Quantum Electronics **30** (11) 1009 – 1013 (2000)

PACS numbers: 42.65.Tg; 42.70.Nq
DOI: 10.1070/QE2000v030n11ABEH001853

Conservation of the angular momentum for multidimensional optical solitons

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Abstract. Analytic expressions are obtained for spin and orbital moments of multidimensional optical solitons — two-dimensional beams and three-dimensional light bullets. It is shown that for given directions of the incidence of light bullets, the time delay in the pulse sequence determines the direction and value of the orbital moment, and can be used as a parameter to control the structure appearing during the mutual capture.

1. Introduction

Optical structures localised in a bulk nonlinear medium, or multidimensional optical solitons, attract increasing attention during the recent decade [1]. An optical beam in the state of waveguide propagation (self-capture) can serve as an example of a spatial soliton. Upon the simultaneous propagation of spatially separated light beams in a nonlinear medium, nonlinearity leads to their interaction and to mutual capture, resulting in the formation of a bound state. A new effect in this field is the so-called spiraling – the rotation of optical beams in space with formation of a double spiral [2 – 8].

The essence of this effect is that a bound state with the formation of spiral structure arises upon mutual capture of *noncoplanar* beams. Many authors [4, 5] noted an analogy between the rotation of beams and the classical problem of two bodies. This allows one to describe the interaction of nonlinear waves within the framework of the well-known problem about the motion of a particle in the effective potential and, in particular, to introduce the concept of the orbital moment of a soliton pair.

Beams with the circular distribution of the field, or vortices, which have the nonzero spin moment, are another example of spatial solitons. It is known that such solitary waves exist formally as stationary solutions of the nonlinear wave equation in media with quadratic or saturating nonlinearity, but they are unstable and collapse during their propagation [5, 9]. The conservation of the angular momentum in the system leads to the transformation of the spin moment in the initial state to the orbital moment of beams that are formed after the vortex collapse.

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Received 13 March 2000; revision received 2 August 2000
Kvantovaya Elektronika **30** (11) 1009 – 1013 (2000)
Translated by A V Uskov

A comparatively new object of studies is the so-called light bullet, or space-time optical soliton [11 – 14]. Analogous behaviour takes place upon the interaction of light bullets in the bimodal space. Namely, this interaction results in the mutual drag of solitons, and the angular momentum is conserved in the system. In this paper we obtain the expression for the orbital moment of a pair of light bullets, which demonstrates that one can choose the direction of the orbital moment for identical solitons, which coincides with the direction of their propagation, and then the bullets will remain in the same transverse plane. If the soliton pair is asymmetric, i.e., the energies of the bullets are different, the formed structure has more complicated shape, because in this case the difference between the projections of group velocities for two pulses plays the role of detuning between the group velocities, and the time delay is nonzero.

2. Interaction potential

The propagation of a light wave in a medium with the focusing Kerr nonlinearity and defocusing fifth-order nonlinearity is described by a parabolic equation, which is a generalisation of the nonlinear Schrödinger equation (NSE):

$$\mathrm{i}\frac{\partial u}{\partial z} + \Delta u + |u|^2 u - |u|^4 u = 0, \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial \tau^2$ is the space-time Laplacian, and u is the slowly varying normalised envelope of the light wave.

Consider the case of a stationary beam (the dimension is $D = 2$) when $u(x, y, \tau, z) = u(\rho, \varphi, z)$; here, the cylindrical coordinate system $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ is used. We seek soliton solutions of Eq. (1) in the form

$$u(\boldsymbol{r}, z) = U(r)\exp(\mathrm{i}kz + \mathrm{i}s\varphi), \tag{2}$$

where s is the topological index of a soliton ($s = 0$ for the soliton of the fundamental mode, and $s = 1$ for a vortex); and $\boldsymbol{r} = (x, y, \tau)$. The function $U(r)$ satisfies the real equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{s^2}{r^2}\right)U - kU + U^3 - U^5 = 0. \tag{3}$$

For light bullets (in this case $D = 3$), Eq. (1) can be rewritten using the spherical coordinates $x = r\sin\vartheta\cos\varphi$, $y = r\sin\vartheta\sin\varphi$, $\tau = r\sin\vartheta$. Here, the angle φ plays the role of the polar angle as in Eq. (2). The function $u(x, y, \tau, z)$ depends on the time τ via the variables ϑ and r. According to [12], we assume that the soliton solution of Eq. (1), which describes a light bullet, has the form

$$u(r,\vartheta,\varphi,z)=V(r,\vartheta)\exp(\mathrm{i}kz+\mathrm{i}s\varphi). \qquad (4)$$

