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Maximum-likelihood estimates of the frequency and other parameters of signals of laser Doppler measuring systems operating in the one-particle-scattering mode

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Abstract. Maximum-likelihood equations are presented for estimates of the Doppler frequency (speed) and other unknown parameters of signals of laser Doppler anemometers and lidars operating in the one-particle-scattering mode. Shot noise was assumed to be the main interfering factor of the problem. The error correlation matrix was calculated and the Rao – Cramer bounds were determined. The results are confirmed by the computer simulation of the Doppler signal and the numerical solution of the maximum-likelihood equations for the Doppler frequency. The obtained estimate is unbiased, and its dispersion coincides with the Rao – Cramer bound.

1. Introduction

Laser Doppler anemometers [1 – 4] and lidars [1, 5] are now firmly established in the industry and research. At the same time, their development continues: both the construction principles and the signals processing techniques are constantly improved. In this work, we derive optimal estimates of the frequency and other unknown parameters of the Doppler signals using the criterion of maximum likelihood. We consider the case when the laser Doppler measuring system (LDMS) operates in the one-particle-scattering mode and shot noise is the main interfering factor. As is known [1, 2], the one-particle-scattering mode is realised in studies of gas flows that have a natural or low dust content. In this case, the probability that there will be two or more scattering particles in the measured volume is extremely small, and the optical Doppler signal becomes a sequence of non-overlapping pulses of light. For an ideal differential LDMS scheme, shown in Fig. 1, the intensity of these pulses is given by

$$I(t) = I_0 \exp\left[-\xi^2\omega_{\mathrm{D}}^2(t-t_0)^2\right][1+\cos\omega_{\mathrm{D}}(t-t_0)], \quad (1)$$

Here, I_0, ω_{D}, and t_0 are the unknown values of the amplitude, the Doppler frequency, which is proportional to the speed of the scattering particle, and the time instant when it enters the centre of the measured volume, respectively; ξ is a known parameter of the optical scheme equal to the inverse number of the interference fringes contained in the measured volume at the e^{-1} level of the maximum light intensity in the objective focal plane. In the case of a differential LDMS scheme, we have

$$\xi = \frac{d}{2\sqrt{2}a}, \quad (2)$$

$$\omega_{\mathrm{D}} = \frac{2V\sin(\theta/2)}{\lambda_{\mathrm{las}}}\cos\alpha, \quad (3)$$

where $2a$ is the distance between the parallel beams in the input plane of the LDMS optical scheme, d is the diameter of these beams at the e^{-1} level of their intensity in the same plane; θ is the angle between probe beams; λ_{las} is the wavelength of the laser radiation; V is the magnitude of the measured velocity vector; and α is the angle between the velocity vector and the direction of the maximum sensitivity of the LDMS.

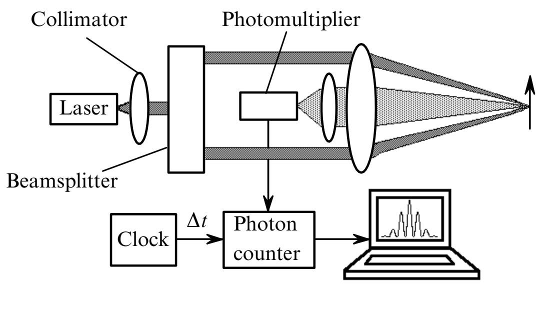

Figure 1. Schematic of the LDMS.

If the main parameters of the optical and electronic parts of the LDMS are chosen properly, shot noise becomes the primary source of errors in estimates of the Doppler signal parameters [1, 4]. A special feature of shot noise is that it is a nonstationary random process: its statistical parameters are not constant but vary in accordance with the instantaneous intensity of the optical signal. We assume that either a photomultiplier or an avalanche photodiode operating in the photoelectron counting mode is used as a photodetector. This detector produces a discrete flux of counts $n_i = n(t_i, \Delta t)$ of photoelectrons emitted during each quantisation interval

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Δt. In this work, we analyse this flux of counts in order to find optimal estimates of the unknown parameters of the Doppler signal and determine the quality of these estimates.

