mathrm{im}} \operatorname{sech}^2 \vartheta_{\mathrm{real}}}\right), \quad (25)$$

where $\gamma = k_0/2\Omega$ and

$$\vartheta_{\mathrm{real}} = \frac{1}{2}k_0\left\{t - \frac{z}{c}\left\langle 1 + \frac{4\alpha'\omega_a\left[k_0^2 + 4(\omega_a^2 + \Omega^2)\right]}{k_0^2 + 8k_0^2(\omega_a^2 + \Omega^2) + 16(\omega_a^2 + \Omega^2)^2}\right\rangle\right\} + \delta',$$

$$\vartheta_{\mathrm{im}} = \Omega\left\{t - \frac{z}{c}\left\langle 1 + \frac{4\alpha'\omega_a\left[4(\omega_a^2 - \Omega^2) - k_0^2\right]}{k_0^2 + 8k_0^2(\omega_a^2 + \Omega^2) + 16(\omega_a^2 + \Omega^2)^2}\right\rangle\right\} + \delta''.$$

Similar to the theory of self-induced transparency for a 0π pulse, the solution to the RMB equations given by Eqns (25) describes a pulse with a zero area. Let us choose parameters k_0 and Ω such that $k_0 \ll \Omega$. Then, expanding expression (25) up to the zeroth order in γ, we find that

$$q\omega_a(t, z) \approx 2k_0 \operatorname{sech} \vartheta_{\mathrm{real}} \cos \vartheta_{\mathrm{im}}.$$

Thus, a McCall – Hahn 2π pulse is the limiting case of a breather corresponding to the set of RMB Eqns (19). Expanding expression (25) as a power series in γ, we can find corrections to the stationary pulse in the theory of self-induced transparency (i.e., corrections to a 2π pulse). In the first order in γ, expression (25) yields

$$q\omega_a(t, z) \approx 2k_0 \operatorname{sech} \vartheta_{\mathrm{real}} \cos[\vartheta_{\mathrm{im}} + \phi(t, z)], \quad (26)$$

where $\phi(t, z) = \gamma \tanh \vartheta_{\mathrm{real}}$. This formula describes a phase-modulated (or chirped) 2π pulse. Defining the instantaneous frequency as $\Delta\omega_{\mathrm{ch}} = \partial\phi/\partial t$, we find that $\Delta\omega_{\mathrm{ch}} = \gamma^2\omega_a \operatorname{sech}^2 \vartheta_{\mathrm{real}}$.

Thus the theory of propagation of extremely short pulses of electromagnetic radiation based on the RMB Eqns (19) is an interpolating theory, which describes the range of parameters stretching from the case of video pulses arising in the form of unipolar transients of the electromagnetic field to the case of ultrashort pulses whose envelopes vary slowly in space and time. The disadvantages of this theory stem from the model of two-level atoms constituting a resonant medium within the framework of this approach.

4. Two-level media under quasi-resonance conditions

If a two-level medium is irradiated with a monochromatic wave with amplitude E_{m}, then the populations of resonant levels under conditions of exact resonance vary periodically in time with the Rabi frequency $\omega_{\mathrm{R}} = dE_{\mathrm{m}}/\hbar$. Generally, when the electromagnetic wave is not monochromatic, we can formally define the instantaneous Rabi frequency following the same approach and understanding E_{m} as the instantaneous strength of an electromagnetic radiation pulse. The Rabi frequency thus defined is not associated with the oscillation frequency of populations in the relevant energy levels, but it serves as a measure of the electric field strength in the radiation wave.

Suppose that the pulse amplitude is such that the Rabi frequency is low compared with the frequency of the resonance transition. Then, a small parameter $\varepsilon = \omega_{\mathrm{R}}/\omega_a$ can be introduced in the theory describing the propagation of such pulses. Now we can try to solve approximately the Bloch equations by representing the solutions to these equations as power-series expansions in ε. Substituting such solutions into the formulas describing the polarisation of the medium, we can derive a nonlinear wave equation governing the evolution of the electric field strength in an electromagnetic pulse.

4.1 The case of a scalar wave

Consider the Bloch Eqns (6.2) governing the evolution of a two-level atom under the action of an electromagnetic pulse characterised by a scalar electric field strength E. Introducing new variables $q = p = \mathrm{i}E/|E_0|$, $B = r_1 + \mathrm{i}r_2$, and $C = r_1 - \mathrm{i}r_2$, we can represent Eqns (6.2) as

$$\omega_{\mathrm{R}} \frac{\partial B}{\partial T} - \mathrm{i}\omega_a B = 2\omega_{\mathrm{R}} q r_3 \,, \qquad \omega_{\mathrm{R}} \frac{\partial C}{\partial T} + \mathrm{i}\omega_a C = -2\omega_{\mathrm{R}} p r_3 \,, \tag{27.1}$$

$$\frac{\partial r_3}{\partial T} = Bp - Cq \,, \tag{27.2}$$

where $\omega_{\mathrm{R}} = d|E_0|/\hbar$; $|E_0|$ is the maximum value of $|E|$, and $T = \omega_{\mathrm{R}} t$. The formal integration of Eqn (27.2) yields

$$r_3 = \sigma + \int_{-\infty}^{T} (Bp - Cq)\, \mathrm{d}T' \,,$$

where $\sigma = -1$ for an absorbing medium. Introducing two-component vectors $\boldsymbol{\chi} = \mathrm{colon}(B, C)$ and $\boldsymbol{\psi} = \mathrm{colon}(q, p)$, we can represent Eqns (27.1) as a single vector equation:

$$\left(\omega_{\mathrm{R}} \hat{R} - \mathrm{i}\omega_a\right)\boldsymbol{\chi} = 2\omega_{\mathrm{R}} \sigma \boldsymbol{\psi} \,, \tag{28}$$

where

$$\hat{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial T} - 2 \begin{pmatrix} q\int p & -q\int q \\ p\int p & -p\int q \end{pmatrix}$$

is the matrix operator, which is also employed in the theory of solitons [55]. The integral operators $u\int v$, involved in the expression for $\hat{R}$, are defined in the following way:

$$\left(u\int v\right) f(T) = u(T)\int_{-\infty}^{T} v(t)\, f(t)\mathrm{d}t \,, \qquad \forall f(t) \,.$$

Using the resolvent operator $\hat{G} = (1 + \mathrm{i}\varepsilon\hat{R})^{-1}$, we can write the solution to Eqn (28) as $\boldsymbol{\chi} = 2\mathrm{i}\sigma\varepsilon\hat{G}\boldsymbol{\psi}$. Since the parameter $\varepsilon = \omega_{\mathrm{R}}/\omega_a$ is small, this operator can be represented as a power series in ε:

$$\hat{G} = \left(1 + \mathrm{i}\varepsilon\hat{R}\right)^{-1} = 1 - \mathrm{i}\varepsilon\hat{R} - \varepsilon^2\hat{R}^2 + \ldots \,.$$

Using the relations that follow from the definition of the operator $\hat{R}$,

$$\hat{R}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \dfrac{\partial q}{\partial T} \\ -\dfrac{\partial p}{\partial T} \end{pmatrix}, \qquad \hat{R}^2\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 q}{\partial T^2} - 2q(pq) \\ \dfrac{\partial^2 p}{\partial T^2} - 2p(qp) \end{pmatrix},$$

with an accuracy up to the second order in ε, we find that

$$B = 2\mathrm{i}\varepsilon\sigma\left[q - \mathrm{i}\varepsilon\frac{\partial q}{\partial T} - \varepsilon^2\frac{\partial^2 q}{\partial T^2} + 2\varepsilon^2 q(qp)\right] \,,$$

$$C = 2\mathrm{i}\varepsilon\sigma\left[p + \mathrm{i}\varepsilon\frac{\partial p}{\partial T} - \varepsilon^2\frac{\partial^2 p}{\partial T^2} + 2\varepsilon^2 p(pq)\right] \,.$$

Then, taking into account that $r_1 = (B + C)/2$ and $q = p$, we derive the expression for the polarisation per single atom: $r_1 = 2\mathrm{i}\varepsilon\sigma(q - \varepsilon^2\partial^2 q/\partial T^2 + 2\varepsilon^2 q^3)$. Thus the expression for the mean polarisation of a single atom,

$$\langle r_1 \rangle = -\left\langle \frac{2d\sigma}{\hbar\omega_a} \right\rangle E + \left\langle \frac{2d\sigma}{\hbar\omega_a^3} \right\rangle \frac{\partial^2 E}{\partial t^2} + \left\langle \frac{4d^3\sigma}{\hbar^3\omega_a^3} \right\rangle E^3 \,. \tag{29}$$

can be substituted into the right-hand side of the wave Eqn (6.1).