In this case, Eq. (1) can be written as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)V+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial V}{\partial\vartheta}\right)$$
$$-\frac{s^2V}{r^2\sin^2\vartheta}-kV+V^3-V^5=0.$$

The approximation that is stronger than the approximation (4) can be obtained, if the azimuthal angle is fixed, $\vartheta=\vartheta_0$, i.e., we assume that the dependence on the normalised time is determined only via the variable r. For the function $U(r)=V(r,\vartheta_0)$, Eq. (1) is reduced to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\tilde{s}^2}{r^2}\right)U-kU+U^3-U^5=0,$$

where $\tilde{s}=s/\sin\vartheta_0$. In the particular case $\vartheta_0=\pi/2$, the vector $\boldsymbol{r}$ lies in the plane normal to the propagation direction of the soliton under study.

If the solution of Eq. (1) is the soliton solution (2) or (4), then the function

$$u(\boldsymbol{r},z)=U(|\boldsymbol{r}-\boldsymbol{r}_0(z)|)\exp(\mathrm{i}kz+\mathrm{i}s\varphi)$$
$$\times\exp\left[\mathrm{i}\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}_0(z))+\mathrm{i}q^2z\right] \qquad (5)$$

can be also its solution if $\boldsymbol{q}=\mathrm{const}$ and $\mathrm{d}\boldsymbol{r}_0/\mathrm{d}z=2\boldsymbol{q}$, where $\boldsymbol{r}_0(z)$ is the coordinate of the soliton centre. This result is a consequence of the Galilean invariance of Eq. (1) and is not related to the choice of the soliton shape.

It has long been known (see, for instance, [15, 16]) that multidimensional solitons of the form (2), (4) with $s=0$ are unstable in purely Kerr nonlinear media. At the same time, under certain conditions they are stable in media with saturating nonlinearity [11, 17–20], in media with quadratic nonlinearity [21, 22] or in media that possess simultaneously quadratic and cubic nonlinearities [23, 24]. If the peak intensity I of a multidimensional soliton is low compared to the saturation intensity, the optical properties of a medium with saturation of the nonlinear permeability $n(I)$ are well described by the expression $n(I)=n_2I-n_4I^2$. (It is this case that is considered in this paper). In such media, multidimensional solitons also exist [12, 25]. They satisfy the Vakhitov–Kolokolov stability criterion [16] and are stable according to numerical calculations [13].

The case of multidimensional solitons with $s=1$ (and $s\geqslant 2$) proved to be more complicated. The stability criterion of Vakhitov–Kolokolov is not applicable in this case. The study [26] demonstrated instability of the two-dimensional soliton with $s=1$ in a medium with saturating nonlinearity with respect to azimuthal perturbations. The stability conditions for such solitons in media with nonlinear properties described by the permeability $n(I)=n_2I-n_4I^2$ with positive coefficients $n_{2,4}$ were found by the variational method [12]. However, the direct numerical modelling [13] has demonstrated that the solitons under these conditions are simply long-lived structures, whereas stable solitons with $s\geqslant 1$ are absent in the general case. Therefore, it is better to say about metastable vortices or spin-solitons. The results of other studies [18, 23, 27] showed that only metastable multidimensional solitons are possible for $s\geqslant 1$. Therefore, attention can be focussed on the study of their decay and (or) formation of the bound states of soliton complexes with the zero topological index.

Consider the interaction of two spatially separated solitons, each of which is described by functions u and v, which are the solutions of the system of equations (the vector NSE) with the additional terms corresponding to the phase cross-modulation of the fifth order [7]:

$$\mathrm{i}\frac{\partial u}{\partial z}+\Delta u+\left(|u|^2+\varepsilon|v|^2\right)u$$
$$-\left(|u|^4+2\alpha|u|^2|v|^2+\alpha|v|^4\right)u=0,$$
$$\mathrm{i}\frac{\partial v}{\partial z}+\Delta v+\left(|v|^2+\varepsilon|u|^2\right)v$$
$$-\left(|v|^4+2\alpha|u|^2|v|^2+\alpha|u|^4\right)v=0.$$

