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Dynamics of ultrashort pulses in birefringent media

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Abstract. A system of equations is proposed that describes the dynamics of a laser pulse containing a few optical cycles (ultrashort pulse) in the transparency region of a medium with induced birefringence. In this case, the approximation of slowly varying envelopes, which is standard in the case of monochromatic signals, is inapplicable. An approximate solution is found for the case when the dispersion spreading length is much smaller than the length of the polarisation ellipse beating. It has the form of a travelling soliton-like bound state of ordinary and extraordinary components. The conditions for the stability of this pulse with respect to self-focusing are determined.

1. Introduction

The advances of the last decade in the generation of optical pulses with the lengths approaching a single optical cycle [1, 2] has led to the appearance of a new research field concerned with the interaction of such pulses with matter. In the already established terminology, such laser signals are called ultrashort pulses [3]. Theoretical studies of ultrashort pulses obviously cannot rely on the approximation of slowly varying amplitudes and phases, which is traditionally used in the optics of monochromatic signals [4]. One should expect that many of the optical effects that are well known in the case of continuous waves and monochromatic pulses would acquire new features in the ultrashort domain.

The possible applications of ultrashort pulses in various optical information systems make it topical to study soliton-like propagation regimes of these signals. It is desirable for many applications that the pulse polarisation remained constant during its propagation in a waveguide [5]. Small birefringent fluctuations in such waveguides are usually suppressed by creating artificial regular birefringence [5]. Constant electric fields (Kerr effect) and static deformations induce the strongest birefringence. In this way, an initially isotropic medium acquires uniaxial optical anisotropy.

The dynamics of ultrashort pulses in uniaxial anisotropic crystals with natural birefringence and quadratic optical nonlinearity was considered in Ref. [6]. The effect of artificial birefringence on the nonlinear dynamics of powerful monochromatic envelope pulses has already been studied in some detail [7 – 9]. The purpose of this work is to derive nonlinear wave equations describing the dynamics of ultrashort pulses in media with induced birefringence and to analyse, in some special cases, their approximate solutions in the form of solitary travelling pulses.

2. Phenomenological equations

When propagating in a birefringent medium, the electric field $\boldsymbol{E}$ of the pulse splits into an ordinary ($\boldsymbol{E}_{\mathrm{o}}$) and an extraordinary ($\boldsymbol{E}_{\mathrm{e}}$) component, which are polarised in mutually orthogonal planes and propagate with the velocities v_{o} and v_{e}, respectively. For $v_{\mathrm{o}} > v_{\mathrm{e}}$, the axis coinciding with $\boldsymbol{E}_{\mathrm{o}}$ is 'fast', and the axis parallel to $\boldsymbol{E}_{\mathrm{o}}$ is 'slow'; the situation reverses for $v_{\mathrm{o}} < v_{\mathrm{e}}$. Suppose that a pulse propagates along the x axis, which is perpendicular to the z axis of the induced optical anisotropy. Then, the wave $\boldsymbol{E}_{\mathrm{o}}$ is polarised along the y axis, which is perpendicular to the plane xz, and the wave $\boldsymbol{E}_{\mathrm{e}}$ is polarised along the z axis.

The Maxwell equation has the same form for the both components:

$$\Delta E_j - \frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P_j}{\partial t^2}, \quad j = \mathrm{e}, \mathrm{o}, \tag{1}$$

where c is the speed of light in vacuum and P_j is the polarisation that corresponds to the ordinary (extraordinary) pulse component and whose dependence on $\boldsymbol{E}$ is specified by constitutive equations.

In the following, we will assume that the frequencies ω of the ultrashort pulse spectrum lie in the medium transparency region related to electronic optical quantum transitions. This condition can be written as [10 – 12]

$$(\omega_0 \tau_{\mathrm{p}})^2 \gg 1, \tag{2}$$

where ω_0 is a characteristic resonance frequency of the medium that corresponds to electronic transitions and τ_{p} is the time scale of the ultrashort pulse.

Inequality (2) corresponds to the slow variation of the pulse profile during its propagation in the medium, when the medium atoms follow this variation adiabatically. The small lag of the polarisation behind $\boldsymbol{E}$ results in a small temporal dispersion. On the other hand, the nonresonant interaction between the pulse and the medium results in a small excitation of the latter. Thus, the nonlinearity and the dispersion are weak and can be taken into account by add-

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ing appropriate terms in the constitutive equations [10–13]. This means that the nonlinear dispersion, i.e., the frequency dependence of nonlinear susceptibilities, can be neglected as a higher-order effect. Thus, if condition (2) is fulfilled, we can assume the linear susceptibility tensor to be independent of the frequency in the transparency region. Its elements are then symmetric with respect to all indices [14].

