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Statistics of an ideal homogeneous Bose gas with a fixed number of particles

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Abstract*. The distribution function $w_0(n_0)$ of the number n_0 of particles is found for the condensate of an ideal gas of free bosons with a fixed total number N of particles. It is shown that above the critical temperature ($T > T_c$) this function has the usual form $w_0(n_0) = (1 - e^{\mu})e^{\mu n_0}$, where μ is the chemical potential in temperature units. In a narrow vicinity of the critical temperature $|T/T_c - 1| \leqslant N^{-1/3}$, this distribution changes and at $T < T_c$ acquires the form of a resonance. The width of the resonance depends on the shape of the volume occupied by the gas and it has exponential (but not the Gaussian) wings. As the temperature is lowered, the resonance maximum shifts to larger values of n_0 and its width tends to zero, which corresponds to the suppression of fluctuations. For $N \to \infty$, this change occurs abruptly. The distribution function of the number of particles in excited states for the systems with a fixed and a variable number of particles (when only a mean number of particles is fixed) prove to be identical and have the usual form.

***Keywords**: Bose–Einstein statistics, condensate, distribution function, canonical ensemble.*

It was shown in papers [1] that the distribution function $w_0(n_0)$ of the number n_0 of particles in the ground state of an ideal Bose gas captured by a trap is described by the distribution function

$$w_0(n_0) = \left(1 - e^{\mu - \varepsilon_0}\right)e^{(\mu - \varepsilon_0)n_0}, \quad (1)$$

found by Einstein [2] only at temperatures above the critical temperature ($T > T_c$), when a mean number of particles in the ground state is small, i.e., a condensate is virtually absent. In (1), μ is the chemical potential in temperature units; $\varepsilon_0 = E_0/T$; E_0 is the ground-state energy of gas particles; T is the temperature in energy units. Hereafter, we measure energies relative to the ground-state energy, i.e., assume that $\varepsilon_0 = 0$.

Below the critical temperature, when a macroscopic number of particles are in the ground state, i.e., a condensate is formed, expression (1) is no longer valid, and the distribution function $w_0(n_0)$ takes a Gaussian shape. When the number N of particles captured by the trap is large, this change in the shape of the distribution function occurs in a very narrow vicinity of the critical temperature (in the case of a parabolic trap, $|T/T_c - 1| \leqslant 1/\sqrt{N}$), i.e., virtually jumpwise. These features of the condensation are described by the distribution [1]

$$w_0(n_0) = S^{-1}\exp\left[\mu n_0 - \frac{(n_0 - \tilde{n}_0)^2}{4D}\right], \ \mu = -\ln\left(1 + \frac{1}{\tilde{n}_0}\right), \quad (2)$$

where $S = \sum_{n_0=0}^{N} w_0(n_0)$ is the normalising factor;

$$\tilde{n}_0 = \left(e^{-\mu} - 1\right)^{-1} \quad (3)$$

is a mean number of particles in the ground state determined by the distribution (1);

$$D = \frac{1}{2}\sum_{k\neq 0}\left(\tilde{n}_k + \tilde{n}_k^2\right);$$

and $\tilde{n}_k = (e^{\varepsilon_k - \mu} - 1)^{-1}$ is a mean number of particles in excited states $k \neq 0$, which is also determined by the Bose–Einstein distribution [2, 3]

$$w_k(n_k) = \left(1 - e^{\mu - \varepsilon_k}\right)e^{(\mu - \varepsilon_k)n_k}, \quad (4)$$

in which $\varepsilon_k = E_k/T$ and the chemical potential (or $\tilde{n}_0$) is determined by the condition [2, 3]

$$\sum_k \tilde{n}_k = N. \quad (5)$$

For $T < T_c$, a mean number of particles in the condensate is large ($\tilde{n}_0 \gg 1$) and distribution (2) has a Gaussian shape. As $\tilde{n}_0$ decreases with increasing temperature, for $T > T_c$, when the condensate virtually disappears, distribution (2) transforms to (1). The change in the shape of the distribution $w_0(n_0)$ at $T < T_c$ is related to the exact fulfilment of the condition

$$n_0 + n_1 + n_2 + \ldots = N, \quad (6)$$

which fixes the number of particles in an ensemble and was used in the derivation of this distribution [1]. This change demonstrates an important difference of the statistical properties of a canonical ensemble from those of a grand cano-

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nical ensemble for which only condition (5) for mean values is fulfilled, resulting in the invariable shape (1) of the function $w_0(n_0)$ at any temperature.

The condensation of an ideal Bose gas captured by a trap is of great practical interest because such condensation has been realised in Ref. [4] and is now being extensively experimentally studied. However, from the fundamental point of view, also is important the case of a free Bose gas (confined only by the vessel walls), whose condensation was predicted by Einstein in 1925 [2] and since then has been theoretically discussed by many authors (see, for example, [3, 5, 6] and references cited in [5, 6]).

