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Collective emission of atoms in photonic bandgap materials

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Abstract. A system of resonance atoms doped into a photonic bandgap material is considered in the case when the resonance transition frequency lies within the band gap. Spontaneous emission of a single atom is suppressed in this situation, which is called the light localisation. It is shown that, if the density of the doped atoms is sufficiently high, the coherent interaction between atoms can appear, resulting in the emission of a single or a series of coherent pulses by a system of atoms.

1. Introduction

There is a class of materials where the electromagnetic waves of a certain frequency range $[\Omega_1, \Omega_2]$ cannot propagate. They are called photonic bandgap materials [1 – 3]. The width of the photonic band gap is $\Delta_{\rm p} \equiv \Omega_2 - \Omega_1$. Depending on the physical nature of the photonic band gap, these materials can be divided into two classes. The more popular one includes the materials with an artificially created periodical structure producing the band gap [1, 2]. It is these structures that are usually called photonic bandgap materials [3].

However, there is another mechanism for the creation of photonic band gaps, namely, the interaction of photons with collective optical excitations of the medium, such as optical phonons, magnons, and excitons [4, 5]. In the latter case, the photonic band gap is called the polariton band gap. Regardless of the physical nature, the two types of band gaps can be described by the same mathematical models [6, 7].

If a resonance atom is doped into a photonic bandgap material and its transition frequency lies within this gap, the spontaneous emission of the atom will be suppressed. The excited atom cannot relax to a lower state by emitting a photon because the photon propagation is forbidden inside the band gap. This effect is called the light localisation. Mathematically, it can be expressed by the condition

$$\lim_{t\to\infty} \langle \sigma^z(t) \rangle = \zeta > 0 ,$$

where σ^z is the Pauli operator and the angle brackets denote statistical averaging.

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It is known that, apart from resonance processes, some nonresonance processes always take place. Although in vacuum the probability of the latter can be significantly lower than that of the former, the atom eventually relaxes to a lower state via nonresonance processes. For these reasons, strictly speaking, the above limit should be understood as the condition

$$\langle \sigma^z(t) \rangle = \zeta > 0 , \qquad T_{\rm res} \ll t \ll T_{\rm non},$$

where $T_{\rm res}$ is the characteristic time of spontaneous emission of the resonance atom in vacuum and $T_{\rm non}$ is the characteristic time of emission via nonresonance processes in a matter. The typical lifetime of most atomic levels is of the order of 10^{-8} s [8], whereas the characteristic times of nonresonance processes are $0.1 - 100$ s. This means that, although the light localisation cannot last infinitely, its lifetime is much larger than $T_{\rm res}$. The light localisation is similar to decay freezing in an atomic system undergoing a specific interaction with the thermal bath [9].

To provide a microscopic description of light localisation, one usually considers stationary models with a single doped atom [6, 7, 10]. The studies of these models revealed that, if the number of doped atoms is sufficiently large, an impurity band can appear inside the forbidden zone, allowing the propagation of electromagnetic waves [10 – 12]. However, to provide an accurate description of the physical processes and the conditions for light localisation, one should consider the actual problem of the nonequilibrium behaviour of the atoms doped into a photonic bandgap material.

The dynamics of a system of resonance atoms whose transition frequency is near a photonic bandgap edge was considered in Refs [13 – 15] using the localised Dicke model. However, the application of this model to the problem of light localisation is not justified. Indeed, the localised model assumes that the emission wavelength is much larger than the sample dimensions. On the other hand, light localisation itself consists in the emission of a photon that propagates a localisation length of the order of several wavelengths. Having been reflected from scatterers, which can represent spatial structures or optical collective excitations depending on the medium type, the emitted photon returns to the atom and is reabsorbed by it.

This is a physical picture of light localisation. Thus, it is obvious that one can speak about light localisation only if the localisation length and, therefore, the wavelength are much smaller than the characteristic sample dimensions. In the case of the localised model, the wavelength is much larger than the sample dimensions and the notion of light localisation becomes senseless.

A one-dimensional model of atoms doped into a photonic bandgap material with Kerr nonlinearity was considered in Ref. [16]. However, as in Refs [13–15], the resonance frequency of the atoms was assumed to lie either near the edge of the photonic band gap or far outside it. The purpose of Ref. [16] was to study the appearance of short solitons.