In most cases, there is no *a priori* information about the statistical properties of the estimated parameters. Therefore, maximisation of the likelihood function for each of the estimated parameters is the best criterion for finding optimal estimates. It is known [6] that, if the dispersion of the estimate of the measured parameter (measurement error) is much lower than the dispersion of its *a priori* distribution, the maximum-likelihood estimate coinsides with the estimates of the maximum of the posterior probability and the optimal Bayes solutions of the problem in the case of a quadratic loss function. Under these conditions, the Kolmogorov – Wigner and Kalman – Bussy methods of linear filtration also yield the same estimates. It is also known that in the absence of *a priori* information about the distribution of the measured parameter, the maximum-likelihood estimate is the rigorous solution to the inverse problem of mathematical statistics. Note that a similar approach has been demonstrated in Refs [7 – 10], and this work develops these studies. The parameters of the single-particle Doppler signal have been also estimated in the Ref. [11]; however, it was assumed that the statistical parameters of the noise are independent of the signal, and the noise itself is a white stationary Gaussian process.

2. Likelihood equations

It is known from the theory of the photoelectric effect [12], that, in the case of varying light intensity, the photodetector output signal is a nonstationary flux of electrons, whose emission rate is proportional to the intensity of the optical signal (in the classical description of light). If we quantise the electron flux uniformly in time and count photoelectrons produced by coherent light sources or even by thermal sources, (the quantisation period being much longer than coherence time), then the probability to receive n_i photoelectron counts during the quantisation interval Δt obeys the Poisson law:

$$P(n_i, \Delta t) = \frac{[\lambda(t_i)\Delta t]^{n_i}}{n_i!} \exp[-\lambda(t_i)\Delta t], \tag{4}$$

Here,

$$\lambda_i = \lambda(t_i) = \frac{I(t_i)k}{hv}$$

is the emission rate of photoelectrons at time t_i; $I(t_i)$ is the light intensity (1) integrated over the detector surface; k is a quantum efficiency of the photodetector; hv is the photon energy. The quantisation period Δt is chosen to be much smaller than the period corresponding to the maximum frequency in the spectrum of $I(t)$, so that $I(t)$ can be treated as a constant on this interval.

In our case, λ_i is determined by the expression

$$\lambda_i = A_0 \exp\left[-\xi^2\omega_{\mathrm{D}}^2(t_i - t_0)^2\right][1 + \cos\omega_{\mathrm{D}}(t_i - t_0)], \qquad A_0 = \frac{I_0 k}{hv}. \tag{5}$$

in accordance with Eqn (1).

The function λ_i is proportional to the photocurrent, but is measured in inverse centimetres rather than amperes. Note that n_i is a random integer dimensionless quantity, equal to the number of photoelectrons counted during the quantisation interval Δt. Given that the optical signal $I(t)$ is proportional to the photoemission rate λ_i, we will for simplicity regard the estimate of A_0 as the estimate of the signal amplitude.

Since the Poisson random variables are independent, the joint probability density of the detected photoelectron flux (the likelihood function) is the product of $P(n_i)$:

$$P(n_1, \ldots, n_N) = \prod_{i=1}^{N} P(n_i, \Delta t), \tag{6}$$

where N is the number of counts received during the detection interval $T = \Delta t N$. As is known, one can find maximum-likelihood estimates by maximising the logarithm of function (6). Taking into account Eqns (4) and (6), we obtain

$$\ln P(n_1, \ldots, n_N) = \sum_{i=1}^{N} [n_i(\ln\lambda_i + \ln\Delta t) - \ln(n_i!) - \lambda_i\Delta t]. \tag{7}$$

Differentiating equation (7) with respect to an unknown parameter x, we derive the likelihood equation in its general form:

$$\frac{\mathrm{d}\ln P(n_1, \ldots, n_N)}{\mathrm{d}x} = \sum_i \left(\frac{n_i\lambda_i'}{\lambda_i} - \lambda_i'\Delta t\right) = 0, \qquad \lambda_i' = \frac{\mathrm{d}\lambda_i(t, x)}{\mathrm{d}x}. \tag{8}$$