In what follows, we will consider several examples of wave equations.

4.2 The nonlinear wave equation

Substituting expression (29) into Eqn (6.1) and taking into account the inhomogeneous broadening of the resonance absorption line, we arrive at the following equation [56]:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(a_1 E^3 + b_1\frac{\partial^2 E}{\partial t^2}\right) \,, \tag{30}$$

where the coefficients a_1 and b_1 and the renormalised velocity V of the electromagnetic radiation pulse are given by

$$a_1 = \left\langle \frac{16\pi n_{\mathrm{at}}\sigma|d|^4}{c^2\hbar^3\omega_a^3} \right\rangle, \qquad b_1 = \left\langle \frac{8\pi n_{\mathrm{at}}\sigma|d|^2}{c^2\hbar\omega_a^3} \right\rangle \,,$$

$$\frac{1}{V^2} = \frac{1}{c^2}\left(1 - \left\langle \frac{8\pi n_{\mathrm{at}}\sigma|d|^2}{\hbar\omega_a} \right\rangle\right) \,.$$

Introducing new variables $\tau = (|b_1|V^4)^{-1/2}z$ and $\zeta = (|b_1|V^2)^{-1/2}t$ and the normalised electric field strength in the radiation pulse $u(\tau, \zeta) = (|a_1|V^2)^{1/2}E(z, t)$, we can represent Eqn (30) as

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2}{\partial \zeta^2}\left(u + \sigma u^3 + \sigma\frac{\partial^2 u}{\partial \zeta^2}\right), \tag{31}$$

where $\sigma = \mathrm{sign}\, a_1 = \mathrm{sign}\, b_1$ determines the sign of the equilibrium population difference between the resonance energy levels. This nonlinear wave equation governs the propagation of extremely short radiation pulses in a dispersive nonlinear medium in the case when the amplitudes of these pulses are small compared with the strength of the atomic field. This approach is free of limitations associated with the use of the slowly varying envelope approximation, but it has some new restrictions, the model of two-level atoms being the main one among them.

We should note that the nonlinear wave Eqn (31) differs from the well-known Boussinesq equation (see Ref. [55], p. 117) by a higher intensity. Therefore the equation considered does not seem to be totally integrable. However, it is not very difficult to find stationary solutions to Eqn (31).

Suppose that u depends on a single variable $y = \zeta \pm \alpha\tau$ and the field vanishes simultaneously with all its variables for $\tau \to \pm\infty$. This requirement can be considered as a boundary condition for a stationary solitary wave with zero asymptotics. Applying Eqn (31), we find that $\mathrm{d}^2u/\mathrm{d}y^2 = \sigma(\alpha^2 - 1)u - u^3$. This equation is often encountered in the theory of nonlinear waves, and the solution to it can be found by using standard methods. The real solution to this equation meeting the boundary conditions exists for $p^2 = \sigma(\alpha^2 - 1) > 0$. Integrating this equation, we arrive at the following solution:

$$u(y) = \frac{p}{\sqrt{2}\cosh[p(y - y_0)]} \,,$$

where y_0 is the integration constant, which can be set equal to zero. Using the initial normalised variables, we can rewrite the solution to Eqn (31) as

$$u(\zeta, \tau) = u_0 \operatorname{sech}\left\{\sqrt{2}u_0\left[\zeta \pm \tau\left(1 + 2\sigma u_0^2\right)^{1/2}\right]\right\} . \quad (32)$$

Different signs of the argument of the function on the right-hand side of Eqn (32) correspond to different propagation directions of the stationary solitary wave. Note that the maximum amplitude of the field strength is bounded in an absorbing medium (i.e., for $\sigma = -1$): $u_0^2 \leqslant 0.5$.

Along with stationary solutions in the form of video pulses (32), Eqn (31) has periodic solutions corresponding to cnoidal waves and solutions with nonzero asymptotics for $\tau \to \pm\infty$, which correspond to dark solitary waves [57, 58]. A class of algebraic solitons can be separated among solitary dark waves. These solitons correspond to the waves whose amplitudes nonexponentially decay at infinity.

For the equation considered, such a solution can be represented as

$$u(y) = u_1 - \frac{4u_1}{1 + [\zeta \pm \tau(1 + 3\sigma u_1^2)^{1/2}]^2} ,$$

where $u_1 = \lim u$ for $\tau \to \pm\infty$. These solutions do not correspond to ultrashort pulses of electromagnetic radiation, and we mention them only to illustrate the diversity of stationary solutions to the nonlinear wave Eqn (31).

4.3 Unidirectional nonlinear scalar waves

To find the polarisation in the reduced wave equation

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi n_{\mathrm{at}} d}{c}\left\langle \frac{\partial r_1}{\partial t} \right\rangle ,$$

which is derived with an assumption that electromagnetic waves propagate in only one of many possible directions [14, 56], we substitute expression (29) for the polarisation into this equation to arrive at the following equation:

$$\frac{\partial E}{\partial z} + \frac{1}{V}\frac{\partial E}{\partial t} + aE^2\frac{\partial E}{\partial t} + b\frac{\partial^3 E}{\partial t^3} = 0 , \quad (33)$$

where the coefficients and the group velocity V are given by

$$a = \left\langle \frac{24\pi n_{\mathrm{at}}\sigma |d|^4}{c\hbar^3\omega_a} \right\rangle, \quad b_1 = \left\langle \frac{4\pi n_{\mathrm{at}}\sigma |d|^2}{c\hbar\omega_a^3} \right\rangle ,$$

$$\frac{1}{V} = \frac{1}{c}\left(1 - \left\langle \frac{4\pi n_{\mathrm{at}}\sigma |d|^2}{\hbar\omega_a} \right\rangle\right) .$$

Eqn (33) written in terms of new variables $\tau = |b|z$, $\zeta = t - z/V$, and $u(\tau, \zeta) = (a/6b)^{1/2}E(z, t)$ has a form of a modified Korteweg–de Vries (mKdV) equation:

$$\sigma\frac{\partial u}{\partial \tau} + 6u^2\frac{\partial u}{\partial \zeta} + \frac{\partial^3 u}{\partial \zeta^3} = 0 .$$

It is well known [59] that this equation is totally integrable, and soliton solutions to it can be obtained with the use of the ISP method [54, 55]. Specifically the one-soliton solution is written as

$$u_{\mathrm{s}}(\tau, \zeta) = u_0 \operatorname{sech}\left(u_0^3\tau - \sigma u_0\zeta + \delta_0\right) , \quad (34)$$

where the parameters u_0 and δ_0 can be determined from the initial conditions within the framework of the ISP method. This solution corresponds to an electromagnetic pulse that can be interpreted as a video pulse. Multisoliton solutions to the mKdV equation describe the propagation and interaction of several video pulses of different polarity, as well as breathers (Fig. 4), which can be interpreted as ultrashort pulses having a carrier wave. Let us consider such a breather solution to the mKdV equation [59],

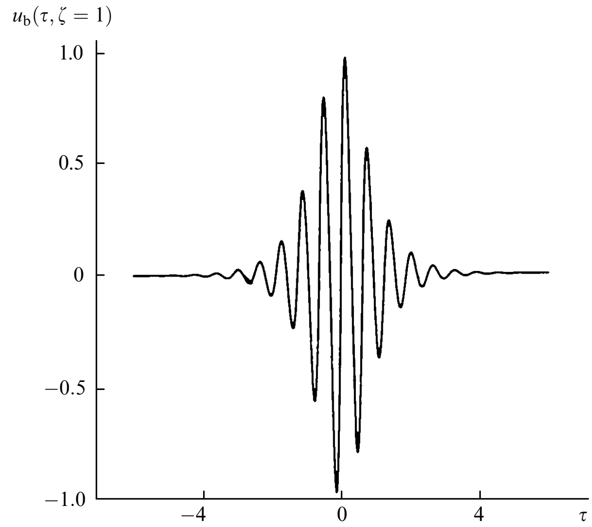

Figure 4. A breather of the mKdV equation as an example of an extremely short pulse of electromagnetic radiation in a quasi-resonance medium.

$$u_{\mathrm{b}}(\tau, \zeta) = -\frac{4u_0}{v_0}\frac{v\cosh\vartheta_1\sin\vartheta_2 - u_0\sinh\vartheta_1\cos\vartheta_2}{\cosh^2\vartheta_1 + (u_0/v_0)\cos^2\vartheta_2} , \quad (35)$$

where $\vartheta_1 = 2u_0\zeta + 8u_0(3v_0^2 - u_0^2)\tau + \delta_1$ and $\vartheta_2 = 2v_0\zeta + 8v_0(v_0^2 - 3u_0^2)\tau + \delta_2$.