The formulation of the physical problem and the purposes of this work are fundamentally different from those of the previously published papers by other authors. First, we consider the actual *three-dimensional dynamic* problem that is based on the microscopic Hamiltonian of a system of atoms doped into a medium. Second, we assume that the emission wavelength is *much smaller* than the sample dimensions, which is a necessary condition for light localisation. Third, the resonance frequency of the atoms is assumed to lie *deep inside* the photonic band gap, so that the spontaneous emission of a single atom is virtually completely suppressed. The purpose of this work is to determine the conditions when the collective nonlinear effects make possible delocalisation of light.

2. Model Hamiltonian

A system of resonance atoms in a medium can be described by the Hamiltonian [17]

$$\hat{H} = \hat{H}_{\mathrm{a}} + \hat{H}_{\mathrm{f}} + \hat{H}_{\mathrm{af}} + \hat{H}_{\mathrm{m}} + \hat{H}_{\mathrm{mf}}, \tag{1}$$

which consists of the following terms:

$$\hat{H}_{\mathrm{a}} = \frac{1}{2}\sum_{i=1}^{N}\omega_0(1+\sigma_i^z) \tag{2}$$

is the Hamiltonian of two-level resonance atoms with the transition frequency ω_0;

$$\hat{H}_{\mathrm{f}} = \frac{1}{8\pi}\int\left(\boldsymbol{E}^2+\boldsymbol{H}^2\right)\mathrm{d}\boldsymbol{r} \tag{3}$$

is the Hamiltonian of the radiation field. Here, $\boldsymbol{E}$ is the electric field, $\boldsymbol{H} = \nabla\times\boldsymbol{A}$ is the magnetic field, and $\boldsymbol{A}$ is the vector potential satisfying the Coulomb calibration $\nabla\boldsymbol{A} = 0$;

$$\hat{H}_{\mathrm{af}} = -\frac{1}{c}\sum_{i=1}^{N}\boldsymbol{j}_i\boldsymbol{A}_i \tag{4}$$

is the Hamiltonian of the interaction between the atoms and the radiation field. Here, $\mathrm{j}_i = \boldsymbol{i}\omega_0(\boldsymbol{d}\sigma_i^+ - \boldsymbol{d}^*\sigma_i^-)$ is the transition current; $\boldsymbol{d}$ is the transition dipole moment; $\sigma_i^\pm$ are the ladder operators; c is the speed of light; and $\boldsymbol{A}_i \equiv \boldsymbol{A}(\boldsymbol{r}_i, t)$;

$$\hat{H}_{\mathrm{mf}} = -\frac{1}{c}\sum_{i=1}^{N_0}\boldsymbol{j}_{\mathrm{m}i}\boldsymbol{A}_i \tag{5}$$

is the Hamiltonian of the interaction between the medium and the radiation field; $\boldsymbol{j}_{\mathrm{m}i}$ is the electric current produced by the medium particles; and $\hat{H}_{\mathrm{m}}$ is the Hamiltonian of the medium, which can be modelled by a system of oscillators. Instead of specifying the concrete form of $\hat{H}_{\mathrm{m}}$, we will model the effective interaction between the atoms and the medium. Below, this method will be used.

3. Equations of motion

To derive the equations of motion, we will use the method based on the elimination of field variables [18]. According to the Maxwell operator equations, the field variables have the form

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{vac}} + \boldsymbol{A}_{\mathrm{rad}} + \boldsymbol{A}_{\mathrm{mat}}, \tag{6}$$

where $\boldsymbol{A}_{\mathrm{vac}}$ is the vector potential of vacuum fluctuations;

$$\begin{aligned}\boldsymbol{A}_{\mathrm{rad}}(\boldsymbol{r}_i, t) &= \sum_{j}^{N}\frac{1}{cr_{ij}}\boldsymbol{j}_j\left(t-\frac{r_{ij}}{c}\right),\\ \boldsymbol{A}_{\mathrm{mat}}(\boldsymbol{r}_i, t) &= \sum_{j}^{N_0}\frac{1}{cr_{ij}}\boldsymbol{j}_{mj}\left(t-\frac{r_{ij}}{c}\right)\end{aligned} \tag{7}$$

are the vector potentials induced by the atoms and the medium, respectively; $r_{ij} \equiv |\boldsymbol{r}_i - \boldsymbol{r}_j|$; and the summation does not include the terms with $j = i$. The joint action of the vacuum and the medium on the atoms is equivalent to the action of the effective vacuum characterised by the vector potential

$$\boldsymbol{A}_{\mathrm{eff}}(\boldsymbol{r}, t) \equiv \boldsymbol{A}_{\mathrm{vac}}(\boldsymbol{r}, t) + \boldsymbol{A}_{\mathrm{mat}}(\boldsymbol{r}, t). \tag{8}$$