The Hamiltonian of this conservative system of equations is

$$H=H_u+H_v+U_{\mathrm{int}}, \qquad (6)$$

where

$$H_u=\int\left(|\nabla u|^2-\frac{1}{2}|u|^4+\frac{1}{3}|u|^6\right)\mathrm{d}\boldsymbol{r};\ H_v=H_u(u\leftrightarrow v);$$
$$U_{\mathrm{int}}=\int\left(-\varepsilon|u|^2|v|^2+\alpha|u|^4|v|^2+\alpha|u|^2|v|^4\right)\mathrm{d}\boldsymbol{r}.$$

As shown in [7], a consideration of the cross-modulation of a higher order than the modulation caused by the Kerr effect results in small additive corrections to the interaction potential in the Kerr medium. Therefore, it is sufficient to consider the interaction potential of the form $U_{\mathrm{int}}=-\varepsilon\int|u|^2|v|^2\mathrm{d}\boldsymbol{r}$, where ε is the coefficient of cubic phase cross-modulation. We will seek a solution of the initial problem (6) in the form (5):

$$\begin{pmatrix}u\\ v\end{pmatrix}=V_n(|\boldsymbol{r}-\boldsymbol{r}_n(z)|)\exp(\mathrm{i}k_nz+\mathrm{i}s_n\varphi)$$
$$\times\exp\left\{\mathrm{i}\frac{1}{2}\frac{\mathrm{d}\boldsymbol{r}_n}{\mathrm{d}z}[\boldsymbol{r}-\boldsymbol{r}_n(z)]+\mathrm{i}\alpha_n(z)\right\}, \qquad (7)$$

where $n=1$, 2, and the function V_n satisfies Eq. (3). Then, the Hamiltonian of the system can be represented in the form

$$\tilde{H}=H-H_u-H_v=\frac{\mu}{4}\left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}z}\right)^2-\varepsilon I(R), \qquad (8)$$

where $I(R)=\int V_1^2(|\boldsymbol{r}-\boldsymbol{r}_1|)V_2^2(|\boldsymbol{r}-\boldsymbol{r}_2|)\mathrm{d}\boldsymbol{r}$ is the overlap integral for the solitons; $\boldsymbol{R}(z)=\boldsymbol{r}_1(z)-\boldsymbol{r}_2(z)$; $\mu=E_1E_2(E_1+E_2)^{-1}$ is the reduced mass of two solitons; and $E_n=\int V_n^2\mathrm{d}\boldsymbol{r}$.

In the bimodal system under study, no energy exchange between solitons can occur. The solitons are deformed in the case of significant overlap but then recover their shape if they are stable. Assuming that the values $H_{u,v}$ differ only weakly from their unperturbed ($\varepsilon=0$) values $H^0_{u,v}$, we can consider $\tilde{H}$ (8) as the Hamiltonian of a classical particle with the mass μ, whose position is determined by the radius vector $\boldsymbol{R}(z)$. This procedure is analogous to the effective particle approximation [10].

It was shown [6, 8] for the beams that the conservation of the quantity

$$\frac{\mu}{2}\left|\boldsymbol{R}\times\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}z}\right| = M = \mathrm{const} \tag{9}$$

is a consequence of the variational equations. Below we will show that the conservation of the orbital moment has a deeper origin and is a consequence of the invariance of the initial equation (1) with respect to the rotation transformation. Using the law of conservation of momentum (9), the Hamiltonian (8) can be written in the well-known form:

$$\tilde{H} = \frac{m}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^2 + \frac{M^2}{2mR^2} - \varepsilon I(R). \tag{10}$$

Here, $m = \mu/2$, i.e., the mass of the particle is two times smaller than the reduced mass of solitons. The main problem in such an approach is the calculation of the dependence $I(R)$ and the determination of the angular momentum for light solitons. Analytic expressions for the interaction potential are known for sufficiently separated solitons [7]. In contrast to single-mode systems [2, 3], stable bound states of mutually rotating solitons (i.e., with the nonzero orbital moment) are possible in the bimodal system.

3. Calculation of the angular momentum

The conservation of the angular momentum is a consequence of the invariance of the parabolic equation (1) to the rotation transformation in the plane perpendicular to the propagation direction. The Lagrangian corresponding to this equation is also invariant to the rotation transformation. The application of the Noether theorem to such a functional for the dimensionality $D = 2 + 1$ demonstrates the conservation of the quantity

$$M = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y\left[x\left(u\frac{\partial u^*}{\partial y} - u^*\frac{\partial u}{\partial y}\right) - y\left(u\frac{\partial u^*}{\partial x} - u^*\frac{\partial u}{\partial x}\right)\right],$$

which can be conveniently written as

$$M = \mathrm{Im}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*|\boldsymbol{r}\times\nabla u|\mathrm{d}x\mathrm{d}y, \tag{11}$$

where $\boldsymbol{r}$ is the radius vector in the x, y plane. This integral is zero for the fundamental mode soliton, while for vortices with the topological index s, $M = sE$. Thus, the spin moment is conserved for a beam with a circular distribution of the field, the topological index being called often as *spin* of a soliton. A soliton with the nonzero spin is also called *a spin-soliton*.