It is shown in this paper that the qualitative change in the distribution function of the number of particles in the condensate of a canonical ensemble of free gas particles at $T < T_c$ is in a whole quite similar to that taking place for the gas in a trap. However, in the former case, the region $|T/T_c - 1| \leqslant N^{-1/3}$ of the jump is different and the distribution $w_0(n_0)$ at $T < T_c$ is not Gaussian and depends on the shape of the gas volume. The distribution function of the number of particles in excited states has the shape (4) at any temperature.

In the case of free gas, as for the gas captured by a trap, the distribution function $w_0(n_0)$ is determined by the summation of the Gibbs distribution

$$w_0(n_0) = S^{-1} \sum_{n_1+n_2+\ldots=N-n_0} \mathrm{e}^{-\varepsilon_0 n_0 - \varepsilon_1 n_1 - \ldots}, \tag{7}$$

where the summation is performed over all positive n_k, except n_0, which satisfy condition (6).

It has been shown in Ref. [1] that condition (6) can be automatically fulfilled if the sum is written in the form (recall that $\varepsilon_0 = 0$)

$$w_0(n_0) = S^{-1} \sum_{n_1,n_2\ldots} \mathrm{e}^{-\varepsilon_1 n_1 - \varepsilon_2 n_2 - \ldots} \frac{1}{2\pi \mathrm{i}} \oint z^{-N+n_0-1+n_1+n_2+\ldots} \mathrm{d}z. \tag{8}$$

The integration contour in (8) is a circle with a centre at the point $z = 0$ and radius $|z| < 1$, which can be conveniently written in the form $z = \mathrm{e}^{\mu}$ for $\mu < 0$. It should be emphasised that the parameter μ introduced in such a way is not generally related to the chemical potential μ appearing in distributions (1) and (4). However, as shown below, it is convenient to choose the parameter μ in (8) for $T > T_c$ as in (1) and (4), by requiring the fulfilment of condition (5).

The summation in (8) is performed over all $n_{k\neq 0} \geqslant 0$, the condition $\mu < 0$ providing the convergence of all the sums. As a result, we obtain

$$w_0(n_0) = S^{-1} \frac{1}{2\pi \mathrm{i}} \oint z^{-N+n_0-1} \mathrm{e}^{G(z)} \mathrm{d}z,$$

$$\mathrm{e}^{G(z)} = \prod_{k\neq 0} \left(1 - z\mathrm{e}^{-\varepsilon_k}\right)^{-1}, \tag{9}$$

$$G(z) = -\sum_{k\neq 0} \ln\left(1 - z\mathrm{e}^{-\varepsilon_k}\right) = \sum_{k\neq 0} \sum_{p=1}^{\infty} \frac{1}{p} z^p \mathrm{e}^{-p\varepsilon_k}.$$

The energy levels of gas particles (in temperature units) in the volume $V = L_x L_y L_z$, which is assumed for simplicity a cube of volume $V = L^3$ (the relevant generalisation will be give below), are determined by the requirement of the periodicity of the wave function:

$$\varepsilon_{\boldsymbol{k}} = \alpha\left(k_x^2 + k_y^2 + k_z^2\right), \quad \alpha = \frac{(2\pi\hbar)^2}{2mTL^2},$$
$$k_i = 0, \pm 1, \ldots, \quad i = x, y, z, \tag{10}$$

where m is the mass of particles.

When the condition $\varepsilon_1 \gg 1$ is satisfied, which can be rewritten in the convenient form

$$t \ll N^{-2/3}, \quad t = \frac{T}{T_c}, \quad T_c = 2\pi\zeta^{-2/3}\left(\frac{3}{2}\right)\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{2/3}, \tag{11}$$

by introducing the usual critical temperature T_c [2, 3], where $\zeta(x)$ is the Riemann zeta function, it follows from (9), as in the case of gas captured by a trap [1], that only two values of the distribution function

$$w_0(n_0 = N) = 1 - 3\mathrm{e}^{-\alpha}, \quad w_0(n_0 = N-1) = 3\mathrm{e}^{-\alpha},$$

$$\alpha = \pi\zeta^{2/3}\left(\frac{3}{2}\right) t^{-1} N^{-2/3}$$

are essential. For $T \to 0$, the distribution function takes the form $w_0(n_0) = \delta_{n_0,N}$, which qualitatively differs from (1).