The action of the effective vacuum on atoms is described by the stochastic variable

$$\xi_i(t) \equiv k_0\boldsymbol{d}\boldsymbol{A}_{\mathrm{eff}}(\boldsymbol{r}_i, t), \tag{9}$$

where $k_0 \equiv \omega_0/c$.

We can eliminate the field variables $\boldsymbol{A}$ and $\boldsymbol{E}$ from the equations of motion by inserting the operator relationships (6)–(8) into the Heisenberg equations for the atomic variables σ_i^z and $\sigma_i^\pm$. In this way, the number of equations is reduced, but new nonlinear terms appear which describe the effective interatomic interaction induced by the radiation field. In the equations for the atomic operators, only the stochastic variable (9) remains. Its modelling is equivalent to the modelling of the properties of the medium with doped atoms. By averaging the equations of motion over the atomic degrees of freedom, we introduce the notation

$$u_i(t) \equiv \langle\sigma_i^-(t)\rangle', \quad s_i(t) \equiv \langle\sigma_i^z(t)\rangle', \tag{10}$$

where the prime means that the averaging is performed over the atomic variables only, excluding the stochastic variable (9). For such partial averages, we will assume that

$$\left\langle\sigma_i^\alpha\sigma_j^\beta\right\rangle' = \langle\sigma_i^\alpha\rangle'\left\langle\sigma_j^\beta\right\rangle' \quad (i \neq j). \tag{11}$$

The retardation of the potentials (7) will be treated within the Born approximation:

$$\left\langle\sigma_j^-\left(t-\frac{r_{ij}}{c}\right)\right\rangle' = \langle\sigma_j^-(t)\rangle'\exp(\mathrm{i}k_0 r_{ij}). \tag{12}$$

The uncoupling (11) is similar to the quasiclassical approximation, but not equivalent to it, since in our case the averaging does not involve the stochastic variable (9). As a result of the uncoupling (11), we obtain a closed system of equations for the variables $\langle\sigma_i^\alpha\rangle'$. However, these equations are stochastic as they explicitly contain the stochastic variable (9). When calculating the values of observables, one should average over this variable. This approach takes into account

local quantum fluctuations, which distinguishes it from the quasiclassical approximation. Therefore, the uncoupling (11) can be called the stochastic mean-field approximation [17].

In order to make the equations of motion more compact, we introduce the notation

$$f_i(t) \equiv k_0 \langle \boldsymbol{dA}_{\rm rad}(r_i, t)\rangle' + \xi_i(t). \tag{13}$$

We will also use the continuous representation where the functions (10) depend on the continuous vector $\boldsymbol{r}$, so that

$$u = u(\boldsymbol{r}, t)\ , \qquad s = s(\boldsymbol{r}, t)\ .$$

Correspondingly, the summation over i from unity to N should be replaced by the integration $\rho \int \mathrm{d}\boldsymbol{r}$, where $\rho \equiv N/V$ is the density of resonance atoms. Finally, introducing the function

$$v = v(\boldsymbol{r}, t) \equiv |u(\boldsymbol{r}, t)|^2, \tag{14}$$

we obtain the following equations of motion

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= -(\mathrm{i}\omega_0 + \gamma_2)u + fs\ , \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -2(u^* f + f^* u) - \gamma_1(s - \zeta), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -2\gamma_2 v + s(u^* f + f^* u). \end{aligned} \tag{15}$$

Since the definition of the function $f = f(\boldsymbol{r}, t)$ (13) contains the stochastic variable $\xi = \xi(r, t)$ defined in Eqn (9), Eqns (15) are stochastic differential equations. They include the terms containing the longitudinal (γ_1) and the transverse (γ_2) relaxation rates, as well as the parameter ζ, which is close to the average initial population inversion,

$$s_0 \equiv \frac{1}{V}\int s(\boldsymbol{r}, 0)\boldsymbol{dr},$$

and which describes the suppression of the spontaneous emission of single atoms.