If the entire signal fits within the detection interval T, the sum $\sum_i \lambda_i'\Delta t$ can be replaced by the integral with the infinite limits. The likelihood equation then assumes the form

$$\sum_i \frac{n_i\lambda_i'}{\lambda_i} = \int_{-\infty}^{\infty} \lambda'\mathrm{d}t. \tag{9}$$

Applying it to each of the unknown parameters A_0, ω_{D}, and t_0, we derive the system of likelihood equations, whose solution yields the following optimal estimates for each of the parameters:

$$A_0 = \frac{\xi\omega_{\mathrm{D}}}{\sqrt{\pi}} \sum_{i=1}^{N} n_i, \tag{10}$$

$$2\xi^2\omega_{\mathrm{D}} \sum_{i=1}^{N} n_i(t_i - t_0)^2 + \sum_{i=1}^{N} n_i(t_i - t_0) \times \tan\frac{\omega_{\mathrm{D}}(t_i - t_0)}{2} = \sum_{i=1} \frac{n_i}{\omega_{\mathrm{D}}}, \tag{11}$$

$$2\xi^2\omega_{\mathrm{D}} \sum_{i=1} N n_i(t_i - t_0) + \sum_{i=1}^{N} n_i \tan\frac{\omega_{\mathrm{D}}(t_i - t_0)}{2} = 0. \tag{12}$$

The system of nonlinear equations (10) – (12) cannot be solved analytically; however, there are many methods for solving it approximately or numerically [13]. Without elaborating on this point, we simply show the logarithmic derivative of the likelihood function as a function of ω_D for the case of a single-particle Doppler signal when the amplitude A_0 and the time t_0 of particle arrival are specified. The computer simulation was performed for signals with amplitudes of 250, 500, and 1000 s^{-1} for $\omega_D = 1$ rad s^{-1}, $\xi = 0.1$, and $t_0 = 0$ (Fig. 2).

One can see from Fig. 2 that, in the case of sufficiently high amplitudes ($A_0 \geqslant 500$), all the curves cross the abscissa axis at a single point, showing that the solution is unique. A more detailed analysis shows that the slopes of these curves near the point $\omega_D = 1$ rad s^{-1}, which determine the accuracy of the Doppler frequency (speed) estimate, is proportional to the signal amplitude.

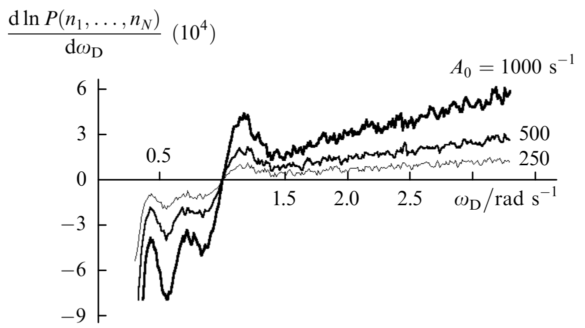


Figure 2. Logarithmic derivative of the likelihood function with respect to the Doppler frequency ω_D.

3. Rao-Cramer bounds

To determine the efficiency of the estimates obtained, we will calculate the Rao – Cramer bounds [6]. The errors of these estimates can be cross-correlated and are described by the correlation matrix R composed of the elements

$$J_{xy} = \mathrm{M}\{(\hat{\chi}_x - \chi_x)(\hat{\chi}_y - \chi_y)\}, \tag{13}$$

Here, M denotes mathematical expectation; $\chi_{x,y}$ and $\hat{\chi}_{x,y}$ are the actual value and the estimate of the unknown parameter respectively; $x, y = 1, \ldots, m$; and m is the number of unknown parameters. In our case, there are three such parameters: A_0, ω_D, and t_0. To determine the error matrix, we will calculate the elements of the Fisher information matrix [6], defined as

$$J_{xy} = \mathrm{M}\left\{\frac{\partial \ln P(\chi)}{\partial \chi_x}\frac{\partial \ln P(\chi)}{\partial \chi_y}\right\} = -\mathrm{M}\left\{\frac{\partial^2 \ln P(\chi)}{\partial \chi_x \partial \chi_y}\right\}. \tag{14}$$