Parameters u_0, v_0, δ_1, and δ_2 are determined from the initial conditions. Provided that $u_0 \ll v_0$ under these conditions, solution (35) can be represented as

$$u_{\mathrm{b}}(\tau, \zeta) \approx -4u_0 \operatorname{sech}\vartheta_1 \sin\vartheta_2 .$$

This expression describes an ultrashort pulse with a hyperbolic secant envelope and a high-frequency (harmonic) carrier wave. Thus we can assume that the breather solution to Eqn (33) describes femtosecond optical pulses in a more adequate way than the solution to the nonlinear Schrödinger equation, which is often employed in nonlinear fibre optics [60].

We should note that some of the solutions to Eqn (33) correspond to dark solitons, which attract considerable attention in nonlinear fibre optics [57, 58]. Algebraic solitons [61, 62], e.g.,

$$u(\tau, \zeta) = u_0 - \frac{4u_0}{1 + 4u_0^2(\zeta - 6u_0^2\tau)^2}$$

are an example of solutions of this class.

Thus Eqn (33) describes optical ultrashort and extremely short pulses of electromagnetic radiation, as well as video pulses in a dispersive nonlinear medium without the use of the approximation of slowly varying complex envelopes in the case of unidirectional wave propagation. This conclusion has also been reached in Ref. [63], where analysis was performed by using of another approach.

4.4 Quasi-resonant propagation of a polarised pulse

To describe the propagation of a pulse of polarised electromagnetic radiation under quasi-resonance conditions, we will employ a model of a two-level medium, which is based on generalised Bloch Eqns (11) and which was considered in Section 3.2. Suppose that the amplitude of the electric field in the pulse is so small that the condition $\varepsilon = \omega_{\mathrm{R}}/\omega_a = |d_{13}|E_0/\hbar\omega_a$ is satisfied, where $E_0 = \max|E^{(\pm 1)}|$ is the constant amplitude of the electric field. The set of

Eqns (11) can be solved by using the resolvent operator method [64] in the same way as was done in the case of scalar waves in Section 4.1. Introducing auxiliary notations

$$B_1 = \rho_{13}\,, \quad B_2 = \rho_{23}\,, \quad C_1 = \rho_{31}\,, \quad C_2 = \rho_{32}\,,$$
$$\omega_R q_1 = i\hbar^{-1} d_{13} E^{(+1)}\,, \quad \omega_R q_2 = i\hbar^{-1} d_{23} E^{(-1)}\,,$$
$$\omega_R r_1 = i\hbar^{-1} d_{31} E^{(+1)}\,, \quad \omega_R r_2 = i\hbar^{-1} d_{32} E^{(-1)}\,,$$
$$T = \omega_R t\,, \quad \omega_R = |d_{13}| E_0/\hbar\,,$$

we can rewrite the generalised Bloch Eqns (11) in the following form:

$$\omega_R \frac{\partial B_1}{\partial T} = i\omega_a B_1 + \omega_R q_1(\rho_{11} - \rho_{33}) + \omega_R q_2 \rho_{12}\,, \quad (36.1)$$

$$\omega_R \frac{\partial B_2}{\partial T} = i\omega_a B_2 + \omega_R q_2(\rho_{22} - \rho_{33}) + \omega_R q_1 \rho_{21}\,, \quad (36.2)$$

$$\omega_R \frac{\partial C_1}{\partial T} = -i\omega_a C_1 - \omega_R r_1(\rho_{11} - \rho_{33}) - \omega_R r_2 \rho_{21}\,, \quad (36.3)$$

$$\omega_R \frac{\partial C_2}{\partial T} = -i\omega_a C_2 - \omega_R r_2(\rho_{22} - \rho_{33}) - \omega_R r_1 \rho_{12}\,, \quad (36.4)$$

$$\frac{\partial \rho_{12}}{\partial T} = (B_1 r_2 - C_2 q_1)\,, \quad \frac{\partial \rho_{21}}{\partial T} = (B_2 r_1 - C_1 q_2)\,, \quad (37.1)$$

$$\frac{\partial}{\partial T}(\rho_{11} - \rho_{33}) = 2(B_1 r_1 - C_1 q_1) + (B_2 r_2 - C_2 q_2)\,, \quad (37.2)$$

$$\frac{\partial}{\partial T}(\rho_{22} - \rho_{33}) = (B_1 r_1 - C_1 q_1) + 2(B_2 r_2 - C_2 q_2)\,. \quad (37.3)$$

Formally solving Eqns (37), we derive

$$\rho_{12} = \hat{I}(B_1 r_2 - C_2 q_1)\,, \quad \rho_{21} = \hat{I}(B_2 r_1 - C_1 q_2)\,, \quad (38.1)$$

$$(\rho_{11} - \rho_{33}) = \sigma + 2\hat{I}(B_1 r_1 - C_1 q_1) + \hat{I}(B_2 r_2 - C_2 q_2)\,, \quad (38.2)$$

$$(\rho_{22} - \rho_{33}) = \sigma + \hat{I}(B_1 r_1 - C_1 q_1) + 2\hat{I}(B_2 r_2 - C_2 q_2)\,, \quad (38.2)$$

where $\sigma = -1$ for an absorbing medium and the integration operator $\hat{I}(f)$ was used. It is convenient to introduce matrices $\hat{J}\mathrm{diag}(1, -1, 1, -1)$, $\chi = \mathrm{colon}(B_1, C_1, B_2, C_2)$, and $\psi = \mathrm{colon}(q_1, r_1, q_2, r_2)$ and an operator $\hat{R} = \hat{J}\,\partial/\partial T - \hat{\Lambda}$, where $\hat{\Lambda}$ is the matrix of integral operators defined in the previous sections:

$$\hat{\Lambda} = \begin{pmatrix} 2q_1 \int r_1 + q_2 \int r_2 & -2q_1 \int q_1 & q_1 \int r_2 & -q_1 \int q_2 - q_2 \int q_1 \\ 2q_1 \int r_1 & -2r_1 \int q_1 - r_2 \int q_2 & r_1 \int r_2 + r_2 \int r_1 & -r_1 \int q_2 \\ q_2 \int r_1 & -q_2 \int q_1 - q_1 \int q_2 & 2q_2 \int r_2 + q_1 \int r_1 & -2q_2 \int q_2 \\ r_2 \int r_1 + r_1 \int r_2 & -r_2 \int r_1 & 2r_2 \int r_2 & -2r_2 \int q_2 - r_1 \int q_1 \end{pmatrix}.$$

Now, Eqns (36) can be represented in the form of a matrix equation with respect to χ: $(1 + i\varepsilon\hat{R})\chi = i\sigma\varepsilon\psi$. Using the resolvent operator $\hat{G} = (1 + i\varepsilon\hat{R})^{-1}$, we can express the solution to this equation in the form of a series $\chi = i\sigma\varepsilon(1 + i\varepsilon\hat{R} - \varepsilon^2\hat{R}^2 + \ldots)\psi$. Restricting our consideration of this series to the terms of the third order in ε, we can write the explicit approximate solution to Eqns (36):

$$\chi = \begin{pmatrix} B_1 \\ C_1 \\ B_2 \\ C_2 \end{pmatrix} = i\sigma\varepsilon \begin{bmatrix} q_1 - i\varepsilon \dfrac{\partial q_1}{\partial T} - \varepsilon^2 \dfrac{\partial^2 q_1}{\partial T^2} + 2\varepsilon^2 q_1(\boldsymbol{q}\cdot\boldsymbol{r}) \\ r_1 + i\dfrac{\partial r_1}{\partial T} - \varepsilon^2 \dfrac{\partial^2 r_1}{\partial T^2} + 2\varepsilon^2 r_1(\boldsymbol{q}\cdot\boldsymbol{r}) \\ q_2 - i\varepsilon \dfrac{\partial q_2}{\partial T} - \varepsilon^2 \dfrac{\partial^2 q_2}{\partial T^2} + 2\varepsilon^2 q_2(\boldsymbol{q}\cdot\boldsymbol{r}) \\ r_2 + i\varepsilon \dfrac{\partial r_2}{\partial T} - \varepsilon^2 \dfrac{\partial^2 r_2}{\partial T^2} + 2\varepsilon^2 r_2(\boldsymbol{q}\cdot\boldsymbol{r}) \end{bmatrix}. \quad (39)$$