Generally speaking, the spontaneous emission of a single atom can be suppressed by one of the two physically different mechanisms. Depending on the relationship between the characteristic times T_1 and $T_{\rm non}$, this is dynamic suppression ($T_1 \ll T_{\rm non}$) or static suppression ($T_1 \sim T_{\rm non}$). In the former case, the equations of motion should contain the relaxation rate γ_1, whereas in the latter case, one should set $\gamma_1 = 0$.

4. The possibility of emission

In our analysis of the evolution equations (15), we will assume that no spatial electromagnetic structures are created in the sample. In this case, one can use the homogeneous approximation, where the functions u, s, and v are independent of the spatial variables. In this approximation,

$$k_0 \langle \boldsymbol{dA}_{\rm rad}\rangle' = \gamma_2\left(G^* u + G u^* \boldsymbol{e}_d^2\right),$$

where

$$G \equiv \frac{3\gamma\rho}{4\gamma_2}\int \frac{\sin(k_0 r) + \mathrm{i}\cos(k_0 r)}{k_0 r}\mathrm{d}\boldsymbol{r}; \tag{16}$$

and $\gamma \equiv 4k_0^3|\boldsymbol{d}|^2/3$ is the natural linewidth. We also introduce the collective transition frequency

$$\Omega \equiv \omega_0 + \gamma_2 g' s \tag{17}$$

and the collective decay rate

$$\Gamma \equiv \gamma_2(1 - gs), \tag{18}$$

where

$$\begin{aligned} g &\equiv \mathrm{Re}G = \frac{3\gamma\rho}{4\gamma_2}\int \frac{\sin(k_0 r)}{k_0 r}\mathrm{d}\boldsymbol{r}; \\ g' &\equiv \mathrm{Im}G = \frac{3\gamma\rho}{4\gamma_2}\int \frac{\cos(k_0 r)}{k_0 r}\mathrm{d}\boldsymbol{r}. \end{aligned} \tag{19}$$

Then, equations (15) take the form

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= -(\mathrm{i}\Omega + \Gamma)u + s\xi + \gamma_2 G s u^* \boldsymbol{e}_d^2, \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -4\gamma_2 g v - 2(u^*\xi + \xi^* u) - \gamma_1(s - \zeta) \\ &\quad -2\gamma_2\left[G(u^*\boldsymbol{e}_d)^2 + G^*(\boldsymbol{e}_d^* u)^2\right], \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -2\Gamma v + s(u^*\xi + \xi^* u) + \gamma_2 s\left[G(u^*\boldsymbol{e}_d)^2 + G^*(\boldsymbol{e}_d^* u)^2\right], \end{aligned} \tag{20}$$

where $\boldsymbol{e}_d \equiv \boldsymbol{d}/d$.

The quantities (16) and (19) are the parameters of atom-atom coupling. The coupling of the atoms to the effective vacuum producing the vector potential (8) is described by the function

$$\alpha \equiv \frac{1}{s\Gamma}\mathrm{Re}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \langle\langle u^*(t)\xi(t)\rangle\rangle \mathrm{d}t, \tag{21}$$

where the double angle brackets denote the averaging over the stochastic variable (9).

Equations (20) can be solved employing the method of scales separation [19, 20], which is the generalisation of the averaging method [21] to the case of stochastic differential equations. The method of scales separation [19, 20] can be applied because there are small parameters

$$\frac{\gamma_1}{\omega_0} \ll 1, \qquad \frac{\gamma_2}{\omega_0} \ll 1, \qquad |\alpha| \ll 1. \tag{22}$$

Inequalities (22) classify the function u as a fast one with respect to the slow functions s and v. Treating the slow functions as quasi-invariants, for the fast function we have

$$u(t) = \left[u_0 + s\int_0^t \mathrm{e}^{(\mathrm{i}\Omega+\Gamma)t'}\ \xi(t')\ \mathrm{d}t'\right]\mathrm{e}^{-(i\Omega+\Gamma)t}, \tag{23}$$

where $u_0 = u(0)$. In the case $\langle\langle\xi\rangle\rangle = 0$, the quantity α (21), which describes the atom-medium coupling, takes the form

$$\alpha = \frac{1}{\Gamma}\mathrm{Re}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \mathrm{d}t\int_0^t \mathrm{e}^{-(\mathrm{i}\Omega+\Gamma)(t-t')}\langle\langle\xi^*(t)\xi(t')\rangle\rangle \mathrm{d}t'. \tag{24}$$