Using the Galilean invariance of the equations, we can show that the angular momentum of the soliton for a solution in the form (5) is the sum of its spin and orbital moments:

$$M = \frac{E}{2}\left|\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}z}\right| + sE. \tag{12}$$

In a bimodal system described by the equations for slowly varying envelopes of two beams u and v, the sum $M = M(u) + M(v)$ is conserved. This sum can be written as

$$M = \frac{\mu}{2}\left|\boldsymbol{R}\times\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}z}\right| + s_1E_1 + s_2E_2. \tag{13}$$

Here, the coordinate origin is chosen in the centre of inertia of a soliton pair

$$E_1\boldsymbol{r}_1 + E_2\boldsymbol{r}_2 = 0. \tag{14}$$

As shown in [6], the condition (14) can be always obtained by choosing the coordinate origin and boundary conditions in a proper way. Physically, such a choice corresponds to the state of rest of the centre of inertia of a soliton pair in a transverse plane and does not restrict the generality of the problem. Then, the explicit expression for the orbital moment of two beams takes the form

$$M = R_0R_0'\tan\beta = R_0(\tan\varphi_1 + \tan\varphi_2)\sin\beta, \tag{15}$$

where $R_0' \equiv (\mathrm{d}R/\mathrm{d}z)|_{z=0} = (\tan\varphi_1 + \tan\varphi_2)\cos\beta$; $\varphi_{1,2}$ are the angles of incidence of the beams on the boundary of a nonlinear medium; β is the angle between the initial directions of vectors $\boldsymbol{R}$ and $\mathrm{d}\boldsymbol{R}/\mathrm{d}z$.

The conservation of momentum for the vortex field is manifested during collapse of such beams. The interaction of two filaments arising after a collapse of vortices was studied in papers [5, 9]. The conservation of momentum resulted in rotation of the filaments both in the case of free spiraling [5] and upon stabilisation of filaments in the field of a beam of the fundamental mode [9]. Diffraction autosolitons propagating in the radiation field with wavefront dislocation exhibit similar behaviour [15].

Recent numerical studies of the stability of a light bullet with the circular field distribution in a medium with the third- and fifth-order nonlinearity showed [13] that identical bullets of the fundamental mode are formed after the collapse of a spin-soliton, and the spin moment is transformed into the orbital moment of the interacting solitons. If the dimension is $D = 3$, the variables x, y, τ enter in Eq. (1) symmetrically. It means that this equation and the corresponding Lagrangian are invariant to the rotation transformation that involves any pair of these variables. The corresponding integrals of motion, i.e., the projections of the orbital moment of a pair of light bullets, can be conveniently written in the form

$$M_z = \frac{\mu}{2}\left|\boldsymbol{R}_t\times\frac{\mathrm{d}\boldsymbol{R}_t}{\mathrm{d}z}\right|, \quad M_y = \frac{\mu}{2}\left(T\frac{\mathrm{d}R_x}{\mathrm{d}z} - R_x\frac{\mathrm{d}T}{\mathrm{d}z}\right),$$

$$M_x = -\frac{\mu}{2}\left(T\frac{\mathrm{d}R_y}{\mathrm{d}z} - R_y\frac{\mathrm{d}T}{\mathrm{d}z}\right), \tag{16}$$

where $\boldsymbol{R}_t = \boldsymbol{e}_xR_x + \boldsymbol{e}_yR_y$ is the projection of the vector, which connects the centres of light bullets, on the transverse plane; $T - \tau_1 - \tau_2$ is the distance between the light bullets on the current time axis, i.e., the time delay between the adjacent pulses. These expressions were derived assuming that two conditions are satisfied. As noted above, the first condition (14) can be considered without loss of generality as a boundary condition under which the centre of inertia is located in the transverse plane. The second condition, $E_1\tau_1(z) + E_2\tau_2(z) = 0$, corresponds to the choice of the coordinate origin at the centre of inertia of a soliton pair. The centre of inertia moves along the z-axis with the velocity having the negative projection:

$$E_1 \frac{d\tau_1}{dz} + E_2 \frac{d\tau_2}{dz} = E_1(\cos\varphi_1 - 1) + E_2(\cos\varphi_2 - 1)$$

$$\approx -\frac{1}{2}\left(E_1\varphi_1^2 + E_2\varphi_2^2\right).$$

Indeed, because a pulse propagates in an isotropic medium with the group velocity in an arbitrary direction, the relative velocity of motion of pulses along the direction z is determined by the projections of the group velocities on this axis. Linear motion along the z-axis with the negative velocity corresponds to the velocity of the centre of inertia that is smaller than the group velocity.