Thus, with an accuracy not exceeding ε^3, the polarisations $P^{(+1)} = d_{13}\rho_{31} + d_{31}\rho_{13}$ and $P^{(-1)} = d_{23}\rho_{32} + d_{32}\rho_{23}$ on the right-hand sides of the wave Eqns (10) are described by the approximate expression

$$P^{(q)} = -\frac{2\sigma|d_{13}|^2}{\hbar\omega_a}\left[E^{(q)} - \frac{1}{\omega_a^2}\frac{\partial^2 E^{(q)}}{\partial t^2} - \frac{2|d_{13}|^2}{(\hbar\omega_a)^2}(\boldsymbol{E}\cdot\boldsymbol{E})E^{(q)}\right]. \quad (40)$$

Substituting Eqn (40) into the equations for the fields, we can write the nonlinear wave equation approximately describing the evolution of the pulse of the electromagnetic wave:

$$\frac{\partial^2 \boldsymbol{E}}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - b\frac{\partial^4 \boldsymbol{E}}{\partial t^4} - a\frac{\partial^2}{\partial t^2}\left[(\boldsymbol{E}\cdot\boldsymbol{E})\boldsymbol{E}\right] = 0\,, \quad (41)$$

where $a = \langle 16\pi n_{at}\sigma|d_{13}|^4/c^2(\hbar\omega_a)^3\rangle$ and $b = \langle 8\pi n_{at}\sigma|d_{13}|^2/c^2\hbar\omega_a^3\rangle$. The velocity of the pulse V, which changes because of the dispersion introduced by the resonance medium, is given by

$$V^{-2} = c^{-2}\left(1 - \left\langle \frac{8\pi n_{at}\sigma|d_{13}|^2}{\hbar\omega_a} \right\rangle\right).$$

Using new normalised variables, we can write $\boldsymbol{W}(\tau, \varsigma) = V\sqrt{|a|}\boldsymbol{E}(x, t)$, $\tau = V^{-2}|b|^{-1/2}$, and $\zeta = V^{-1}|b|^{-1/2}t$. The nonlinear wave equation can then be written in a more elegant form:

$$\frac{\partial^2 \boldsymbol{W}}{\partial \tau^2} = \frac{\partial^2}{\partial \zeta^2}\left[\boldsymbol{W} + \sigma(\boldsymbol{W}\cdot\boldsymbol{W})\boldsymbol{W} + \sigma\frac{\partial^2 \boldsymbol{W}}{\partial \zeta^2}\right]. \quad (42)$$

This expression is a simple extension of the nonlinear wave equation from Ref. [56] to the case of polarised radiation.

Solution in the form of a running wave is often the simplest solution to nonlinear evolution equations. It would be natural to find such a solution for Eqn (42). Let the normalised field $\boldsymbol{W}(\tau, \zeta)$ depend on a single variable $y = \zeta \pm \alpha\tau$. Then, Eqn (42) yields the following equation for the complex field $Z = W_1 + iW_2$:

$$\frac{d^2 Z}{dy^2} = p^2 Z - |Z|^2 Z\,, \quad (43)$$

where $p^2 = \sigma(\alpha^2 - 1) > 0$.

Reducing Eqn (42) to Eqn (43) we took into account that the electromagnetic field vanishes at infinity, which means that the integration constants should be equal to zero. Setting $Z = W\exp(i\Phi)$, we can employ Eqn (43) to derive a set of equations for real variables:

$$\frac{d^2 W}{dy^2} - W\left(\frac{d\Phi}{dy}\right)^2 = p^2 W - W^3\,, \quad 2\frac{d\Phi}{dy}\frac{dW}{dy} + W\frac{d^2\Phi}{dy^2} = 0\,. \quad (44)$$

Multiplying the second equation in set (44) by W and integrating the resulting expression, we obtain the integral of motion, $W^2 \mathrm{d}\Phi/\mathrm{d}y = \text{const} = 0$. The choice of the integration constant is determined by the boundary conditions for the electric field at infinity. Since the solution to Eqns (44) with $W = 0$ is of no interest, we have to impose the requirement $\mathrm{d}\Phi/\mathrm{d}y = 0$ for all the values of the variable $y = \varsigma \pm \alpha\tau$.

Thus, the stationary solutions to Eqn (42) describing the propagation of a short pulse of electromagnetic radiation can be derived with the assumption that the polarisation state (the vector $\boldsymbol{e} = \boldsymbol{E}/|\boldsymbol{E}| = \{e_1 = W\cos\Phi,\ e_2 = W\sin\Phi\}$) of the radiation wave remains unchanged in the process of radiation propagation. In such a situation, Eqn (42) or the first equation in set (44) is reduced to Eqn (16) from Ref. [56]. Using the well-known results, we can readily write the relevant solutions.

For zero-boundary conditions with $\tau \to \pm\infty$, the stationary solution (9) can be represented as

$$\boldsymbol{W}(\zeta \pm \alpha\tau) = W_0 \boldsymbol{e}\,\mathrm{sech}\left\{\sqrt{2}W_0[\zeta \pm \tau(1+2\sigma W_0^2)^{1/2} - \zeta_0]\right\}, \quad (45)$$

where the integration constant ζ_0 can be set equal to zero.

For nonzero boundary conditions and with a fixed polarisation vector, the stationary solutions have the form of a solitary dark wave, of algebraic solitary waves similar to those considered in Section 4.2, and of nonlinear periodic waves [56]. The analysis of solutions of this class falls beyond the scope of this paper.

Let us examine whether the solution derived above is consistent with our assumption that the Rabi frequency is much lower than the characteristic frequency $\bar{\omega}$ of quantum transitions. Using nonnormalised (physical) variables, we can write the solution (45) in the following way:

$$\boldsymbol{E}(z,t) = E_0 \boldsymbol{e}\,\mathrm{sech}\left[t_{\mathrm{s}}^{-1}\left(t \pm \frac{\alpha z}{V} - t_0\right)\right], \quad (46)$$

where $t_{\mathrm{s}} = E_0^{-1}(|b|/2|a|)^{1/2}$ and $\alpha = (1 + 2\sigma E_0^2 V^2 |a|)^{1/2}$. The spectral width of Eqn (46) can be estimated as $\sim t_{\mathrm{s}}^{-1}$. Consequently, the resonance approximation remains applicable if $t_{\mathrm{s}}^{-1} \ll \bar{\omega}$. Employing the definition of the parameters $|a|$ and $|b|$ involved in Eqn (41), we can estimate the ratio of these parameters as $|a|/|b| \approx 2|d_{13}|^2\bar{\omega}^2\hbar^{-2}$ (where we neglected inhomogeneous broadening and all the frequencies ω_a were replaced by $\bar{\omega}$). Thus the condition of the resonance approximation coincides with the condition of the smallness of the amplitude: $\omega_{\mathrm{R}} \ll \bar{\omega}$.