When performing the integration in Eqn (24), one should treat the slow variables as constants. In the next section we will give a specific example of calculating the quantity α (24). By substituting the fast function (23) into the equations for the slow variables and averaging the right-hand sides of the equations over time and the stochastic variable, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -4g\gamma_2\left(v - \alpha s^2\right) - \bar{\gamma}_1\left(s - \bar{\zeta}\right),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2\gamma_2(1 - gs)(v - \alpha s^2)\,, \tag{25}$$

where

$$\bar{\gamma}_1 \equiv \gamma_1 + 4\gamma_2\alpha; \qquad \bar{\zeta} = \frac{\gamma_1}{\bar{\gamma}_1}\zeta\,.$$

We will first consider the more general case when $\gamma_1 \neq 0$. To find out the conditions for light localisation, we have to consider the behaviour of the solutions s and v at $t \to \infty$, that is, the stationary points of equations (25). In the case $g\bar{\zeta} < 1$, the stationary solutions are

$$s_1^* = \bar{\zeta}\,, \qquad v_1^* = \alpha\bar{\zeta}^2.$$

Hereafter, we assume for siplicity that the coupling parameter $|\alpha| \approx$ const. Since $|\alpha| \ll 1$, we have $\bar{\zeta} \approx \zeta$ and there is virtually no light delocalisation.

For $g\bar{\zeta} > 1$, the solutions of equations (25) tend to the stationary point

$$s_2^* = \frac{1}{g}\,, \qquad v_2^* = \frac{\bar{\gamma}_1(g\bar{\zeta} - 1)}{4\gamma_2 g^2} + \frac{\alpha}{g^2}.$$

In this case, partial light delocalisation takes place because $g^{-1} < \zeta$. The light leaks out either as a single broad pulse or, in the case

$$g\bar{\zeta} > 1 + \frac{\bar{\gamma}_1}{8\gamma_2},$$

as a series of narrow coherent pulses. At $t \to \infty$, the time period between two neighbouring pulses is

$$T_\infty = \frac{4\pi}{\left[8(g\bar{\zeta} - 1)\bar{\gamma}_1\gamma_2 - \bar{\gamma}_1^2\right]^{1/2}}.$$

Thus, the light delocalisation via collective emission of atoms is possible only if the coherent interatomic interaction is sufficiently strong, so that $g\bar{\zeta} > 1$, which is equivalent to

$$\frac{g\zeta\gamma_1}{\gamma_1 + 4\alpha\gamma_2} > 1. \tag{26}$$

Given that $\zeta \approx s_0$ and $|\alpha| \ll 1$, the last equality reduces to the condition

$$gs_0 > 1. \tag{27}$$

Consider now the case when the longitudinal relaxation is completely suppressed, so that $\gamma_1 = 0$. It follows from equations (25) that

$$s(t) = -\frac{\gamma_0}{g\gamma_2}\tanh\left(\frac{t - t_0}{\tau_0}\right) + \frac{1}{g},$$

where

$$\gamma_0 \equiv 2g|s_0|\gamma_2\sqrt{\alpha_\mathrm{c} - \alpha} \equiv \frac{1}{\tau_0};$$

$$\alpha_\mathrm{c} \equiv \left|\frac{gs_0 - 1}{2gs_0}\right|^2 + \left|\frac{u_0}{s_0}\right|^2;$$

$$t_0 \equiv \frac{\tau_0}{2}\,\ln\left|\frac{\gamma_0 - \gamma_2(1 - gs_0)}{\gamma_0 + \gamma_2(1 - gs_0)}\right|.$$

Light delocalisation implies the inequality $s(\infty) < s_0$; in addition, the delocalisation should take place at times $t_0 \ll T_1$. Since

$$s(\infty) = -2|s_0|\sqrt{\alpha_\mathrm{c} - \alpha} + \frac{1}{g},$$

the condition for light delocalisation takes the form

$$g\left(s_0 + 2|s_0|\sqrt{\alpha_c - \alpha}\right) > 1. \tag{28}$$

The most interesting is the case when the atoms do not experience any external coherent driving at the initial time instant, $u_0 = 0$. Then, if we assume that $\alpha \ll \alpha_\mathrm{c}$, inequality (28) transforms to condition (27). If the interatomic coupling parameter is large, $g \gg 1$, then we have $\alpha_\mathrm{c} \simeq 1/4$ and $s(\infty) \simeq -|s_0|$, i.e., virtually all light is released.

