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On the normalisation of the observed spectral gain line profile with increasing optical thickness of a substance layer

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Abstract. It is shown that a spectral gain line narrows down and its profile is normalised with increasing optical thickness of a layer of the amplifying medium. As a result, the gain line profile always becomes Gaussian, independently of the true form factor of the line, when the thickness of the active medium layer is sufficiently large. The normalisation of the line profile is demonstrated for the lines with Lorentzian, Gaussian, and 'time-of-flight' profiles.

Consider a layer of substance of thickness z that amplifies the light intensity with the gain $\alpha(\omega)$. If this gain is caused by the population inversion of the levels involved in the spectral transition at the frequency ω_0 and the form factor of the spectral line $g(\Omega)$ ($\omega = \omega_0 + \Omega$), we have [1, 2]

$$\alpha(\omega) = \alpha(\omega_0 + \Omega) = \alpha_0 g(\Omega). \tag{1}$$

Here, the true form factor $g(\Omega)$ of the spectral line is normalised to unity at the maximum [$g(0) = 1$] and $\alpha_0 = \alpha(\omega_0)$ is the gain at the line centre. Then, the light intensity transfer coefficient of a layer $G(\omega, z) = \exp[\alpha(\omega)z]$ has the form [1]

$$G(\omega, z) = G(\Omega, \xi) = \exp[\xi g(\Omega)], \tag{2}$$

where $\xi = \alpha_0 z$ is the optical thickness of the layer.

It is obvious that the line shape observed in the absence of nonresonance absorption is described by the function

$$\Phi(\Omega, \xi) = G(\Omega, \xi) - 1 = \exp[\xi g(\Omega) - 1]$$

$$= [\exp(\xi) - 1]\gamma(\Omega, \xi), \tag{3}$$

where

$$\gamma(\Omega, \xi) \equiv \frac{\Phi(\Omega, \xi)}{\Phi(0, \xi)} = \frac{\exp[\xi g(\Omega)] - 1}{\exp(\xi) - 1} \tag{4}$$

is the observed form factor of the line normalised to unity at the maximum and equal to zero away from the line centre ($\gamma(0, \xi) = 1$, $\gamma(\pm\infty, \xi) = 0$ for any ξ).

It is well known [1, 2] that because the transfer coefficient $G(\Omega,\xi)$ of a layer depends exponentially on the true form factor $g(\Omega)$ of the line, the observed form factor $\gamma(\Omega,\xi)$ (4) will coincide with the true one only for a small optical thickness of the layer ($\xi \ll 1$) and, therefore, for a weak gain in the layer. Then, by neglecting the terms of the order of ξ, we obtain

$$\gamma(\Omega, \xi) = g(\Omega). \tag{5}$$

When the optical density of the layer is not small ($\xi \simeq 1$ or $\xi \gg 1$) and, hence, the gain $\Phi(\Omega,\xi)$ is large, the observed form factor $\gamma(\Omega,\xi)$ will differ from the true one. In particular, upon collision broadening of the spectral line (the Lorentzian line shape) and $\xi \gg 1$, the observed width of the spectral line will decrease as $\sim \xi^{-1/2}$ with increasing optical thickness ξ [1].

The aim of this work is to emphasise the fact that an increase in the optical thickness of a layer causes both the narrowing of the spectral gain line and its normalisation. As a result, the line shape tends to a Gaussian whose parameters are determined by the optical thickness of the layer ξ and the sharpness of the true form factor $g(\Omega)$ of the line at the line centre $g''(0)$. The observed form factor $\gamma(\Omega,\xi)$ does not depend on the other parameters of the line in the limit $\xi \to \infty$.

Indeed, by expanding the function $\ln\gamma(\Omega,\xi)$ into a series in Ω near the line centre ($\Omega = 0$) and retaining only the first two terms of the expansion (quadratic approximation), we easily obtain the following Gaussian approximation for the form factor $\gamma(\Omega,\xi)$ (4)

$$\gamma(\Omega, \xi) = \exp\left[-\left(\frac{2\Omega}{\Omega_0}\right)^2 \frac{\xi}{1 - \exp(-\xi)}\right] = \beta^{-[2\Omega/\Delta\Omega_{1/\beta}(\xi)]^2}, \tag{6}$$

where $\Omega_0^2 \equiv -8/g''(0)$ and

$$\Delta\Omega_{1/\beta}(\xi) = \Omega_0\{\ln\beta[1 - \exp(-\xi)]/\xi\}^{1/2} \tag{7}$$

is the width of the observed form factor of the line at the $1/\beta$ level of its maximum. In particular, the full width at half maximum (FWHF) $\gamma(\Omega,\xi)$ of the observed form factor is

$$\Delta\Omega_{1/2}(\xi) = \Omega_0 f(\xi),\ f(\xi) \equiv \{\ln(2)[1 - \exp(-\xi)]/\xi\}^{1/2}. \tag{8}$$

For $\xi \ll 1$, we have $f(\xi) = (\ln 2)^{1/2} \simeq 1$, and for $\xi \gg 1$, $f(\xi) = [(\ln 2)/\xi]^{1/2} \ll 1$.