By substituting expression (7) into (14), we obtain the following generalised formula for an arbitrary matrix element:

$$J_{xy} = -\mathrm{M}\left\{\sum_i \left[\frac{n_i(\lambda''_{xy}\lambda - \lambda'_x\lambda'_y)}{\lambda^2} - \lambda''_{xy}\Delta t\right]\right\}. \tag{15}$$

By substituting expression (5) into (15), we find all the elements of the Fisher matrix. Under the condition $\xi \ll 1$, which holds for most LDMSs, the resulting Fisher matrix has the form

$$J = \begin{pmatrix} \dfrac{\sqrt{\pi}}{A_0\xi\omega_D} & -\dfrac{\sqrt{\pi}}{\xi\omega_D^2} & 0 \\ -\dfrac{\sqrt{\pi}}{\xi\omega_D^2} & \dfrac{A_0\sqrt{\pi}}{2\xi^3\omega_D^3} & 0 \\ 0 & 0 & \dfrac{A_0\omega_D\sqrt{\pi}}{\xi} \end{pmatrix}. \tag{16}$$

The columns from left to right and the rows from top to bottom correspond to A_0, ω_D, and t_0, respectively. The Rao – Cramer inequality for the lower bound of the correlation matrix of estimation errors has the form

$$R \geqslant J^{-1}, \tag{17}$$

where J^{-1} is the error matrix, which is inverse to the Fisher information matrix. Calculating the inverse of matrix (16) and taking into account that $\xi \ll 1$, we derive the correlation matrix of errors

$$R = \begin{pmatrix} \dfrac{A_0\xi\omega_D}{\sqrt{\pi}} & \dfrac{2\xi^3\omega_D^2}{\sqrt{\pi}} & 0 \\ \dfrac{2\xi^3\omega_D^2}{\sqrt{\pi}} & \dfrac{2\xi^3\omega_D^3}{A_0\sqrt{\pi}} & 0 \\ 0 & 0 & \dfrac{\xi}{A_0\omega_D\sqrt{\pi}} \end{pmatrix}. \tag{18}$$

The diagonal elements of matrix (18) are respectively the dispersions of the estimates of the amplitude, the Doppler frequency, and the particle's time of arrival to the centre of the measured volume. The nondiagonal elements define the covariances of these estimates (with an accuracy to the sign). Thus, the frequency and amplitude estimates are mutually interrelated, but do not correlate with the particle's time of arrival to the centre of the measured volume, as indicated by the zero values of the corresponding matrix elements.

The lower bounds for the dispersions of estimates of the amplitude, Doppler frequency, and time instant t_0 are given by

$$\sigma_{A_0}^2 = \frac{A_0\xi\omega_D}{\sqrt{\pi}},\ \sigma_{\omega_D}^2 = \frac{2\xi^3\omega_D^3}{A_0\sqrt{\pi}},\ \sigma_{t_0}^2 = \frac{\xi}{A_0\omega_D\sqrt{\pi}}. \tag{19}$$

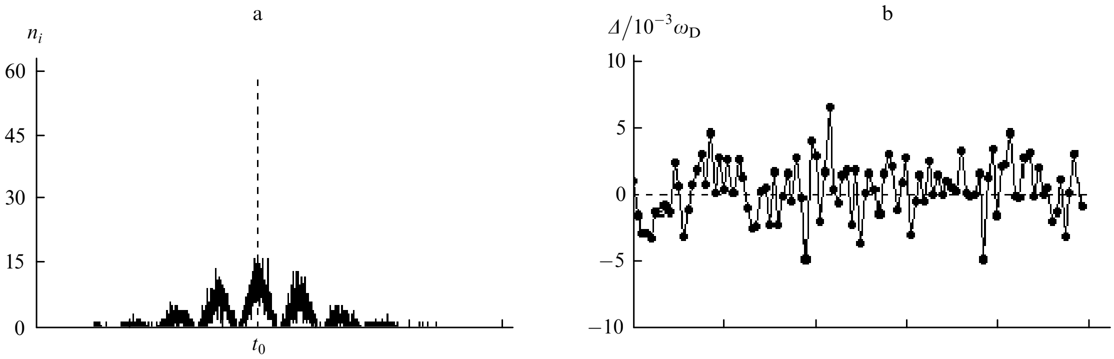
The corresponding minimum relative root-mean-square deviations have the form