It is interesting to compare the conditions of delocalisation to the conditions of superradiance [22]. The latter takes place if $\tau_0 < T_2$, which in our case is equivalent to

$$2g|s_0|\sqrt{\alpha_\mathrm{c} - \alpha} > 1. \tag{29}$$

For $u_0 = 0$ and $\alpha \ll \alpha_\mathrm{c}$, this inequality reduces to

$$gs_0 > 2. \tag{30}$$

Thus, the meanings of the delocalisation condition (27) and the superradiance condition (30) are close but not equivalent.

5. Polariton band gap

To clarify the situation, we have to explain how the coupling constant α (24) is calculated. To be specific, we will consider a material with a polariton forbidden gap. The polariton effect is well pronounced in many dielectrics and semiconductors [4, 5, 23], for example, in CuCl, CuBr, CdSe, ZnSe, GaAs, GaSb, InAs, AlAs, and SiC. When there is a polariton band gap, the polariton spectrum consists of two branches:

$$0 \le \omega_{q1} < \Omega_1\,, \qquad \Omega_2 \le \omega_{q2} < \infty\,, \tag{31}$$

where q is the wave number. The width of the gap is $\Delta_\mathrm{p} \equiv \Omega_2 - \Omega_1 > 0$. The frequencies of the lower and the upper gap edges are related by the expression $\varepsilon_0\Omega_1^2 = \varepsilon_\infty\Omega_2^2$, where ε_0 and ε_∞ are the dielectric constants for static and high-frequency fields, respectively. The polariton spectrum of a cubic crystal consists of the two branches

$$\omega_{qs}^2 = \frac{1}{2}\left\{\Omega_2^2 + c^2q^2 \pm \left[(\Omega_2^2 + c^2q^2)^2 - 4\Omega_1^2c^2q^2\right]^{1/2}\right\}, \tag{32}$$

where c is the speed of light in the medium, which is related to the speed of light in vacuum by $c = c_0/\sqrt{\varepsilon_\infty}$. When the

polariton decay rate $\Gamma_{\rm p} \sim 10^{12}$ s^{-1} is introduced, the dispersion relationship takes the form

$$\frac{c^2k^2}{\omega^2} = \frac{\Omega_2^2 - \omega^2}{\Omega_1^2 - \omega^2 + {\rm i}\omega\Gamma_{\rm p}}. \tag{33}$$

When calculating the averages of the stochastic variable $\xi(t)$ appearing in Eqn (24), one should take into account the sample geometry. In the following, we will assume that the sample is cylindrical with the radius R, the length L ($R \ll L$), and the volume $V = \pi R^2 L$. The cylinder axis corresponds to the seed mode ${\rm e}^{{\rm i}kz}$ with the wave number $k \approx k_0 \equiv \omega_0/c$. Then,

$$\xi(t) = \frac{1}{N_0}\sum_{i=1}^{N_0} \xi_i(t){\rm e}^{-{\rm i}kz_i}, \tag{34}$$

where $\xi_i(t)$ is defined by Eqn (9). The effective vector potential

$$\boldsymbol{A}_{\rm eff}(\boldsymbol{r}_i, t) = \sum_j^{N_0} \frac{1}{cr_{ij}} \boldsymbol{J}_{\rm eff}\left(\boldsymbol{r}_j, t - \frac{r_{ij}}{c}\right)$$

is determined by the medium current $\boldsymbol{J}_{\rm eff}(r_j, t) = (e/m)\boldsymbol{p}_j(t)$, where e and m are the ion charge and mass. The moment $\boldsymbol{p}_j$ is defined by the second-quantised form

$$\boldsymbol{p}_j(t) = -{\rm i}\sum_{q,s}\left(\frac{m\omega_{qs}}{2N_0}\right)^{1/2}\left(b_{qs}\,{\rm e}^{-{\rm i}\omega_{qs}t} - b_{-qs}^{+}\,{\rm e}^{{\rm i}\omega_{qs}t}\right)\boldsymbol{e}_{qs}{\rm e}^{{\rm i}\boldsymbol{q}\boldsymbol{r}_j},$$

where the summation over s includes the summation over the two spectrum branches and two transverse polarisations.