We will also assume that in the case of crystals, the axis of the induced optical anisotropy coincides with one of the crystallographic axes. Then, the linear susceptibility tensor is diagonal, and the tensor of the corresponding cubic optical nonlinearity has the simplest form. For example, in a cubic crystal, the axes x, y, and z should be directed along the mutually orthogonal axes of the fourth-order symmetry [15]. In originally isotropic media with induced birefringence, these conditions are fulfilled automatically.

For the chosen geometry of pulse propagation, the constitutive relations should be invariant with respect to the transformations $y \to -y$ and $z \to -z$. This leads to the following dependence of the medium-polarisation vector components on the electric field of the ultrashort pulse

$$P_{\rm o} = \chi_{\rm o}^{(1)} E_{\rm o} + \chi_{\rm o}^{(3)} E_{\rm o}^3 + \chi_{\rm eo}^{(3)} E_{\rm e}^2 E_{\rm o} - \frac{1}{2}\chi_{\rm o}'' \frac{\partial^2 E_{\rm o}}{\partial t^2}, \tag{3}$$

$$P_{\rm e} = \chi_{\rm e}^{(1)} E_{\rm e} + \chi_{\rm e}^{(3)} E_{\rm o}^3 + \chi_{\rm eo}^{(3)} E_{\rm o}^2 E_{\rm e} - \frac{1}{2}\chi_{\rm e}'' \frac{\partial^2 E_{\rm e}}{\partial t^2}. \tag{4}$$

Here, $\chi_{\rm o}^{(1)} \equiv \chi_{yy}^{(1)}(0)$ and $\chi_{\rm e}^{(1)} \equiv \chi_{zz}^{(1)}(0)$ are the components of the linear low-frequency susceptibility tensor for the ordinary and extraordinary waves; the parameters $\chi_{\rm o}^{(3)} \equiv \chi_{yyyy}^{(3)}(0,0,0)$, $\chi_{\rm e}^{(3)} = \chi_{zzzz}^{(3)}(0,0,0)$, $\chi_{\rm eo}^{(3)} = 3\chi_{yzzy}^{(3)}(0,0,0) = 3\chi_{yyzz}^{(3)}(0,0,0) = 3\chi_{yzyz}^{(3)}(0,0,0) = 3\chi_{zyyz}^{(3)}(0,0,0) = 3\chi_{zyzy}^{(3)}(0,0,0) = 3\chi_{zzyy}^{(3)}(0,0,0)$ are expressed in terms of the components of the third-order nonlinear susceptibility tensor; and the coefficients $\chi_{\rm e,o}'' = (\partial^2\chi_{\rm e,o}/\partial\omega^2)_{\omega=0}$ describe the small temporal dispersion of the linear nonresonant response of the medium.

Inserting expressions (3) and (4) into Eqn (1) and using the unidirectional propagation approximation [13] for the x axis, we obtain the system

$$\frac{\partial E_{\rm o}}{\partial x} - \mu\frac{\partial E_{\rm o}}{\partial\tau} + 3\beta_{\rm o}E_{\rm o}^2\frac{\partial E_{\rm o}}{\partial\tau} + \alpha\frac{\partial}{\partial\tau}\left(E_{\rm e}^2E_{\rm o}\right) + \delta_{\rm o}\frac{\partial^3E_{\rm o}}{\partial\tau^3} = \frac{\upsilon_{\rm m}}{2}\Delta_\perp\int_{-\infty}^{\tau}E_{\rm o}\,{\rm d}\tau', \tag{5}$$

$$\frac{\partial E_{\rm e}}{\partial x} + \mu\frac{\partial E_{\rm e}}{\partial\tau} + 3\beta_{\rm e}E_{\rm e}^2\frac{\partial E_{\rm e}}{\partial\tau} + \alpha\frac{\partial}{\partial\tau}\left(E_{\rm o}^2E_{\rm e}\right) + \delta_{\rm e}\frac{\partial^3E_{\rm e}}{\partial\tau^3} = \frac{\upsilon_{\rm m}}{2}\Delta_\perp\int_{-\infty}^{\tau}E_{\rm e}\,{\rm d}\tau', \tag{6}$$