If we consider the propagation of a pulse of electromagnetic radiation in the approximation of unidirectional waves, then Eqns (10) should be replaced by a pair of reduced Maxwell equations:

$$\frac{\partial E^{(\pm1)}}{\partial z} + \frac{1}{c}\frac{\partial E^{(\pm1)}}{\partial t} = -\frac{2\pi n_{\mathrm{at}}}{c}\frac{\partial}{\partial t}\left\langle P^{(\pm1)}\right\rangle, \quad (47)$$

where the polarisations $P^{(\pm1)}$ are defined by expressions (40). Substituting Eqn (40) into Eqn (47) for the fields, we arrive at the nonlinear equation describing approximately the evolution of a pulse of the electromagnetic wave:

$$\frac{\partial \boldsymbol{E}}{\partial z} + \frac{1}{V}\frac{\partial \boldsymbol{E}}{\partial t} + b\frac{\partial^3 \boldsymbol{E}}{\partial t^3} + a\frac{\partial}{\partial t}\left[(\boldsymbol{E}\cdot\boldsymbol{E})\boldsymbol{E}\right] = 0, \quad (48)$$

where the parameters a and b are two times smaller than the corresponding parameters in Eqn (41), and the propagation velocity $V^{-1} = c^{-1}(1 - \langle 4\pi n_{\mathrm{at}}\sigma|d_{13}|^2/\hbar\omega_a\rangle)$.

Using new normalised variables $\tau = |b|z$, $\zeta = t - z/V$, and $\psi(\tau,\zeta) = \sqrt{a/b}[E^{(+1)}(z,t) + \mathrm{i}E^{(-1)}(z,t)]$, we can represent the nonlinear Eqn (48) as

$$\sigma\frac{\partial\psi}{\partial\tau} + \frac{\partial}{\partial\zeta}\left(\frac{\partial^2\psi}{\partial\zeta^2} + |\psi|^2\psi\right) = 0. \quad (49)$$

This equation, which is referred to as the complex mkdV equation, was considered in Ref. [65]. If we assume that the phase ψ of the complex field remains unchanged in the process of propagation, then expression (49) is reduced to the conventional (real) mKdV equation. In principle, the latter equation can be solved by using the ISP method [59], and soliton solutions to this equation can be written in the explicit form. The mKdV equation and solutions to this equation were discussed earlier in Section 4.3, where the 1-soliton and breather solutions were given by formulas (34) and (35).

We have no solution for a more general situation, when Eqn (49) is not reduced to a real mKdV equation. We may anticipate that the nonsoliton character of solutions to this equation should be manifested most clearly when separate pulses interact with each other. Note that Eqn (49) may have solutions describing the propagation of extremely short pulses (and video pulses as a particular case of such pulses) and pulses of polarised electromagnetic radiation enveloping an arbitrary number of field cycles. Such signals are not represented as quasi-harmonic waves in this case.

5. The propagation of electromagnetic pulses in media with a cubic nonlinearity

Consider the propagation of a pulse of electromagnetic radiation in the approximation of a unidirectional wave. Suppose that resonance transitions are absent in the nonlinear medium. Such a model was analysed by Vuzhva [66], who studied self-induced transparency in ion crystals within the framework of the Duffing model for a nonlinear nonresonant medium. The Maxwell equation for slowly varying envelopes in this case has the form of Eqn (5). It would be instructive to consider two cases of the Duffing model describing a particular situation of a medium with a cubic nonlinearity. The approximation of a medium with a cubic nonlinearity should be understood as an assumption that the third-order susceptibility is the first nonzero nonlinear susceptibility determining the nonlinear polarisation of a medium in a given field.

5.1 The scalar Duffing model

Let us supplement Eqn (5) for the electric field strength with an equation describing the polarisation $P = n_{\mathrm{at}}d_a P_1$:

$$\frac{\partial^2 P_1}{\partial t^2} + \omega_a^2 P_1 + \beta_a P_1^3 = \frac{e^2}{md_a}E, \quad (50)$$

where P_1 is the coordinate related to normal oscillations of an anharmonic one-dimensional oscillator. We assume that all the oscillators have equal frequencies ω_a of natural oscillations and ignore the relaxation of oscillations. Introducing dimensionless variables

$$\zeta = \frac{z}{ct_{\mathrm{p0}}}, \quad \tau = \left(t - \frac{x}{c}\right)t_{\mathrm{p0}}^{-1},$$
$$E = E_0 q(\zeta, \tau), \quad p = 2\pi n_{\mathrm{at}}d_a E_0^{-1}P_1,$$
$$Q = 2\pi n_{\mathrm{at}}d_a E_0^{-1}\frac{\partial P_1}{\partial t}$$

we can represent the set of equations (5) and (50) in the following form:

$$\frac{\partial q}{\zeta} = -Q\,, \quad \frac{\partial p}{\partial \tau} = Q\,, \quad \frac{\partial Q}{\partial \tau} + v_a^2 p + 2b_a p^3 = \alpha_a q\,, \quad (51)$$

where $\alpha_a = (2\pi e^2 n_{\mathrm{at}} d_a t_{\mathrm{p0}}^2/m)$, $b_a = (\beta_a/2)(E_0 t_{\mathrm{p0}}/2\pi n_{\mathrm{at}} d_a)^2$, and $v_a = \omega_a t_{\mathrm{p0}}$.

To find the stationary solution to this set of equations, we assume that the variables q, p, and Q depend on a single variable $\eta = \tau - \zeta/V$ (V is the velocity of the stationary pulse). Then, Eqns (51) can be rewritten as

$$\frac{\partial q}{\partial \eta} = VQ\,, \quad \frac{\partial p}{\partial \eta} = Q\,, \quad \frac{\partial Q}{\partial \eta} + v_a^2 p + 2b_a p^3 = \alpha_a q\,. \quad (52)$$

The first two equations of this set give $q(\eta) = Vp(\eta)$. The set (52) also allows us to find the relation between $Q(\eta)$ and $p(\eta)$: $Q^2 = (\alpha_a V - v_a^2)p^2 - b_a p^4$. In the case when $\alpha_a V < v_a^2$, the equality $Q = p = 0$ is the only solution to the considered equation, given that p and Q are real. Consequently, a nontrivial solution to Eqn (52) exists only for $\alpha_a V > v_a^2$, which implies that the velocity V of the stationary pulse is always higher than the critical velocity $V_{\mathrm{c}} = v_a^2/\alpha_a = 2\omega_a^2/\omega_{\mathrm{p}}^2$, where $\omega_{\mathrm{p}} = (4\pi e^2 n_a/m)^{1/2}$ is the frequency of plasma oscillations.

We should note that the stationary pulse considered in terms of variables z and t propagates with the velocity $V_{\mathrm{st}} = cV(1+V)^{-1}$, which meets the condition

$$c > V_{\mathrm{st}} \geqslant c\frac{2\omega_a^2}{\omega_a^2 + \omega_{\mathrm{p}}^2}\,.$$

The shape of the stationary pulse is determined by set (52):

$$q(\eta) = V\left(\frac{\alpha_a V - v_a^2}{b_a}\right)^{1/2} \operatorname{sech}\left[\eta\left(\alpha_a V - v_a^2\right)^{1/2}\right]\,. \quad (53)$$

This solution describes an extremely short pulse of electromagnetic radiation arising in the considered model of a nonlinear medium. Such a pulse has no carrier wave and can be treated as a video pulse. A more general case of propagation of short (not only stationary) pulses was investigated numerically in Ref. [67]. This study has shown that the propagation of a bipolar video pulse enveloping several cycles of oscillations of the electric field strength around its zero value is accompanied by the appearance of high-frequency oscillations on the trailing edge of the pulse. Fig. 5 shows one of the plots obtained in Ref. [67], illustrating such a behaviour of an extremely short pulse.

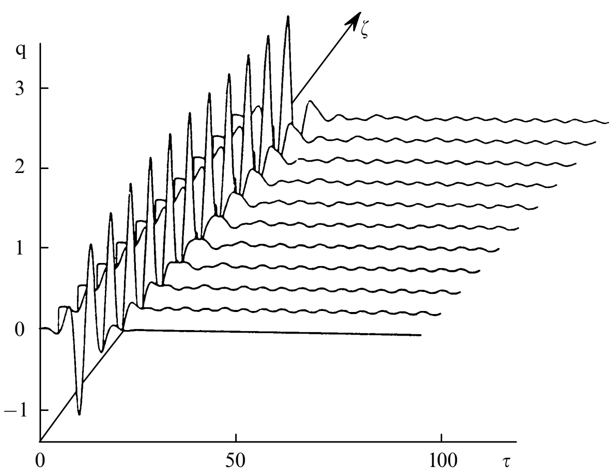

Figure 5. Nonstationary propagation of a video pulse in a medium with a cubic nonlinearity, manifested in the formation of oscillations on the trailing edge of the pulse ($b_a = 5$, $v_a = 0.2$).