$$\frac{\sigma_{A_0}}{A_0} = \left(\frac{\xi\omega_D}{A_0\sqrt{\pi}}\right)^{1/2},\ \frac{\sigma_{\omega_D}}{\omega_D} = \left(\frac{2\xi\,3\omega_D}{A_0\sqrt{\pi}}\right)^{1/2},$$
$$\frac{\sigma_{t_0}}{t_0} = \left(\frac{\xi}{t_0^2 A_0\omega_D\sqrt{\pi}}\right)^{1/2}. \tag{20}$$

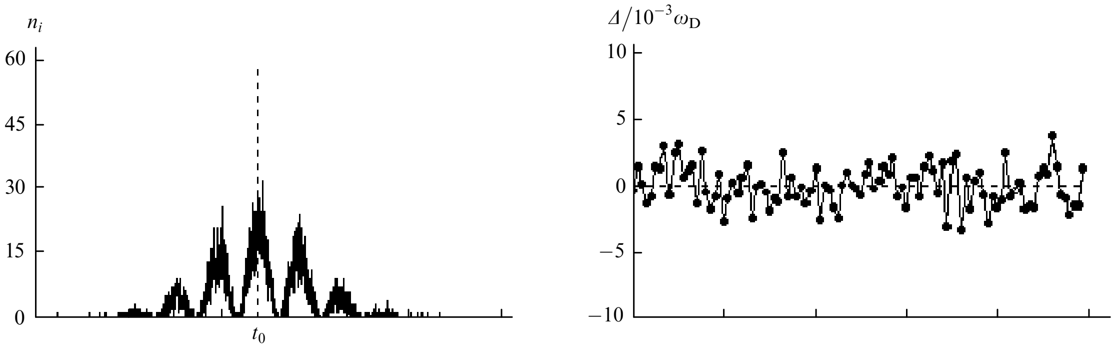
4. Numerical solution of the likelihood equation for the Doppler frequency

As we already mentioned, the derived likelihood equations cannot be solved analytically; however, we can solve them numerically. To demonstrate this and to evaluate the quality of the estimates obtained, we performed a number of model numerical simulations. First, we had to model the Doppler signal produced at the photodetector output. Using a subroutine that generated random numbers uniformly distributed over the interval 0 – 1, we formed a nonstationary Poisson process whose parameter was the photoemission rate (5) for $\omega_D = 1$ rad s^{-1} and $t_0 = 0$. Fig. 3a shows the simulation results for the three amplitudes $A_0 = 250$, 500, and 1000 s^{-1}. The abscissa axis corresponds to time, and the ordinate axis corresponds to the random number of photoelectrons detected during consecutive intervals Δt. Then, we inserted the resulting flux of counts n_i into the likelihood equation for the Doppler frequency (12) and solved this equation using a combination of the Newton and secant methods [13].