where $\alpha = 2\pi\chi_{\rm eo}^{(3)}\upsilon_{\rm m}/c^2$; $\beta_j = 2\pi\chi_j^{(3)}\upsilon_{\rm m}/c^2$; $\delta_j = -\pi\upsilon_{\rm m}\chi_j''/c^2$ (j = e, o); the parameter $\mu = (\upsilon_{\rm m}/4)(\upsilon_{\rm e}^{-2} - \upsilon_{\rm o}^{-2})$ describes the mismatch between the velocities $\upsilon_{\rm e} = c/(1+4\pi\chi_{\rm e})^{1/2}$ and $\upsilon_{\rm o} = c/(1+4\pi\chi_{\rm o})^{1/2}$; $\upsilon_{\rm m}$ is some average velocity that lies between $\upsilon_{\rm e}$ and $\upsilon_{\rm o}$ and is defined by the expression $\upsilon_{\rm m}^{-2} = (\upsilon_{\rm o}^{-2} + \upsilon_{\rm e}^{-2})/2$; $\tau = t - x/\upsilon_{\rm m}$; and $\Delta_\perp$ is the transverse Laplacian.

Consider the system (5), (6) in the limit of monochromatic pulses. For this purpose, we will use the representation

$$E_j = \Psi_j \exp({\rm i}\omega\tau - q_jx) + {\rm c.c.}, \quad j = {\rm e, o}, \tag{7}$$

where Ψ_j are slowly varying complex amplitudes ($|\partial\Psi_j/\partial\tau| \ll |\omega\Psi_j|$, $|\partial\Psi_j/\partial x| \ll |q_j\Psi_j|$) and q_j are the wave numbers in the moving reference frame. Using the one-dimensional approximation (vanishing right-hand sides of Eqns (5) and (6)), we obtain the well-known system of equations describing the dynamics of a monochromatic envelope pulse in a birefringent medium [5, 7, 8]

$${\rm i}\frac{\partial\Psi_{\rm o}}{\partial x} - {\rm i}\nu\frac{\partial\Psi_{\rm o}}{\partial T} - \tilde\delta_{\rm o}\frac{\partial^2\Psi_{\rm o}}{\partial T^2} - 3\left(\tilde\beta_{\rm o}|\Psi_{\rm o}|^2 + \frac{2}{3}\tilde\alpha|\Psi_{\rm e}|^2\right)\Psi_{\rm o} - \tilde\alpha\Psi_{\rm e}^2\Psi_{\rm o}^*\exp[2{\rm i}(q_{\rm o} - q_{\rm e})x] = 0, \tag{8}$$

$${\rm i}\frac{\partial\Psi_{\rm e}}{\partial x} + {\rm i}\nu\frac{\partial\Psi_{\rm e}}{\partial T} - \tilde\delta_{\rm e}\frac{\partial^2\Psi_{\rm e}}{\partial T^2} - 3\left(\tilde\beta_{\rm e}|\Psi_{\rm e}|^2 + \frac{2}{3}\tilde\alpha|\Psi_{\rm o}|^2\right)\Psi_{\rm e} - \tilde\alpha\Psi_{\rm o}^2\Psi_{\rm e}^*\exp[-2{\rm i}(q_{\rm o} - q_{\rm e})x] = 0, \tag{9}$$

where $T = \tau - 3(\delta_{\rm o} + \delta_{\rm e})\omega^2x/2$; $\nu = \mu + 3(\delta_{\rm e} - \delta_{\rm o})\omega^2/2$; $\tilde\alpha = \omega\alpha$; $\tilde\beta_j = \omega\beta_j$; $\tilde\delta_j = 3\omega\delta_j$; and $q_{\rm e} = \mu\omega + \delta_{\rm e}\omega^3$, $q_{\rm o} = -\mu\omega + \delta_{\rm o}\omega^3$ are wave numbers.

The system (8), (9) was studied earlier in many papers. In particular, Maimistov et al. [8] used a variational method to find an approximate solution in the form of solitary envelope pulses with elliptical polarisation. If the length of the birefringent beating of the polarisation ellipse [4] is $L_{\rm b} = 2\pi/|q_{\rm o} - q_{\rm e}| \gg l_{\rm d} = 1/(\tilde\delta_{\rm o}\omega^2)$, where $l_{\rm d}$ is the diffraction spreading length of the wave packet, the last terms in Eqns (8) and (9) can be neglected. If, in addition, $\tilde\beta_{\rm o} = \tilde\beta_{\rm e} = 2\tilde\alpha/3$, equations (8), (9) transform to the Manakov system [7].

Since the system (8), (9) can be derived from the system (5), (6) in the limit of long monochromatic signals, equations (5) and (6) can be said to be more general: They describe the dynamics of both envelope pulses and ultrashort pulses.