Thus, the stochastic variable can be modelled by the expression

$$\xi(t) = -{\rm i}\sum_{q,s} kk_0\boldsymbol{d}\boldsymbol{e}_{qs}\left(\frac{r_{\rm e}\omega_{qs}}{2N_0}\right)^{1/2} \times\left(\varphi_{qs}b_{qs}{\rm e}^{-{\rm i}\omega_{qs}t} - \varphi_{-qs}b_{qs}^{+}{\rm e}^{{\rm i}\omega_{qs}t}\right), \tag{35}$$

where $r_{\rm e} \equiv e^2/mc^2$,

$$\varphi_{qs} \equiv \frac{1}{N_0}\sum_{i\neq j}^{N_0}\frac{1}{kr_{ij}}\exp\left({\rm i}\frac{\omega_{qs}}{c}r_{ij} + {\rm i}\boldsymbol{q}\boldsymbol{r}_j - {\rm i}kz_i\right); \tag{36}$$

$$\varphi_{-qs} \equiv \frac{1}{N_0}\sum_{i\neq j}^{N_0}\frac{1}{kr_{ij}}\exp\left(-{\rm i}\frac{\omega_{qs}}{c}r_{ij} - {\rm i}\boldsymbol{q}\boldsymbol{r}_j - {\rm i}kz_i\right). \tag{37}$$

For the averages of the Bose operators b_{qs}, we have

$$\langle\langle b_{qs}^{+}b_{q's'}\rangle\rangle = \delta_{qq'}\delta_{ss'}n_{qs}, \qquad \langle\langle b_{qs}b_{q's'}\rangle\rangle = 0,$$

where $n_{qs} = ({\rm e}^{\beta\omega_{qs}} - 1)^{-1}$. Using these equalities, we find the correlator

$$\langle\langle\xi^*(t)\xi(t')\rangle\rangle = \sum_{q,s}\frac{kr_{\rm e}k_0^3}{2N_0}|\boldsymbol{d}\boldsymbol{e}_{qs}|^2\omega_{qs}\Big[|\varphi_{qs}|^2 n_{qs}\,{\rm e}^{{\rm i}\omega_{qs}(t-t')} + |\varphi_{-qs}|^2(1+n_{qs}){\rm e}^{-{\rm i}\omega_{qs}(t-t')}\Big]. \tag{38}$$

Inserting Eqn (38) into Eqn (24), we obtain

$$\alpha = \sum_{q,s}\frac{kr_{\rm e}k_0^3}{2N_0}|\boldsymbol{d}\boldsymbol{e}_{qs}|^2\omega_{qs}\left[\frac{n_{qs}|\varphi_{qs}|^2}{(\Omega-\omega_{qs})^2 + \Gamma^2} + \frac{(1+n_{qs})|\varphi_{-qs}|^2}{(\Omega+\omega_{qs})^2 + \Gamma^2}\right]. \tag{39}$$

Passing from summation to integration in Eqn (36), (37),we make use of the equality

$$\frac{1}{V}\int {\rm e}^{{\rm i}(\boldsymbol{q}-\boldsymbol{k})\boldsymbol{r}}{\rm d}\boldsymbol{r} = \delta_{qk}\,,$$

which yields

$$\varphi_{qs} = \delta_{qk}\varphi_{ks}\,, \qquad \varphi_{-qs} = \delta_{-qk}\varphi_{ks}^*\,,$$

where

$$\varphi_{ks} \equiv \int\frac{\rho_0}{kr}{\rm e}^{{\rm i}(k_s r - kz)}{\rm d}\boldsymbol{r}\,; \tag{40}$$