5.2 The vector Duffing model

Let us assume that an ultrashort pulse propagates in a nonlinear medium that can be represented as a set of molecules whose internal degrees of freedom are described by the potential field

$$U(x, y) = \tfrac{1}{2}\omega_1 x^2 + \tfrac{1}{2}\omega_2 y^2 + \tfrac{1}{2}\varkappa_2 x^2 y^2 + \tfrac{1}{4}\varkappa_4\left(x^4 + y^4\right)\,. \quad (54)$$

If the coupling parameter is $\varkappa_2 = 0$, then this potential describes the scalar Duffing model for an anharmonic oscillator that may be involved in two independent oscillations along two orthogonal directions. Expression (54) is an elementary generalisation of this model. The polarisation of a molecule is determined by the expression $\boldsymbol{p} = ex\boldsymbol{e}_1 + ey\boldsymbol{e}_2$, whereas the total polarisation is given by the product of the atomic density n_{at} and the polarisation of a single molecule.

The propagation of an electromagnetic pulse will be considered in the approximation of a unidirectional wave. The wave equations for the electric field strengths corresponding to different polarisation components of an ultrashort pulse are written as

$$\frac{\partial E_1}{\partial z} + \frac{1}{c}\frac{\partial E_1}{\partial t} = -\frac{2\pi n_{\mathrm{at}} e}{c}\frac{\partial x}{\partial t}\,, \quad \frac{\partial E_2}{\partial z} + \frac{1}{c}\frac{\partial E_2}{\partial t} = -\frac{2\pi n_{\mathrm{at}} e}{c}\frac{\partial y}{\partial t}\,.$$

The equations of motion for the oscillator considered can be derived from the classical Newton equation and can be represented as

$$\frac{\partial^2 x}{\partial t^2} + \omega_1^2 x + \varkappa_2 x y^2 + \varkappa_4 x^3 = \frac{e}{m}E_1(z, t)\,,$$

$$\frac{\partial^2 y}{\partial t^2} + \omega_2^2 y + \varkappa_2 y x\,2 + \varkappa_4 y^3 = \frac{e}{m}E_2(z, t)\,.$$

Let us introduce new independent variables $\zeta = z/ct_{\mathrm{p0}}$, $\tau = (t - z/c)/t_{\mathrm{p0}}$, $p_j = \partial q_j/\partial\tau$, $q_1 = (2\pi e n_{\mathrm{at}})x$, and $q_2 = (2\pi e n_{\mathrm{at}})y$. Thus, the two-component (vector) Duffing model is governed by the following set of equations:

$$\frac{\partial E_j}{\partial \zeta} = -p_j\,, \quad \frac{\partial q_j}{\partial \tau} = p_j\,, \quad i = 1, 2\,, \quad (55)$$

$$\frac{\partial p_1}{\partial \tau} = aE_1 - v_1^2 q_1 - b_2 q_1 q_2^2 - b_4 q_1^3\,,$$

$$\frac{\partial p_2}{\partial \tau} = aE_2 - v_2^2 q_2 - b_2 q_2 q_1^2 - b_4 q_2^3\,. \quad (56)$$

where $a = 2\pi n_{\mathrm{at}} e^2 t_{\mathrm{p0}}^2/m$, $b_{2,4} = \varkappa_{2,4}(2\pi e n_{\mathrm{at}} t_{\mathrm{p0}}^2)^{-2}$, and $v_{1,2} = \omega_{1,2} t_{\mathrm{p0}}$.

Stationary solutions can be found for this set of equations. With the assumption that the fields depend only on $\eta = \tau - \zeta/V$, Eqns (55) subject to boundary conditions for the electric field strength in the pulse and the polarisation of molecules vanishing at infinity yield $E_j = Vq_j$. Substituting these expressions into Eqns (56), we derive

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}\eta^2} + \left(b_4 q_1^2 + b_2 q_2^2\right)q_1 = \left(aV - v_1^2\right)q_1\,, \quad (57.1)$$

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}\eta^2} + \left(b_2 q_1^2 + b_4 q_2^2\right)q_2 = \left(aV - v_2^2\right)q_2\,. \quad (57.2)$$

We have already encountered these equations in the set (14) of Section 3.2. Therefore we can readily find the solution to these equations. Let us assume that the eigenfrequencies ω_1 and ω_2 are equal to each other. In this case, a particular solution to the set (57) can be written as

$$q_1(\eta)=q_2(\eta)=\left[\frac{2(aV-v^2)}{b_2+b_4}\right]^{1/2}\operatorname{sech}\left[\left(aV-v^2\right)^{1/2}(\eta-\eta_0)\right] ,$$

where η_0 is the integration constant. This solution describes a linearly polarised video pulse propagating in the medium under study without any distortions.

A more interesting solution can be found when the oscillation frequencies ω_1 and ω_2 differ from one another, but the anharmonicity coefficients are equal to each other ($b_1 = b_2$). In this case, the set (57) coincides with the set of equations considered in Section 3.2, and we can simply employ the solutions derived in this section:

$$q_1(\eta) = \frac{2\sqrt{2b_2^{-1}}\mu_1 \exp(\theta_1)[1+\exp(2\theta_2+\mu_{12})]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\exp(2\theta_1+2\theta_2+\mu_{12})} ,$$

$$q_2(\eta) = \frac{2\sqrt{2b_2^{-1}}\mu_2 \exp(\theta_2)[1+\exp(2\theta_1+\mu_{12})]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\exp(2\theta_1+2\theta_2+\mu_{12})} , \qquad (58)$$

where $\exp(\mu_{12}) = (\mu_1-\mu_2)/(\mu_1+\mu_2)$, $\mu_{1,2}^2 = (aV - v_{1,2}^2)$, and $\theta_{1,2} = \mu_{1,2}(\eta-\eta_{1,2})$, $\eta_{1,2}$ are the integration constants. Solutions (58) describe the stationary propagation of a polarised video pulse whose polarisation components change asynchronously. One of the components of this pulse represents a unipolar transient of the electric field, whereas the second component corresponds to a sign-alternating solitary wave.

Considering these examples, we assumed that an electromagnetic wave propagates in one direction. However, if we restrict our analysis to the propagation of a stationary pulse, this approximation is of no importance. If the reduced Maxwell equation is replaced by the wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi n_{\rm at} e}{c^2}\frac{\partial^2 x}{\partial t^2}$$

(or the vector generalisation of this equation), then the assumption that the field strength depends only on $\eta = \tau - \zeta/V$ gives the relationship $E = 4\pi n_{\rm at} e V^2 (c^2 - V^2)^{-1} x$, which eventually brings us to the set of equations of the form of set (52). Following this procedure, we arrive at the result known from Ref. [20].

6. The propagation of electromagnetic pulses in media with quadratic nonlinearities

To describe a nonlinear medium in this section, we will use a model of an anharmonic oscillator, which is often employed to investigate parametric processes in nonlinear optics [68, 69]. The calculation of polarisation in the field of a monochromatic wave in such a model gives the second-order nonlinear susceptibility, which is characteristic of the parametric frequency summation or subtraction in a pair of interacting monochromatic waves. Such a representation of interacting waves is inapplicable when we consider video pulses, since the carrier wave is absent in this case. It would be appropriate therefore to examine the propagation of extremely short pulses enveloping one or several cycles of the electric field within the framework of simple models without invoking the concepts of harmonic analysis.

6.1 The scalar model of an anharmonic oscillator

We start by considering the scalar model of an anharmonic oscillator. The electromagnetic wave in this model is described by the wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2} , \qquad (59.1)$$

where the polarisation P is related to the generalised coordinate x of the anharmonic oscillator by the formula $P = n_{\rm at} e x$. The Newton equation for the considered oscillator is written as

$$\frac{\partial^2 x}{\partial t^2} + \omega_0^2 x + \varkappa_2 x^2 = \mathscr{L}\frac{e}{m}E(z,t) , \qquad (59.2)$$

where ω_0 is the eigenfrequency of the oscillator, $\varkappa_2$ is the anharmonicity coefficient, $\mathscr{L} = (\varepsilon+2)/3$ is the Lorentz factor, and ε is the dielectric constant. We can include the parameter $\mathscr{L}$ in the mass m by introducing the effective mass $m_{\rm eff}$.