3. Soliton-like pulses in the case of weak birefringence

The general analysis of Eqns (5), (6) seems to be rather involved and probably can be performed only numerically. In the one-dimensional approximation, these equations become a coupled system of modified Korteweg-de Vries equations (MKdV). If in this case we represent a solution of equations (5), (6) in the form of a two-component stationary travelling wave, then, after a single integration, these equations will transform to a system describing two coupled cubic-nonlinear oscillators. Assuming also that $\alpha = \beta_{\rm o} = \beta_{\rm e} = \beta$ and $\delta_{\rm o} = \delta_{\rm e} = \delta$, we obtain a special case ($n = 2$) of the Garneer system [16], describing n nonlinear oscillators with different eigenfrequencies.

The system of equations (5) and (6) was solved in Ref. [16]; its generalised version was integrated in quadratures in Ref. [17]. For $n = 2$, Maimistov [18, 19] used the Hirota method to find a localised solution that, in the case of an infinitesimal parameter $\varepsilon^2 = |\mu|\tau_{\rm p}^2/(2|\delta|)$, is similar to the solution found in Ref. [20]:

$$E_{\rm o,e} = E_{\rm m}\frac{\theta(\pm\mu)\cosh\zeta - \theta(\mp\mu)\sinh\zeta}{1 + \varepsilon^2\cosh^2\zeta}, \tag{10}$$

where $\theta(\mu)$ is the Heaviside step function; $\zeta = (t - x/\upsilon)/\tau_{\rm p}$; $E_{\rm m} = 2(\mu/\beta)^{1/2}$; and the velocity υ in the laboratory reference frame is related to the pulse duration $\tau_{\rm p}$ by the expression $1/\upsilon = 1/\upsilon_{\rm m} + \delta/\tau_{\rm p}^2$.

Consider the physical meaning of the parameter ε^2. In the case of elliptical birefringence and monochromatic pulses, the characteristic beating length L_b is defined by the expression $L_b = 2\pi/|q_o - q_e| = 2\pi/(\omega|1/v_o - 1/v_e|)$. The analogous parameter L_{bs} for ultrashort pulses, which do not have any carrier frequency, is defined by making the substitution $2\pi/\omega \to 4\tau_p$ in the expression for L_b (the factor 4 is introduced for the sake of convenience). Then, we have $L_{bs} = 4\tau_p/|1/v_o - 1/v_e|$.

One can see from Eqns (5), (6) that, in the case $\delta_e = \delta_o = \delta$, the dispersion spreading length l_{ds} of the ultrashort pulse can be defined as $l_{ds} = \tau_p^3/\delta$. Taking into account the closeness of the velocities v_e and v_o in the case of $\varepsilon^2 \ll 1$, we find that $|\mu| \approx |v_o - v_e|(2v_o v_e)^{-1}$. Performing simple algebraic transformations, we finally obtain $\varepsilon^2 = l_{ds}/L_{bs}$. Thus, the condition $\varepsilon^2 \ll 1$ means that the dispersion spreading length is much smaller than the beating length of the polarisation ellipse.

We will now derive a soliton-like solution of the system (5), (6) when $l_{ds} \ll L_{bs}$ and the nonlinearity coefficients α, β_o, and β_e are different. If the birefringence is weak, χ_e and χ_o differ only slightly, and the quantity μ is correspondingly small. The dispersion terms in Eqns (5) and (6) have a higher order of smallness with respect to the parameter $(\omega_0\tau_p)^{-2}$; therefore, for $\varepsilon^2 \ll 1$, we can neglect the difference between δ_e and δ_o and set $\delta_e = \delta_o = \delta$.

We will use the average variational principle of the Ritz–Whitham [21, 22]. The Lagrangian density for the system (5), (6) can be written as

$$\Lambda = \frac{1}{2}\frac{\partial\Phi_o}{\partial x}\frac{\partial\Phi_o}{\partial\tau} + \frac{1}{2}\frac{\partial\Phi_e}{\partial x}\frac{\partial\Phi_e}{\partial\tau} + \frac{\mu}{2}\left(\frac{\partial\Phi_e}{\partial\tau}\right)^2 - \frac{\mu}{2}\left(\frac{\partial\Phi_o}{\partial\tau}\right)^2$$
$$- \frac{\beta_o}{4}\left(\frac{\partial\Phi_o}{\partial\tau}\right)^4 - \frac{\beta_e}{4}\left(\frac{\partial\Phi_e}{\partial\tau}\right)^4 - \frac{\alpha}{2}\left(\frac{\partial\Phi_o}{\partial\tau}\right)^2\left(\frac{\partial\Phi_e}{\partial\tau}\right)^2 \quad (11)$$
$$+ \frac{\delta}{2}\left(\frac{\partial^2\Phi_o}{\partial\tau^2}\right)^2 + \frac{\delta}{2}\left(\frac{\partial^2\Phi_e}{\partial\tau^2}\right)^2 - \frac{v_o}{4}(\nabla_\perp\Phi_o)^2 - \frac{v_e}{4}(\nabla_\perp\Phi_e)^2,$$

where potentials Φ_o and Φ_e are defined by the relations $E_o = \partial\Phi_o/\partial\tau$ and $E_e = \partial\Phi_e/\partial\tau$.