It follows from the method of obtaining approximation (6) that it can be used for any optical thickness ξ but only for $\Omega \to 0$ (with an accuracy to the terms quadratic in Ω). Nevertheless, the Gaussian approximation (6) becomes asymptotically exact for very large optical thickness of the layer ξ.

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To prove this fact, note that it completely coincides with the 'non-probability' part of the proof of the central limit theorem. The 'non-probability' part of the central limit theorem can be formulated as a statement that the product of a sufficiently large number of the characteristic functions of independent random quantities (upon summation of the independent random quantities, their characteristic functions are multiplied) tends to a Gaussian function, which represents the characteristic function of their normally distributed sum [3].

Now it is sufficient to note that $G(\Omega,\xi) = [G(\Omega,1)]^{\xi}$, where the function $G(\Omega,1)$ can be treated as a characteristic function of one random quantity, while the function $G(\Omega,\xi)$ can be treated as a normalised characteristic function of a sum ξ of random quantities. It follows from this that the condition for the asymptotic accuracy of approximation (6) upon an infinite increase in ξ is the existence and negativity of the second derivative of the true form factor at the spectral line centre (a similar condition in the central limit theorem is the existence of the dispersion of random quantities being summed).

Leaving aside an analogy with the central limit theorem, the asymptotic accuracy of (6) follows simply from the known fact that the spectral gain line narrows down with increasing gain. In this case, the observed form factor of the spectral line $\gamma(\Omega,\xi)$ becomes substantially narrower than the true form factor $g(\Omega)$ and its shape is controlled only by the central part of the form factor $g(\Omega)$ rather than by its wings, resulting, taking into account (4), in the Gaussian approximation (6). One should not expect, for example, the normalisation of the observed line shape in the case of the rectangular, trapezoid, or triangular form factor of the line.

Let us specify expressions (6) and (7) for the three most common true form factors: the Lorentzian (L), which is caused by the collision or radiative broadening of the line, the time-of-flight (T), which appears due to the limited time of interaction of light with a substance, and the Gaussian form factor caused by the Doppler broadening of the line [1, 2]. In these cases, the true form factors and the parameter $\Omega_0 \equiv (-8/g''(0))^{1/2}$ have the form

$$g_{\rm L}(\Omega) = \left[1 + (2\Omega/\Delta\Omega_{\rm L})^2\right]^{-1},\ \Omega_0^{\rm L} = \Delta\Omega_{\rm L},$$

$$g_{\rm T}(\Omega) = {\rm sinc}^2\left[x_0(2\Omega/\Delta\Omega_{\rm T})\right],\ \Omega_0^{\rm T} = (3^{1/2}/x_0)\Delta\Omega_{\rm T}, \qquad (9)$$

$$g_{\rm G}(\Omega) = \exp\left[-\ln(2)(2\Omega/\Delta\Omega_{\rm G})^2\right],\ \Omega_0^{\rm G} = (\ln 2)^{-1/2}\Delta\Omega_{\rm L},$$

where $x_0 \approx 1.39$ is the root of the equation ${\rm sinc}^2 x_0 = 1/2$ (${\rm sinc}\,x \equiv (\sin x/x)$); $\Delta\Omega_{\rm L}$, $\Delta\Omega_{\rm T}$, and $\Delta\Omega_{\rm G}$ are the FWHM of the true form factors $g_{\rm L}$, $g_{\rm T}$, and $g_{\rm G}$. The dependence of these parameters on the properties of a medium is presented, for example, in Refs [1, 2].

Taking into account (8) and (9), the FWHMs of the observed spectral gain line for the Lorentzian, time-of-flight, and Gaussian true line shapes are

$$\Delta\Omega_{1/2}^{\rm L}(\xi) = \Delta\Omega_{\rm L} f(\xi),$$

$$\Delta\Omega_{1/2}^{\rm T}(\xi) = (3^{1/2}/x_0)\Delta\Omega_{\rm T} f(\xi), \qquad (10)$$

$$\Delta\Omega_{1/2}^{\rm G}(\xi) = (\ln 2)^{-1/2}\Delta\Omega_{\rm G} f(\xi),$$

where the function $f(\xi)$ is defined in (8).