At the first step of our analysis, we search for the stationary solution to the set of Eqns (59). The electric field strength and the polarisation corresponding to the stationary solution depend only on the variable $\tau = t \pm z/V$. Taking this into account, we can rewrite the wave Eqn (59.1) as $\mathrm{d}^2E/\mathrm{d}\tau^2 = \beta \mathrm{d}^2 x/\mathrm{d}\tau^2$, where $\beta = 4\pi n_{\rm at} e V^2 (c^2 - V^2)^{-1}$. Integrating this equation allowing for the relevant boundary conditions characteristic of a solitary wave vanishing at infinity, we find the relation between the electric field strength and the coordinate of the oscillator: $E = \beta x$. Now, Eqn (59.2) can be rewritten as a nonlinear equation for an anharmonic oscillator:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + (\omega_0^2 - v\omega_{\rm p}^2)x + \varkappa_2 x^2 = 0 , \qquad (60)$$

where $v = V^2(c^2-V^2)^{-1}$ and $\omega_{\rm p}^2 = 4\pi n_{\rm at} e^2/m_{\rm eff}$ is the frequency of plasma oscillations.

The first integral of Eqn (60) can be found in a standard way:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + (\omega_0^2 - v\omega_{\rm p}^2)x^2 + \frac{2}{3}\varkappa_2 x^3 = 0 , \qquad (61)$$

where we take into account that the sought-for solution represents a solitary wave vanishing at infinity. Introducing the notation

$$p^2 = v\omega_{\rm p}^2 - \omega_0^2 > 0 , \qquad (62)$$

we find that the solution to Eqn (60) describes a nonsingular function meeting the boundary conditions. In this case, the integration of Eqn (61) yields

$$y = \operatorname{sech}^2\left[\frac{p}{2}(\tau-\tau_0)\right] ,$$

where $y = (2\varkappa_2/3p^2)x$, and τ_0 is the integration constant. Switching back to the initial physical variables we can write the following expression for the electric field strength in the video pulse:

$$E(\tau) = E_0 \operatorname{sech}^2\left[\tfrac{1}{2}(v\omega_{\rm p}^2 - \omega_0^2)^{1/2}(\tau-\tau_0)\right] , \qquad (63)$$

where $E_0 = 6\pi n_{\rm at}\varkappa_2^{-1} e v(v\omega_{\rm p}^2 - \omega_0^2)$.

Inequality (62) imposes a limitation on the velocity of the stationary pulse thus determined. This limitation can be written as

$$\frac{\omega_0^2}{\omega_0^2+\omega_{\rm p}^2} < \frac{V^2}{c^2} < 1 .$$

We can simplify the set of Eqns (59) by considering the propagation of a pulse in the approximation of a unidirectional wave. In this case, expression (59) can be replaced by the formulas

$$\frac{\partial E}{\partial z}+\frac{1}{c}\frac{\partial E}{\partial t}=-\frac{2\pi n_{\text{at}}e}{c}\frac{\partial x}{\partial t}\,,$$
$$\frac{\partial^2 x}{\partial t^2}+\omega_0\,2x+\varkappa_2 x^2=\frac{e}{m_{\text{eff}}}E(z,\,t). \tag{64.1}$$

This set of equations can be rewritten as

$$\frac{\partial \mathscr{E}}{\partial \zeta}=-\frac{\partial q}{\partial \tau}\,,\qquad \frac{\partial^2 q}{\partial \tau^2}+q+q^2=\mathscr{E}\,, \tag{64.2}$$

where the normalised variables are defined by the following expressions:

$$\zeta=z\frac{\omega_{\text{p}}^2}{2c\omega_0}\,,\qquad \tau=\omega_0\left(t-\frac{z}{c}\right),\qquad \mathscr{E}=\frac{e\varkappa_2}{m_{\text{ef}}\omega_0^4}E\,,$$
$$q=\frac{\varkappa_2}{\omega_0^2}x.$$

The numerical solution to the set of Eqns (64.2) has shown that a small-amplitude video pulse decays in the process of propagation (Fig. 6), whereas in the case of high-power pulses (beginning with some threshold power), dispersion-induced distortions of the video pulse are suppressed [the set of Eqns (64.2) was numerically solved by E V Kazantseva (Moscow Engineering Physics Institute), who was kind enough to give her permission to represent her results here, see Figs. 6 and 7].

Fig. 7 illustrates the interaction of two video pulses corresponding to stationary solutions of the set (64). Owing to a difference in their amplitudes, these pulses have different group velocities, which allows us to investigate collisions of these pulses in the approximation of unidirectional waves. If an amplitude modulation in the form of a weak harmonic wave is put on the initial stationary video pulse (63), then this modulation is filtered out as the stationary pulse propagates through the medium. This process changes only slightly the parameters of the pulse. Unfortunately, these results are insufficient for us to refer to stationary video pulses arising in this model as solitons.

6.2 The vector model of an anharmonic oscillator

The propagation of a pulse of polarised radiation in a medium with a quadratic nonlinearity can be analysed by

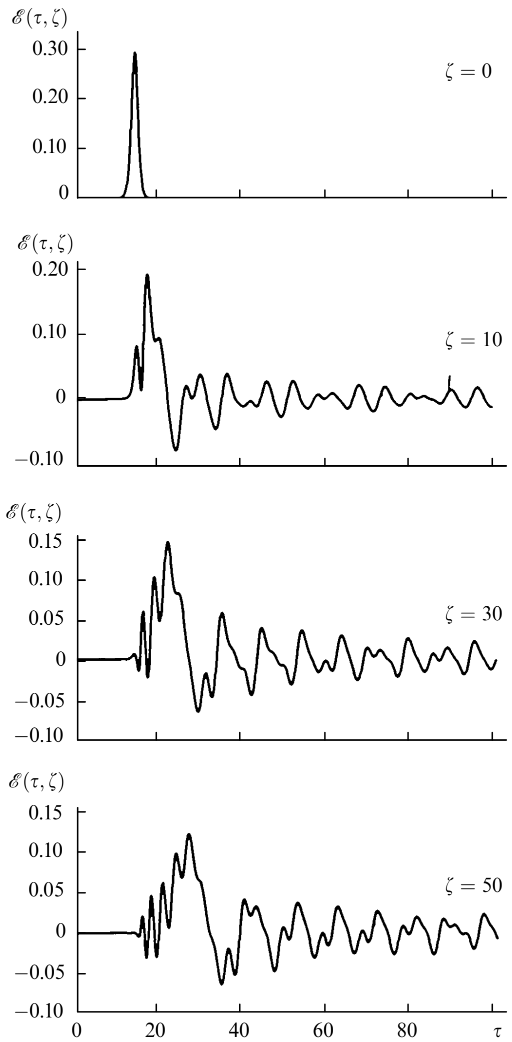

Figure 6. Dependence of spherical components of a video pulse on the dimensionless time $y=\sqrt{2}\omega_a(t\pm z/V)$ for (a) $q_0^2 n_{01}=4$ and $q_0^2 n_{02}=7$ and (b) $q_0^2 n_{01}=9$ and $q_0^2 n_{02}=8$.