Taking into account expression (10), we represent a trial soliton-like solution for Φ_o and Φ_e in the form

$$\Phi_{o,e} = A(x, \boldsymbol{r}_\perp)[\theta(\pm\mu)\arctan(\eta\sinh\xi)]$$
$$-\theta(\mp\mu)\arctan(\eta\cosh\xi)], \quad (12)$$

where $\xi = [\tau - f(x, \boldsymbol{r}_\perp)]/\tau_p(x, \boldsymbol{r}_\perp)$; $A(x, \boldsymbol{r}_\perp)$, $\tau_p(x, \boldsymbol{r}_\perp)$, $\eta(x, \boldsymbol{r}_\perp)$ are slow functions of their arguments; and $f(x, \boldsymbol{r}_\perp)$ is a fast function of its arguments. In addition, η depends on ξ and satisfies the condition $\eta^2(x, \boldsymbol{r}_\perp) \ll 1$. Then, the expressions for E_o and E_e coincide with expression (10) up to the replacement $\varepsilon \to \eta$.

After inserting Eqn (12) into Eqn (11), we retain the x- and $\boldsymbol{r}_\perp$-derivatives of only the fast function $f(x, \boldsymbol{r}_\perp)$. The subsequent integration over τ produces the averaged Lagrangian

$$L\left(\tau_p, A, \eta, \frac{\partial f}{\partial x}, \nabla_\perp f\right) \equiv \int_{-\infty}^{+\infty} \Lambda \, d\tau.$$

Employing L, we obtain the following Euler–Lagrange equations for the variables τ_p, A, ε, and f in the one-dimensional approximation

$$f = \left(\frac{1}{v} - \frac{1}{v_m}\right)x, \quad \frac{1}{v} = \frac{1}{v_m} + \frac{\delta}{\tau_p^2},$$
$$A = A_0 \equiv 4\left(\frac{\delta}{\beta_o + \beta_e + 2\alpha}\right)^{1/2}, \ \eta = 2e^{-\kappa},$$
$$\kappa = \frac{1}{4\varepsilon^2}\left[1 + \frac{11\beta_o - 21\beta_e - 10\alpha}{3(\beta_o + \beta_e + 2\alpha)}\right], \quad (13)$$

where the pulse duration τ_p is a free parameter. If the expression appearing in the square brackets in the expression for κ is positive, the condition $\varepsilon^2 \ll 1$ automatically ensures that the parameter η^2 is small. According to Eqns (12) and (13), the expressions for E_o and E_e can be written in the form Eqn (10), where $E_m = A\varepsilon/\tau_p$, $\zeta = t - x/v_o$.

Thus, in the case $l_{ds} \ll L_{bs}$, the ordinary and the extraordinary components of the ultrashort pulse can propagate together in the form of a stationary localised state. In this state, the component polarised along the fast axis is a unipolar pulse of the electric field strength, whereas the component parallel to the slow axis is a bipolar pulse.

As it passes through a fixed plane $x = \text{const}$, the end of the vector $\boldsymbol{E}$ of the ultrashort pulse describes a closed curve, whose equation can be easily derived in the case $\mu > 0$ from Eqn (10). Taking into account the substitution $\varepsilon \to \eta$, we obtain

$$Y^2 - Z^2 = \left[(1 + \eta^2)Y^2 - Z^2\right]^2, \quad (14)$$

where $Z = E_e/E_m$ and $Y = E_o/E_m$. The analogous expression for the case $\mu < 0$ can be obtained by exchanging $Z \leftrightarrow Y$.