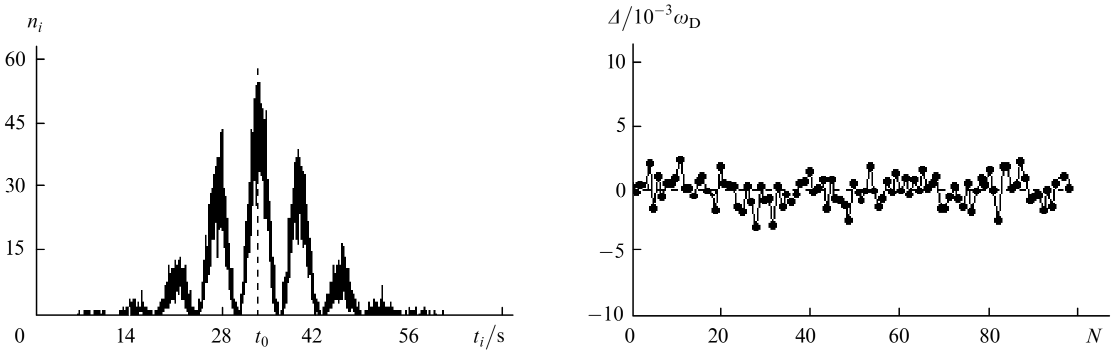
The results of this procedure, repeated 100 times for each amplitude, are displayed in Fig. 3b. This figure shows the relative deviation Δ/ω_D of the Doppler frequency estimate from its actual value as well as its relative root-mean-square deviation and the root-mean-square Rao – Cramer bound. Analysing these results, we can draw the following conclusions: Relative deviations of the mean estimates from the actual frequency $(\langle\hat{\omega}_D\rangle - \omega_D)/\omega_D$ have the order of 10^{-5}. This is significantly less than the relative root-mean-square deviation of the estimates $\sigma_{\omega_D}/\omega_D$; therefore, we can consider the obtained solutions to be unbiased estimates of the Doppler frequency. The root-mean-square deviations of the frequency estimates and the root-mean-square Rao – Cramer bounds σ_{RK} (element R_{22} of matrix (18)) are very close to each other, allowing us to conclude that the obtained estimates are efficient.



$A_0 = 250$ s^{-1}, $\dfrac{\langle\hat{\omega}_D\rangle - \omega_D}{\omega_D} = 0.030\,\%$, $\dfrac{\sigma_{\omega_D}}{\omega_D} = 0.212\,\%$, $\dfrac{\sigma_{RK}}{\omega_D} = 0.212\,\%$



$A_0 = 500$ s^{-1}, $\dfrac{\langle\hat{\omega}_D\rangle - \omega_D}{\omega_D} = 0.004\,\%$, $\dfrac{\sigma_{\omega_D}}{\omega_D} = 0.156\,\%$, $\dfrac{\sigma_{RK}}{\omega_D} = 0.150\,\%$



$A_0 = 1000$ s^{-1}, $\dfrac{\langle\hat{\omega}_D\rangle - \omega_D}{\omega_D} = 0.005\,\%$, $\dfrac{\sigma_{\omega_D}}{\omega_D} = 0.110\,\%$, $\dfrac{\sigma_{RK}}{\omega_D} = 0.106\,\%$

Figure 3. Simulation of the single-particle Doppler electric signal (a) and the relative deviation of the maximum-likelihood estimates of the Doppler frequency from its actual value (b).

5. Conclusions

With the help of the derived likelihood equations, one can obtain the most accurate compatible estimates of the Doppler frequency, the signals amplitude, and the time of arrival of the scattering particle to the centre of the measured volume. We have solved this problem in the case of laser Doppler anemometers and lidars operating in the single-particle-scattering mode with a photon counter serving as a photodetector. We have also found the Rao – Cramer bounds, which approximate the minimum possible dispersions of the estimates. The computer simulation of the Doppler signals and the numerical solution of the likelihood equation have shown that the obtained estimates are unbiased and efficient. The results of this work offer new possibilities for high-accuracy studies of gas flows with the help of LDMSs.

Based on the results obtained, we can draw an important conclusion about the optimal reception of optical signals in the regime of analogue photodetection. This regime is realised when the intensity of the optical signal is sufficiently large so that single-electron pulses overlap due to the photodetector's inertia and form a continuous analogue signal at the output. If the inertia of the photodetector is described by a time constant τ, the periodically measured analogue signal will be proportional to the number of photoelectrons emitted by the cathode during the measurement period τ. This is explained by the integrating action of the equivalent RC circuit. Given the above likelihood equation, the optimal estimation procedures and the Rao – Cramer bounds will be similar to those obtained in this work. Obviously, the time constant characterising the photodetector inertia should be much smaller than the period corresponding to the maximum modulation frequency of the optical signal.