One can see from (8) that for $\xi \gg 1$, the observed width of the spectral line is determined not by the FWHM of its true form factor $g(\Omega)$ but by its sharpness at the line centre, i. e., by the parameter Ω_0. In particular, it follows from (10) that for the same half-widths of the true form factor, the observed collision line broadening will be approximately 20 – 25% lower than the Doppler or time-of-flight broadening. This appears quite natural because the Lorentzian form factor is sharper at the line centre.

To estimate an actual degree of the normalisation of the observed shape of the spectral line, we compared in Fig. 1 the Gaussian approximation (6) with exact form factors observed for the Lorentzian, time-of-flight, and Gaussian spectral line profiles, which were calculated directly from expressions (4) and (9). Fig. 1 presents functions $\gamma^{\rm L,T,G}(\Omega,\xi)$ for $\xi = 0$ [when the observed form factor γ coincides with the true form factor g (see (5)], the intermediate optical thickness $\xi = 2$, 4, 8, and for $\xi = \infty$ (when the observed form factor is completely normalised). The FWHM $\Delta\Omega_{1/2}$ of the form factor in Fig. 1 expressed in terms of the dimensionless frequency $2\Omega/\Delta\Omega_{1/2}(\xi)$ is determined by expressions (10) and depends on the line broadening mechanism.

An appropriate choice of the scale provides the same 'calculated' width for all the plots, i. e., it ensures automatically the account for the spectral line narrowing with increasing optical thickness ξ of a substance layer according to (8) and for the difference in the observed line widths for different broadening mechanisms according to (10). This ensures a coincidence of all the curves in Fig. 1 at the line centre (at $\Omega = 0$), which allows one to focus attention on their shape (in which they only differ).

Fig. 2 compares the widths of the observed gain line at the levels $1/\beta = 1/2$, $1/10$, and $1/100$ of its maximum for different line broadening mechanisms. Here, $2\Delta\Omega_{1/\beta}/\Omega_0$ is the width of the observed gain line normalised to the sharpness of the true form factor Ω_0. The parameter Ω_0 for the three type of broadening L, T, and G is determined by expressions (9), as before, and the observed line widths $\Delta\Omega_{1/\beta}$ were calculated directly from exact expression (4). The curves corresponding to different broadening mechanisms are indicated in Fig. 2 by letters L, T, and G. The letter A refers to the approximation (6).

An appropriate choice of the ordinate axis scale in Fig. 2, as in Fig. 1, provides the account for the natural difference between collision, time-of-flight, and Doppler line broadening, leaving for analysis only the dependence of 'normalised' widths on the optical thickness of the layer (otherwise, for each type of the broadening, its own asymptotic curve would be required).

Figs 1 and 2 show that in all the cases under study, the normalisation of the observed line profile does occur with increasing optical thickness of the layer. In particular, the asymptotics of expression (7) is confirmed for any choice of the level for the line width $1/\beta$ measurement. At the same time, one can see that for small values of $1/\beta$ (when the width of the line is measured at its 'base'), the region of real applicability of the approximation (6), (7) shifts to a greater optical thickness ξ.

In addition, one can see that in the case of collision broadening (L), the normalisation of the observed spectral gain line occurs most slowly. In this case, the base of the observed line profile is always broader than that in the Gaussian approximation, and the 'almost' Lorentzian line profile passes to the 'almost' Gaussian profile quite lately, at $\xi \approx 4$, which corres-