The Fig. 1 shows the curve (14) for $\mu > 0$. It follows from Eqn (10) that $\tan\varphi \equiv Z/Y = -\tanh\zeta$, and therefore $-45° < \varphi < 45°$ (see Fig. 1). This means that the vector $\boldsymbol{E}$ of the soliton-like pulse cannot deflect from the fast axis by more than 45°. The maximum angles ±45° are reached at the leading and the trailing edges of the pulse, whereas in the middle of the pulse $\varphi = 0$. The parameter η describes the influence of the birefringence on the propagation of the signal. When this

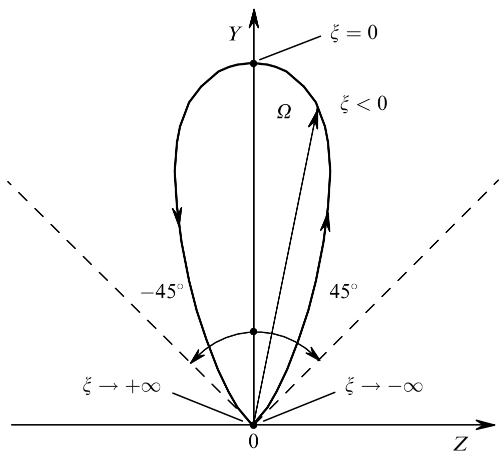

Figure 1. Trajectory described by the end of the electric field vector of a soliton-like ultrashort pulse as it passes through a fixed plane $x = \text{const}$ in the case $\mu > 0$. The ordinary component is polarised along the fast axis y, the extraordinary one, along the slow axis z. $\Omega = (E_o\boldsymbol{e}_y + E_e\boldsymbol{e}_z)/E_m$, where $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are the corresponding unit vectors.

parameter increases, the amplitude of the component polarised along the fast axis varies insignificantly (given that in our case $\eta^2 \ll 1$). On the contrary, the amplitude of the slow component decreases as $\sim 1/\eta$. As a result, the curve shown in the Fig. 1 shrinks along the slow axis. With the further increase in the anisotropy parameter η, the slow component virtually disappears and only the fast component remains in the pulse. This conclusion also follows from the exact solution found in Refs [18,19] for the case $\beta_o = \beta_e = \alpha$. In the limit $\varepsilon^2 \gg 1$, this solution is a unipolar pulse of the fast component and the slow component is almost completely suppressed. This soliton is virtually indistinguishable from a conventional MKdV soliton polarised along the fast axis.

It follows from the expressions for β_o, β_e, α, δ_o, δ_e, and A, that a one-dimensional soliton-like bound state can be created in a focusing medium ($\chi_e^{(3)}, \chi_o^{(3)}, \chi_{eo}^{(3)} > 0$) with the anomalous group dispersion ($\chi_e'' = \chi_o'' < 0$) or in a defocusing medium ($\chi_e^{(3)}, \chi_o^{(3)}, \chi_{eo}^{(3)} < 0$) with the normal dispersion of the group velocity ($\chi_e'' = \chi_o'' > 0$). This agrees well with the known results of the theory of optical solitons [23]. The fast dispersion spreading is compensated by the nonlinear increase in the steepness of the wave profile, resulting in the creation of a soliton-like pulse. The effect of the slow birefringent beating is reduced to the asynchronous dependence of the pulse polarisation along its profile, which manifests itself in the asynchronous dependence of the components E_o and E_e on ξ (see Eqn (10)). This is probably the primary distinction between the effect that the birefringence has on ultrashort pulses and the analogous effect on monochromatic signals, which have a well-defined carrier frequency.

It is known [5] that in the latter case the polarisation ellipse experiences continues periodic deformations accompanied by its rotation. In our case, the profile of the propagating ultrashort pulse is stationary. However, the direction of the electric field varies along the pulse in the co-moving reference frame, deflecting slightly to the left or to the right (due to the changing sign of the slow component) from the polarisation of the central part of the pulse.

We will analyse the stability of the soliton-like pulse with respect to transverse perturbations (self-focusing) using a variational method, as in Ref. [22]. According to this method, it is sufficient to assume that $\nabla_\perp f \neq 0$ in the expression for L. This approach imposes important restrictions on the character of the transverse perturbations. The fact that A, τ_p, and η vary slowly as functions of the arguments x and $\boldsymbol{r}_\perp$ means that the transverse perturbations weakly distort the pulse wave front, i.e., the wave fronts differ only slightly from the planes $z - vt = \text{const}$.