$k_s \equiv \omega_{ks}/c$, and $\rho_0 \equiv N_0/V$. Equation (39) then transforms to

$$\alpha = \sum_s\frac{kr_{\rm e}}{8N_0}\gamma_{ks}|\varphi_{ks}|^2\omega_{ks}\left[\frac{n_{ks}}{(\Omega-\omega_{ks})^2 + \Gamma^2} + \frac{1+n_{ks}}{(\Omega+\omega_{ks})^2 + \Gamma^2}\right], \tag{41}$$

where

$$\gamma_{ks} \equiv 3\gamma\,|\boldsymbol{e}_d\boldsymbol{e}_{ks}|^2. \tag{42}$$

Since $k \approx k_0$ and $\omega_{ks} \sim \omega_o$, we have

$$\frac{|k_s - k|}{k} \ll 1. \tag{43}$$

Performing the integration over the cylinder volume in expression (40) in the same way as it was done for similar integrals in Refs [24, 25], we obtain

$$\varphi_{ks} = \frac{\pi L}{k^2a^3}\left(\pi F - 2F{\rm Si}\,2F + 1 - \cos 2F + {\rm i}\sin 2F - 2{\rm i}F{\rm Ci}\,2F\right), \tag{44}$$

where $F \equiv \pi R^2/\lambda L$ is the Fresnel number; $\rho_0 a^3 = 1$; and Si x and Ci x are respectively the sine and cosine integral functions. In the limits of large and small Fresnel numbers, we have respectively

$$\varphi_{ks} \simeq \frac{\pi L}{k^2a^3}F(\pi + 2{\rm i}\ln 2F) \qquad (F \ll 1),$$
$$\varphi_{ks} \simeq \frac{\pi L}{k^2a^3}\left(1 + {\rm i}\,\frac{\cos 2F}{2F}\right) \qquad (F \gg 1). \tag{45}$$

To estimate the atom-polariton coupling constant (41), we will use the parameters typical for the materials mentioned at the beginning of the section: $\Omega_1 \sim \Omega_2 \sim 10^{14}$ s^{-1}, $\Delta_{\rm p} \sim 10^{13}$ s^{-1}, $\lambda \sim 10^{-3}$ cm, and $\gamma \sim 10^8 - 10^9$ s^{-1}. We will also assume that the resonance frequency $\omega_0 \sim 10^{14}$ c^{-1} lies

deep inside the polariton band gap and use the fact that $a \sim 10^{-8}$ cm and $r_e \sim 10^{-13}$ cm. The room temperature corresponds to the frequency 10^{13} s^{-1}; therefore, $\beta\omega_k \sim 10$ and $n_k \simeq e^{-\beta\omega_k} \sim 10^{-5}$. For the atom-polariton coupling constant (41), we finally obtain $\alpha < 10^{-3}$. This justifies the assumption about the smallness of α made in Sec. (4) during the analysis of the evolution equations.

6. Conclusions

We have studied the possibility of *collective* emission by atoms doped into a bandgap material, in the situation when the resonance emission frequency lies *deep inside* the forbidden gap so that the spontaneous emission of single atoms is suppressed. We have considered a realistic three-dimensional system where the radiation wavelength is much smaller than the sample dimensions. The emission by doped atoms becomes possible if their density exceeds a certain critical value, when the nonlinear effects related to the collective interaction between the atoms begin to play the dominant role.

The situation when a single atom cannot emit and the light is localised, but an ensemble of atoms can emit, can be reasonably termed *collective release of light*. The critical atom density, at which the collective release occurs, depends on the initial conditions, the atom-atom coupling parameter (19), and the medium-atom coupling parameter (24). It was shown in the preceding section that $\alpha \ll 1$; therefore, the renormalisation of the characteristic parameters γ_1 and ζ in Eqns (25) is insignificant: $\overline{\gamma}_1 \approx \gamma_1$ and $\overline{\zeta} \approx \zeta \approx s_0$. It follows from the analysis of Section 4 that the release of light becomes possible for $gs_0 > 1$.

The atom-atom coupling parameter g defined by Eqn (19) depends on the sample geometry. To estimate the critical density of doped atoms ρ_c that triggers the collective emission, one can set $g \approx \rho\lambda^3$. The critical density is then $\rho_c = 1/s_0\lambda^3$. For a polariton bandgap material with $\lambda \sim 10^{-3}$ cm and the completely inverted initial state, $s_0 = 1$, the critical density is $\rho_c \sim 10^9$ cm^{-3}. If the atom density exceeds ρ_c, the *collective release of light* should take place. The prediction of this effect is the main result of this paper.

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