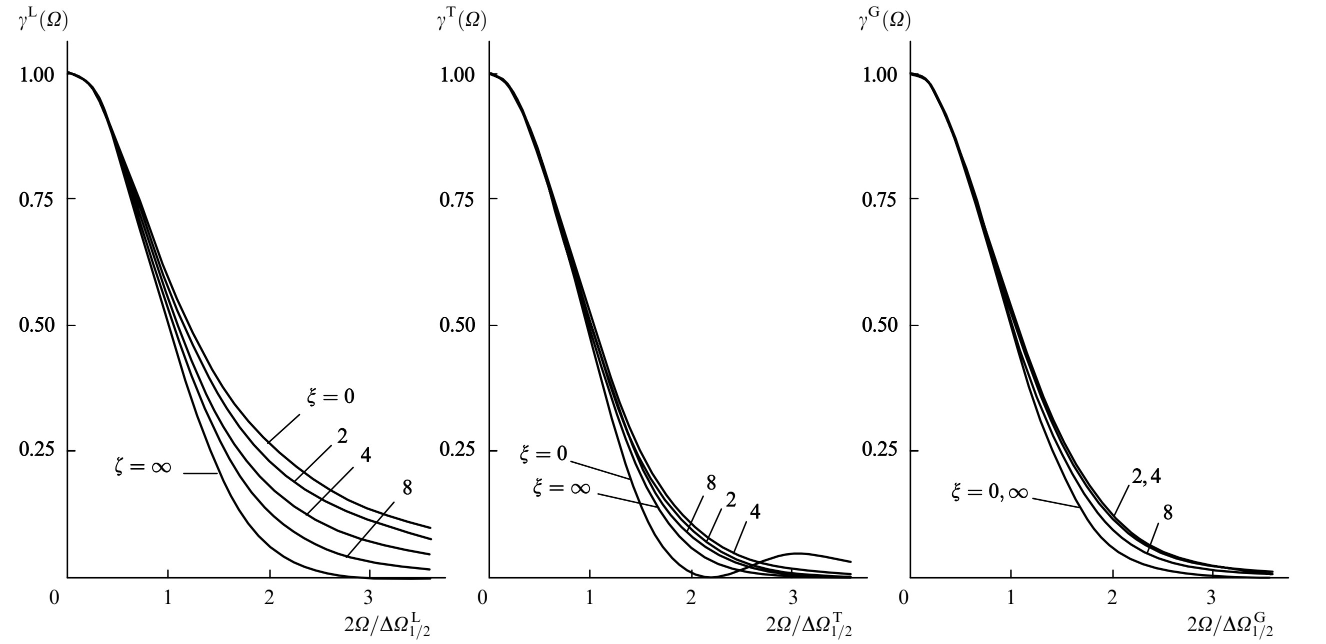

Figure 1. Form factors of the spectral gain line observed at the different optical thickness of a substance layer ξ for Lorentzian (L), Gaussian (G), and time-of-flight (T) true form factors of the line.

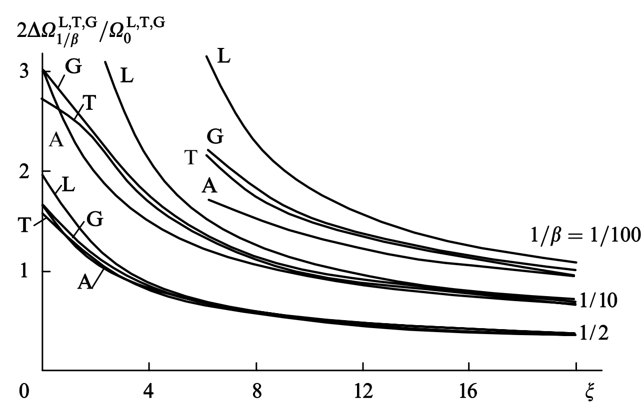

Figure 2. Dependences of the width of the observed spectral gain line at the 1β level of the optical thickness of a substance layer ξ for different parameters $1/\beta$ for Lorentzian (L) , Gaussian (G), and time-of-flight (T) true form factors of the line and their common analytic approximation (A).

ponds to the intensity gain at the line centre $\Phi(0,\xi) \approx 50$. In the case of time-of-flight (T) or Doppler (G) broadening, the observed line shape is normalised much faster. In these cases, the maximum difference (not very large) of the observed line profile from a Gaussian is reached already for $\xi \approx 2$, which corresponds to the gain at the line centre $\Phi(0,\xi) \approx 6$. As the optical thickness of the layer increases, the Gaussian asymptotics (6) is quite rapidly achieved.

Note that for $\xi \geqslant 1$, the shape of the spectral line observed in the case of the time-of-flight or Doppler broadening proves to be virtually the same (which is not the case for the true line shape; see Fig. 1 for $\xi = 0$). In the case of the time-of-flight broadening, the shape of the spectral line that is observed at a small optical thickness ξ substantially differs from a Gaussian; however, already for $\xi \approx 0.5$, it is closer to a Gaussian than to its initial shape. The approach to the Gaussian asymptotics ($\xi \gg 1$) occurs not from the side of the thin-layer asymptotics ($\xi \ll 1$) but from the side that is opposite to the deviation of the Gaussian asymptotics from the true profile at a small optical thickness (see Figs 1 and 2).

In the case of the Doppler line broadening, the approximation (6), (7) is valid not only for $\xi \to \infty$ but also for $\xi \to 0$. Although this circumstance is accidental, it makes the approximation (6), (7) universal for a Gaussian profile of the true form factor of the spectral line, i. e., it is valid for any optical thickness ξ of a substance layer. The error of this approximation is small and reaches a maximum for $\xi \approx 3$, which corresponds to the gain at the line centre $\Phi(0,\xi) \approx 20$.