It is therefore reasonable to consider quasi-one-dimensional perturbations propagating at small angles with respect to the x axis. The more so as, at large angles, the form of the constitutive equations (3) and (4) should change, because in this case, the symmetry with respect to the operations $y \leftrightarrow -y$, $z \leftrightarrow -z$ is violated. As a result, the wave equations (5), (6) are also modified. At the same time, it follows from the form of the trial function (12) that these perturbations do not change the polarisation structure of the one-dimensional soliton-like pulse. This means that these perturbations, as the one-dimensional pulse (10), are polarised in the cone spanning angles from -45 to 45° with respect to the fast axis. The Euler–Lagrange equations for the case $\eta^2 \ll 1$ then assume the form of the equations describing the hydrodynamics of an ideal liquid, with the coordinate x serving as time:

$$\frac{\partial \rho}{\partial x} + \nabla_\perp(\rho \boldsymbol{V}) = 0, \tag{15}$$

$$\frac{\partial \boldsymbol{V}}{\partial x} + (\boldsymbol{V}\nabla_\perp)\boldsymbol{V} = -\frac{1}{\rho}\nabla_\perp p, \tag{16}$$

$$p = -\frac{\beta_o + \beta_e + 2\alpha}{24A_0^2}\rho^3 + \text{const}, \tag{17}$$

where $\boldsymbol{V} \equiv v_m(\nabla_\perp f)$; $\rho = A^2/\tau_p$; and A_0 is defined in Eqn (13).

The stability of the soliton under study with respect to transverse perturbations is obviously equivalent to the stability of the ideal liquid flow described by Eqns (15)–(17); both are given by the condition $\partial p/\partial \rho > 0$. Employing the equation of state (17), which expresses the dependence of the pressure p on the density ρ, we obtain

$$\beta_o + \beta_e + 2\alpha < 0. \tag{18}$$

The physical meaning of equation (18) is quite clear: for positive nonlinear third-order susceptibilities, the medium is focusing, whereas for negative ones, it is defocusing.

It is much more interesting to consider the case of a nonlinear medium with mixed properties, when the diagonal elements of the tensor $\hat{\chi}^{(3)}$ ($\chi_e^{(3)}$ and $\chi_o^{(3)}$) are positive, whereas the nondiagonal elements ($\chi_{eo}^{(3)}$) are negative. It then follows from Eqn (18) and the expressions for β_o, β_e, and α that the stationary bound state is stable if $2|\chi_{eo}^{(3)}| > \chi_e^{(3)} + \chi_o^{(3)}$. The pulses that are polarised strictly along the fast or the slow axis will experience self-focusing in the medium. Only the pulses whose polarisation plane forms a certain angle with these axes can be stable.

We will now make some numerical estimates. Obviously, $|\chi_{e,o}''| \sim |\chi_{e,o}|/\omega_0^2$. Then, according to Eqns (12) and (13), the electric field amplitude of the ultrashort pulse is

$$E_m \sim \frac{\varepsilon}{\omega_0\tau_p}\left(\left|\frac{\chi_o}{\chi_o^{(3)}}\right|\right)^{1/2}.$$

Here, we assume that the parameters $\chi_o^{(3)}$, $\chi_e^{(3)}$, and $\chi_{eo}^{(3)}$ have the same order of magnitude and the same sign. Taking the following parameters for a solid focusing dielectric, such as quartz glass, $\omega_0 \sim 10^{15}$ s^{-1}, $\chi_0 \sim 0.1$, $\chi_o^{(3)} \sim 10^{-14}$ CGSE units [24], we obtain $E_m \sim 10^7$ V cm^{-1}, which corresponds to the intensity $I \approx cE_m^2/4\pi \sim 10^{14}$ W cm^{-2}. Only CO_2 lasers can produce pulses of such intensities. Note, however, that, since $\chi_o^{(3)}, \chi_o^{(3)}, \chi_{eo}^{(3)} > 0$, these pulses are unstable with respect to self-focusing.

In gases, liquids, and some polymers, the principal mechanism of the cubic optical nonlinearity in the transparency region is thermal anharmonicity [25]; $\chi_o^{(3)}, \chi_e^{(3)}, \chi_{eo}^{(3)}$ are negative in this case. Taking for a polymer solution, such as dimethylformaldehyde, $\chi_0 \sim 0.1$ and $|\chi_o^{(3)}| \sim 10^{-11}$ CGSE units [26], we find $E_m \sim 10^6$ V cm^{-1} and $I \sim 10^{12}$ W cm^{-2}. Because $\omega_0 \sim 10^{15}$ s^{-1}, the ultrashort pulse duration $\tau_p \sim 10^{-14}$ s satisfies condition (2). The fact that the nonlinear susceptibilities are negative ensures that the considered soliton-like bound state of the ordinary and the extraordinary pulse components is stable in this polymer with induced anisotropy.

4. Conclusions

Thus, the proposed system of equations (5) and (6) describes the nonlinear dynamics of the ordinary and the extraordinary components of an ultrashort pulse that propagates perpendicularly to the optical axis in a medium with induced anisotropy. In the limit of monochromatic pulses, equations (5) and (6) transform to the well-known system of coupled equations for slowly varying amplitudes.