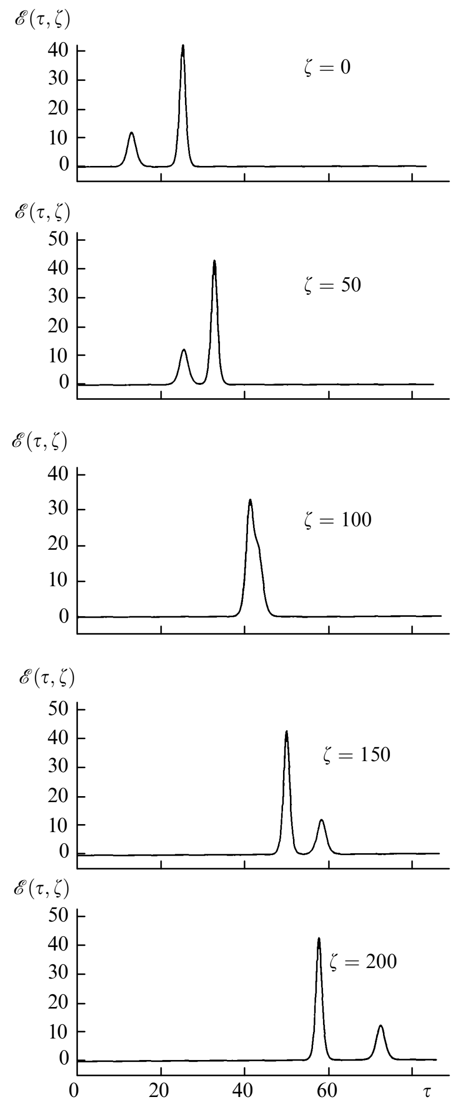

Figure 7. Dependence of spherical components of a video pulse on the dimensionless time $y=\sqrt{2}\omega_a(t\pm z/V)$ for (a) $q_0^2 n_{01}=4$ and $q_0^2 n_{02}=7$ and (b) $q_0^2 n_{01}=9$ and $q_0^2 n_{02}=8$.

using a model of an ensemble of two-component anharmonic oscillators. As an elementary generalisation of the model considered above, we can employ the following equations in such an analysis:

$$\frac{\partial^2 E_1}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_1}{\partial t^2} = \frac{4\pi n_{\text{at}} e}{c^2}\frac{\partial^2 x}{\partial t^2}\,,$$
$$\frac{\partial^2 E_2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_2}{\partial t^2} = \frac{4\pi n_{\text{at}} e}{c^2}\frac{\partial^2 y}{\partial t^2}\,, \qquad (65)$$

$$\frac{\partial^2 x}{\partial t^2} + \omega_1^2 x + \varkappa_{12} xy + \varkappa_{11} x^2 = \frac{e}{m} E_1(z,\,t)\,, \qquad (66.1)$$

$$\frac{\partial^2 y}{\partial t^2} + \omega_2^2 y + \varkappa_{12} xy + \varkappa_{22} y^2 = \frac{e}{m} E_2(z,\,t)\,. \qquad (66.2)$$

The electric field strength and the polarisation corresponding to the stationary solution depend only on the variable $\tau = t \pm z/V$. Similar to the scalar model, we can integrate the equations for the fields in this case, arriving at the relations for $E_{1,2} = \beta x, \beta y$. Eqns (66) are reduced to the following set of nonlinear equations:

$$\frac{\partial^2 x}{\partial \tau^2} + \left(\omega_1^2 - v\omega_{\text{p}}^2\right)x + \varkappa_{12} xy + \varkappa_{11} x^2 = 0\,, \qquad (67.1)$$

$$\frac{\partial^2 y}{\partial \tau^2} + \left(\omega_2^2 - v\omega_{\text{p}}^2\right)x + \varkappa_{12} xy + \varkappa_{22} y^2 = 0\,. \qquad (67.2)$$

If both the eigenfrequencies of normal oscillations and the anharmonicity constants are equal to each other (a model of an isotropic anharmonic oscillator), then the problem considered can be reduced to the case of a scalar model. A model of an anisotropic oscillator is much more interesting.

Consider a more general case when the potential energy of small oscillations of a molecule or an electron around the equilibrium position can be written in an arbitrary form, $U(x,y)$. Assuming that the generalised coordinates correspond to small deviations of the system under study from the equilibrium position, we can expand this potential as a Taylor series:

$$U(x,\,y) = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}x^2 + \frac{1}{2}\frac{\partial^2 U}{\partial y^2}y^2 + \frac{\partial^2 U}{\partial x \partial y}xy + \frac{1}{6}\frac{\partial^3 U}{\partial x^3}x^3 + \frac{1}{6}\frac{\partial^3 U}{\partial y^3}y^3 + \frac{1}{2}\frac{\partial^3 U}{\partial x^2 \partial y}x^2 y + \frac{1}{2}\frac{\partial^3 U}{\partial y^2 \partial x}y^2 x + \ldots\,.$$

Restricting our analysis to the cubic terms in this expansion, we can replace Eqns (66) by the Newton equations for the anharmonic oscillator considered:

$$\frac{\partial^2 x}{\partial t^2} + \omega_1^2 x + \omega_{12}^2 y + 2\varkappa_{12} xy + 3\varkappa_{11} x^2 + \varkappa_{21} y^2 = \frac{e}{m} E_1(z,t)\,, \qquad (68.1)$$

$$\frac{\partial^2 y}{\partial t^2} + \omega_2^2 y + \omega_{21}^2 x + 2\varkappa_{21} xy + 3\varkappa_{22} y^2 + \varkappa_{12} x^2 = \frac{e}{m} E_2(z,\,t)\,, \qquad (68.2)$$

The coefficients involved in these equations are related to the coupling constants of the potential $U(x,y)$.

We can then once again determine the relation between the electric field strength and the generalised coordinate, $E_{1,2} = \beta x, \beta y$, and derive the final equations in a form similar to Eqns (67):

$$\frac{\partial^2 x}{\partial t^2} - p_1^2 x + \omega_{12}^2 y + 2\varkappa_{12} xy + 3\varkappa_{11} x^2 + \varkappa_{12} y^2 = 0\,, \qquad (69.1)$$

$$\frac{\partial^2 y}{\partial t^2} - p_2^2 y + \omega_{21}^2 x + 2\varkappa_{21} xy + 3\varkappa_{22} y^2 + \varkappa_{12} x^2 = 0\,. \qquad (69.2)$$

To find particular solutions to the set (69) we can assume that the normal oscillations are proportional to each other, $x \sim y$, and the potential $U(x,y)$ has a symmetric shape. It would be much more important to find solutions to the set (69) without imposing limitations on the potential. However, such solutions have so far not been found.

7. Conclusions

Thus we have considered some cases of model media that allow the explicit temporal dependence of the electric field to be determined in the analytical form for an arbitrarily short pulse of the electromagnetic field. We did not make any assumptions concerning the harmonic carrier of the wave or the variation rate of the field in the pulse. In most cases, only particular solutions describing the stationary propagation of video pulses can be found. Such solutions correspond to sufficiently strong electromagnetic fields where the dispersion inherent in the medium is suppressed by nonlinear processes.

The estimates presented above demonstrate that the amplitude of the field strength in stationary video pulses is close to the strength of the atomic field in a medium. Consequently, ionisation processes and multiphoton absorption should be taken into account for an accurate description of the models considered. In the case of weaker and nonstationary pulses, the dispersion of a nonlinear medium leads to the broadening of pulses enveloping one or several cycles of oscillations of the electric field around its zero value and results in modulation of the trailing edges of such pulses. We may anticipate that such pulses should gradually evolve into quasi-harmonic waves.

A comprehensive review of all the results obtained in the considered area of research is beyond the scope of this paper. However, it seems useful to mention several studies. In particular, we should note that an analysis beyond the framework of the model of two-level atoms was performed. A multifrequency ultrashort pulse involved in a cascade process of stimulated Raman scattering was investigated in Refs [19, 70]. The derived stationary solutions to generalised Maxwell – Bloch equations represent a nonlinear superposition of individual single-frequency Lorentzian 'bright solitons'. The interference of these solitons may give rise to the formation of a sequence of easily resolvable high-power pulses with a duration of about 0.2 fs.

In a review [71] Shvartsburg discusses the method that can be employed to model video pulses with an arbitrary steepness of pulse edges and an asymmetric shape. It is proposed to represent such pulses as a superposition of waves with an envelope described by orthogonal polynomials rather than as wave packets consisting of harmonic waves. In particular, the Laguerre optics of video pulses is considered. This analysis demonstrates that video pulses and quasi-harmonic waves display different behaviour in the linear case.

Since the notion of the carrier wave is not defined for video pulses, nonlinear optical phenomena involving such pulses also display some new features. Harmonic generation and parametric frequency summation or subtraction are manifested in the spectral width of a pulse rather than in the change of well-resolved frequency components of radiation (e.g., ω_0, $2\omega_0$, $3\omega_0$, etc.). Consequently, the duration of a video pulse will decrease. On the other hand, material dispersion should impede such shortening of a video pulse, which implies that we deal with two competing mechanisms, determining the duration of an extremely short pulse of electromagnetic radiation.

Similar to many other works, plane-wave pulses were discussed in this paper. An obvious extension of this approach should be associated with the investigation of electromagnetic waves with a more complex geometry, e.g., moving bunches of electromagnetic radiation ('optical bullets' [72], bubbles [20], etc.).

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