In a more realistic statement of the problem, one should take into account, along with resonance amplification of light at the frequencies close to the spectral line frequency, nonresonance absorption of light, which weakly depends on the frequency near the resonance frequency ω_0. In this case, instead of (1), the gain is described by the expression

$$\alpha(\omega) = \alpha_0 g(\Omega) - \alpha_1, \tag{11}$$

where α_1 is the nonresonance absorption coefficient, and instead of (2), we have

$$G(\Omega, \xi) = \exp[\xi g(\Omega) - \xi_1] = \exp(-\xi_1)[1 + \Phi(\Omega, \xi)], \tag{12}$$

where $\xi_1 = \alpha_1 z$.

One can easily verify that the replacement of expressions (1) and (2) by (11) and (12) does not lead to any changes in the subsequent discussion except one substantial feature. In this case, the normalisation and narrowing of the spectral line are not necessarily observed only when the gain in the layer is large. Indeed, the normalisation is controlled by the parameter $\xi \gg 1$ and it does not depend on the additional nonresonance attenuation of a signal. This means that for $\xi \approx \xi_1 \gg 1$, the spectral gain line can in principle narrow down to any degree and can be normalised at a moderate gain (or absorption) at the line centre.

It is interesting that such a 'filter' is not an interference filter, in contrast to spectral analysers of the Fabry – Perot interferometer type, because the narrowness of its transmission (or amplification) band is caused simply by the frequency selection of photons propagating through the substance layer rather than by the fulfilment of some phase relations.

Note that an attempt to narrow down a spectral line by increasing the effective optical thickness of the active medium layer in the absence of nonresonance absorption inevitably results in an exponential increase in the signal power and in the necessity to consider the gain saturation [1, 2], which is ignored in this paper.

The results obtained above can be used to take into account not only a 'distributed' filter represented by a layer of the active substance but also 'localised' filters such as semitransparent mirrors, lenses, etc. One can easily verify that the appearance of such elements does not change substantially the results presented above.

If the number of localised filters is finite, their influence can be accounted for by a simple multiplication of the Gaussian transfer coefficient of a layer by the transfer coefficients of localised filters. The result of this multiplication will be determined by the narrowest filter. For a sufficiently thick layer, this is always a Gaussian filter, while the influence of other elements is manifested in a general amplification or attenuation of a signal.

In a more interesting case of the infinite increase in the number of localised filters of the same type with increasing optical thickness of a substance layer (a lens line or an open resonator), the transfer characteristics of additional can be introduced into (2). This will only change the function $\alpha(\omega)$, which will be now determined not only by the true form factor $g(\Omega)$ of the spectral line (1) and the nonresonance absorption coefficient α_1 (11) but also by the parameters of localised filters. In the case of a strong frequency dependence of the transfer characteristics of the filters, this 'renormalisation' will result in the shift in the central frequency of the transmission band from the spectral line centre ω_0 and in the change in the parameter Ω_0. However, it will affect neither an analogy with the central limit theorem noted above nor the normalisation of the transfer characteristic of a substance layer. Note that this analogy will be even more direct because an artificial separation of a substance layer into 'sublayers' of the unit thickness playing the role of individual filters will disappear and one can consider simply the multiplication of 'single-pass' transfer functions upon repeated passages through a filter.

The main result of this study is that the observed profile of the spectral gain line and the transfer function of a substance layer are normalised with increasing optical thickness ξ of the substance independently of the shape of the true form factor of the spectral line and the mechanism of the spectral line broadening. In any case, the observed line shape tends to a Gaussian with increasing optical thickness ξ and the width of the spectral line decreases as $\xi^{-1/2}$. The line shape and its width measured at different levels are described by universal expressions (6) and (7).

This means, in particular, that a noise signal is always normalised when the thickness of the active layer is sufficient [3], the correlation time increasing as $\xi^{1/2}$. The duration of the determinate signal propagating through the layer increases as $\xi^{1/2}$ (which is substantially slower than the usual dispersion spreading of a wave packet [4]). In this case, the signal energy increases as $\xi^{-1/2}\exp\xi$ and its maximum intensity increases as $\xi^{-1}\exp\xi$. Of course, these conclusions are valid only when the correlation time of the noise being filtered or the initial duration of the determinate is sufficiently short. However, because of the narrowing of a spectral line with increasing optical thickness of a substance layer, these conditions are always fulfilled for any noise or signal when the layer is sufficiently thick.

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