The solution of the system (5), (6), derived in the form of a soliton-like bound state of ordinary and extraordinary components can be realised in weakly birefringent media if the beating length L_{bs} of the polarisation ellipse is much larger than the dispersion spreading length l_{ds}. Under certain conditions, this solution is stable with respect to small wave front perturbations.

It would also be interesting to consider the soliton-like solution whose polarisation ellipse undergoes nonstationary beating in the case $L_{bs} \ll l_{ds}$. To perform the variational averaging in these case, one should choose appropriate trial functions for Φ_o and Φ_e. We plan to solve this problem in the future.

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References

1. Becker P C, Fragnito H L, Bigot J Y, Brito-Crus C H, Shank C V *Phys. Rev. Lett.* **63** 505 (1989)
2. Darrow J T, Hu B B, Chuang X C, Auston D H *Opt. Lett.* **15** 323 (1990)
3. Maimistov A I *Kvantovaya Elektron. (Moscow)* **30** 287 (2000) [*Quantum Electron.* **30** 287 (2000)]
4. Akhmanov S A, Vysloukh V A, Chirkin A S *Optika femtosekundnykh lasernykh impul'sov* [*Optics of femtosecond laser pulses*] (Moscow: Nauka, 1988)
5. Agrawal G P *Nonlinear fiber optics* (San Diego, CA: Academic, 1995)
6. Dubrovskaya O B, Sukhorukov A P *Izv. Ross. Akad. Nauk, Ser. Fiz.* **56** no. 12, p. 184 (1992) [*Bull. RAS Phys.* **56** 1991 (1992)]
7. Menyuk C R *IEEE J. Quantum. Electron.* **25** 2674 (1989)
8. Maimistov A I, Elyutin S O *J. Mod. Opt.* **39** 2193 (1992)
9. Makarov V A, Petrov K P *Kvantovaya Elektron. (Moscow)* **20** 1011 (1993) [*Quantum Electron.* **23** 880 (1993)]
10. Belenov E M, Nazarkin A V *Pis'ma Zh. Eksp. Teor. Fiz.* **51** 252 (1990) [*JETP Lett.* **51** 288 (1990)]
11. Belenov E M, Nazarkin A V, Ushchapovskii V A *Zh. Eksp. Teor. Fiz.* **100** 762 (1991) [*Sov. Phys. JETP* **73** 422 (1991)]
12. Kozlov S A, Sazonov S V *Zh. Eksp. Teor. Fiz.* **111** 404 (1997) [*JETP* **84** 221 (1997)]
13. Sazonov S V *Zh. Eksp. Teor. Fiz.* **107** 20 (1995) [*JETP* **80** 10 (1995)]
14. Klyshko D N *Physical foundations of quantum electronics* (Moscow: Nauka, 1986)
15. Sirotin Yu I, Shaskol'skaya M P *Basics of crystal physics* (Moscow: Nauka, 1979)
16. Garneer R *Rend. Sirc. Matem. Palermo* **43** 155 (1919)
17. Zaitsev A A *Mat. Zametki* **60** 924 (1996) [*Mat. Notes* **60** 698 (1996)]
18. Maimistov A I *Kvantovaya Elektron. (Moscow)* **24** 963 (1997) [*Quantum Electron.* **27** 935 (1997)]
19. Maimistov A I *Opt. Spektrosk.* **87** 104 (1999) [*Opt. Spectrosc.* **87** 96 (1999)]
20. Sazonov S V, Yakupova L S *J. Phys. B* **27** 369 (1994)
21. Anderson D *Phys. Rev. A* **27** 3135 (1983)
22. Zhdanov S K, Trubnikov B A *Quasi-gaseous unstable media* (Moscow: Nauka, 1991)
23. Vederko A V, Dubrovskaya O B, Marchenko S F, Sukhorukov A P *Vestn. Mosk. Gos. Univ., Ser. 3: Fiz., Astron.* **33** (3) 4 (1992)
24. Azarenkov A N, Altshuler G B, Belashenkov N R, Kozlov S A *Kvantovaya Elektron. (Moscow)* **20** 733 (1993) [*Quantum Electron.* **23** 633 (1993)]
25. Reintjes J F *Nonlinear optical parametric processes in liquids and gases* (New York.: Academic, 1983)
26. Chemla D S, Zyss J (Eds) *Nonlinear optical properties of organic molecules and crystals* (New York: Academic, 1987, vol. 2).