Appendix

1. Derivation of the likelihood equations for A_0

By substituting Eqn (5) and the expression for $\partial\lambda(t_i, A_0)/\partial A_0$ into (8), we obtain

$$\frac{\mathrm{d}\ln P(n_i,\ldots,n_N)}{\mathrm{d}A_0} = \sum_i \left\{ \frac{n_i}{A_0} - \exp\left[-\xi^2\omega_\mathrm{D}^2(t_i-t_0)^2\right] \times[1+\cos\omega_\mathrm{D}(t_i-t_0)]\Delta t \right\}. \quad \text{(A1)}$$

Replacing the second part of the sum in Eqn (A1) by the integral with infinite limits, we arrive at

$$\frac{\mathrm{d}\ln P(n_1,\ldots,n_N)}{\mathrm{d}A_0} = \sum_i \left(\frac{n_i}{A_0}\right) - \int_{-\infty}^{\infty} \exp\left[-\xi^2\omega_\mathrm{D}^2(t-t_0)^2\right][1+\cos\omega_\mathrm{D}(t-t_0)]\mathrm{d}t$$

$$= \sum_i \left(\frac{n_i}{A_0}\right) - \frac{\sqrt{\pi}}{\xi\omega_\mathrm{D}}\left[1+\exp\left(-\frac{1}{4\xi^2}\right)\right]. \quad \text{(A2)}$$

We can neglect the exponential term in (A2), because when $\xi \ll 1$, which is realised in practice, this term is much less than unity. Therefore, the likelihood equation for the amplitude A_0 would takes the final form:

$$\sum_i \left(\frac{n_i}{A_0}\right) = \frac{\sqrt{\pi}}{\xi\omega_\mathrm{D}} \;{}^9\ddot{o}^9\; A_0 = \frac{\xi\omega_\mathrm{D}}{\sqrt{\pi}}\sum_i n_i. \quad \text{(A3)}$$

2. Derivation of the likelihood equation for ω_D

Inserting Eqn (5) and the expression for $\partial\lambda(t_i, \omega_\mathrm{D})/\partial\omega_\mathrm{D}$ into equation (8) and acting as above, we obtain

$$\frac{\mathrm{d}\ln P(n_i,\ldots,n_N)}{\mathrm{d}\omega_\mathrm{D}} = 2\xi^2\omega_\mathrm{D}\sum_i\left[-n_i(t_i-t_0)^2\right]$$

$$-\sum_i\left[\frac{n_i(t_i-t_0)\sin\omega_\mathrm{D}(t_i-t_0)}{1+\cos\omega_\mathrm{D}(t_i-t_0)}\right]$$

$$+\int_{-\infty}^{\infty} A_0\exp\left[-\xi^2\omega_\mathrm{D}^2(t-t_0)^2\right]\left\{2\xi^2\omega_\mathrm{D}(t-t_0)^2\right.$$

$$\left.\times[1+\cos\omega)\mathrm{D}(t-t_0)] + (t-t_0)\sin\omega_\mathrm{D}(t-t_0)\right\}\mathrm{d}t. \quad \text{(A4)}$$

Evaluating the integral and neglecting the exponential term as before, we obtain the likelihood equation

$$2\xi^2\omega_\mathrm{D}\sum_i n_i(t_i-t_0)^2 + \sum_i n_i(t_i-t_0)\tan\frac{\omega_\mathrm{D}(t_i-t_0)}{2} = \frac{\sqrt{\pi}A_0}{\xi\omega_\mathrm{D}^2}. \quad \text{(A5)}$$

3. Derivation of the likelihood equation for t_0

Inserting Eqn (5) and $\partial\lambda(t_i, A_0)/\partial t_0$ into Eqn (8), we obtain

$$\frac{\mathrm{d}\ln P(n_i,\ldots,n_N)}{\mathrm{d}t_0} = 2\xi^2\omega_\mathrm{D}^2\sum_i n_i(t_i-t_0) + \sum_i n_i\omega_\mathrm{D}\tan\frac{\omega_\mathrm{D}(t_i-t_0)}{2}. \quad \text{(A6)}$$

Thus, the likelihood equation for the time instant t_0 takes the form

$$2\xi^2\omega_\mathrm{D}\sum_i n_i(t_i-t_0) + \sum_i n_i\tan\frac{\omega_\mathrm{D}(t_i-t_0)}{2} = 0. \quad \text{(A7)}$$

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