The projections of the momentum depend on the initial velocity of the relative motion of the pulses:

$$\left.\frac{dT}{dz}\right|_{z=0} = \frac{d\tau_1(0)}{dz} - \frac{d\tau_2(0)}{dz} = \cos\varphi_1 - \cos\varphi_2. \tag{17}$$

The relation $E_1 \tan\varphi_1 = E_2 \tan\varphi_2$ [6] is valid if the boundary conditions (14) are satisfied, and the expression (17) can be written as

$$\left.\frac{dT}{dz}\right|_{z=0} = \frac{E_1^2 - E_2^2}{2}\left(\frac{\varphi_i}{E_{3-i}}\right)^2, \quad i = 1, 2. \tag{18}$$

The relative velocity of light bullets (18) is zero for the case of identical solitons ($E_1 = E_2 = E$, $\varphi_1 = \varphi_2 = \varphi$), and the expressions for the momentum and its projections (16) can be written as:

$$M^2 = E^2 \tan^2\varphi\left(R_t^2(0)\sin^2\beta + T^2(0)\right),$$

$$M_t = T(0)\frac{E\tan\varphi}{2}, \quad M_z = \frac{\mu}{2}\left|\boldsymbol{R}_t \times \frac{d\boldsymbol{R}_t}{dz}\right|. \tag{19}$$

The projection M_t of the momentum on the transverse plane depends only on the initial time delay between pulses. This allows one to propose a new interesting opportunity to control process of spiraling.

When the momentum is conserved, the motion occurs, as in the problem of two bodies, in the plane that is normal to the momentum direction. If pulses are initially located in the same transverse plane ($T(0) = 0$), then the momentum is directed along the z-axis, and the motion takes place in the transverse plane. In this case, the trajectories of light bullets form a spiral, which is analogous to the spiral shown in [8] for beams. By changing the magnitude of $T(0)$, one can change the direction of the momentum (in the longitudinal plane), i.e., to change the position of the plane in which the solitons will rotate. In other words, one can change the spatial parameter of the spiral by changing the time delay between pulses.

Let us assume that a train of identical pulses (light bullets) with a specified spacing is chosen as a reference wave, the delay between pulses being large enough to neglect any interaction between them. Let a signal wave be a train of the same pulses but with the modulated spacing, i.e., the time delay between the adjacent pulses serves as a unit of information. The signal detection is performed during an elementary act of the interaction between two pulses — the reference and signal ones. Then, according to (19), the structure, which is formed upon the mutual capture of the reference and signal pulses, will be determined by the time delay.

In other words, the modulation of the pulse train will transform to the modulation of the spatial position of pulses. To estimate the time of such switching, note that a light bullet is an ultrashort pulse with the duration of about ten femtoseconds and with the transverse size of about several microns [14]. As the interaction length, we can choose the period of spiraling, which is equal to the diffraction length [8] and is about 1 mm for the light bullet. Then, the switching time is equal to a few picoseconds, which corresponds to the clock frequency higher than 1 THz. These estimates coincide with the results of direct numerical modelling performed in [14] where it was noted that the switching energy can be ~ 25 pJ.

In conclusion, note that the interaction of light bullets with different energies is more complicated. In this case, the inclination angles of the trajectories of two solitons to the z-axis are not equal to each other, and therefore the relative velocity of pulses (17) is nonzero. One can draw the analogy with the problem on the interaction of orthogonally polarised pulses in a birefringent fibre when the group velocity mismatch becomes important [29]. It is known that such pulses can form a bound state called a vector soliton, in which a periodic modulation of the relative velocity takes place. In the case of light bullets, such a capture will substantially complicate the picture of interaction, making it more plentiful.

Acknowledgements. The authors thank Yu Kivshar for the references to the LANL archives [9] and B Malomed for placing a reprint of paper [13] at our disposal before its publication. We thank also N N Rozanov for his important comments during the preparation of this paper. This work was supported by the INTAS Grant No. 96-0339.

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