At the temperature that is still much lower than the critical temperature, the condition $t \gg N^{-2/3}$ comes into play, which is equivalent to the condition $\alpha \ll 1$. In this case, $\varepsilon_{\boldsymbol{k}} = \alpha \boldsymbol{k}^2 \ll 1$ up to very large values of $\boldsymbol{k}$, and to study the distribution function $w_0(n_0)$, it is convenient to make the change of variables $z = \mathrm{e}^{\mu + \mathrm{i}x}$ in (8). Then, we find

$$w_0(n_0) = S^{-1} \mathrm{e}^{\mu n_0} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(N-n_0)x + F(x,\mu)} \mathrm{d}x,$$
$$F(x,\mu) = \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{e}^{(\mu+\mathrm{i}x)p} \sum_{\boldsymbol{k}\neq 0} \mathrm{e}^{-\alpha p \boldsymbol{k}^2}. \tag{12}$$

The sum over $\boldsymbol{k} \neq 0$ in the definition of $F(x,\mu)$ exponentially decreases at $p \to \infty$, providing the convergence of the sum over p for $\mu = 0$, i.e., the function $F(x,\mu)$ is continuous at $\mu = 0$. So far, the parameter μ was restricted by the condition $\mu < 0$, being arbitrary in all other respects. The continuity of the function $F(x,\mu)$ at $\mu = 0$ allows us to calculate the integral determining $w_0(n_0)$ in (12) at $\mu = 0$. Thus, we assume that $\mu = 0$ in (12) and introduce the notation $F(x) = F(x, \mu = 0)$. By differentiating $F(x)$ two times with respect to x, we obtain

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -\sum_{p=1}^{\infty} p\mathrm{e}^{\mathrm{i}px} f(\alpha p), \quad f(z) = \sum_{\boldsymbol{k}\neq 0} \mathrm{e}^{-z\boldsymbol{k}^2}. \tag{13}$$

To calculate $w_0(n_0)$ at temperatures below the critical temperature ($N^{-2/3} \ll t \leqslant 1$) and in a narrow vicinity above the critical temperature ($0 \leqslant t - 1 \ll 1$), it is necessary to study the behaviour of the function $F(x)$ at small $|x| \ll 1$. It is important that the function $f(z)$ tends to two limiting values at large and small z:

$$f(z) = \begin{cases} \left(\dfrac{\pi}{z}\right)^{3/2}, & z \ll 1, \\ 6\mathrm{e}^{-z}, & z \gg 1. \end{cases} \tag{14}$$

Therefore, for $|x| \ll 1$, we can pass from summation over p in (13) to integration in the region from 0 to $+\infty$, because the resulting integral, as one can see from (14), converges:

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = -\int_0^{\infty} p\mathrm{e}^{\mathrm{i}px} f(\alpha p)\mathrm{d}p = -\frac{1}{\alpha^2}\int_0^{\infty} zf(z)\mathrm{e}^{\mathrm{i}(x/\alpha)z}\mathrm{d}z$$

$$= -\frac{1}{\alpha^2}\sum_{\boldsymbol{k}\neq 0}\left(\boldsymbol{k}^2 - \mathrm{i}\frac{x}{\alpha}\right)^{-2}. \qquad (15)$$

Then, by integrating this expression two times with respect to x and taking into account that for $\alpha \ll 1$,

$$F(0) = \sum_{p=1}^{\infty} p^{-1} f(\alpha p) \approx \left(\frac{\pi}{\alpha}\right)^{3/2}\zeta\left(\frac{5}{2}\right),$$

$$\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)_{x=0} = \mathrm{i}\sum_{p=1}^{\infty} f(\alpha p) \approx \mathrm{i}\left(\frac{\pi}{\alpha}\right)^{3/2}\zeta\left(\frac{3}{2}\right) = \mathrm{i}Nt^{3/2},$$

we obtain

$$F(x) = F(0) + \mathrm{i}Nt^{3/2}x + g(x/\alpha),$$

$$g(u) = -\sum_{\boldsymbol{k}\neq 0}\left[\ln\left(1 - \frac{\mathrm{i}u}{\boldsymbol{k}^2}\right) + \frac{\mathrm{i}u}{\boldsymbol{k}^2}\right]. \qquad (16)$$

The value of $F(0)$ enters into the normalisation upon the substitution into the integral in (12) and can be discarded below.

For small values of u, the function $g(u)$ can be represented by the series

$$g(u) = \sum_{n=2}^{\infty} c_n(\mathrm{i}u)^n, \quad c_n = \frac{1}{n}\sum_{\boldsymbol{k}\neq 0}\frac{1}{(\boldsymbol{k}^2)^n}, \quad |u| < 1, \qquad (17)$$

whose radius of convergence is restricted by the condition $|u| < 1$. In another limiting case, $u \gg 1$, we can pass from summation over $\boldsymbol{k}$ in (16) to integration, which gives

$$g(u) = -\frac{4}{3}\pi^2\mathrm{e}^{\mathrm{i}\pi/4}u^{3/2}, \quad u \gg 1. \qquad (18)$$

The use of (16) in the calculation of the integral in (12) after the change of variables $x/\alpha = u$ yields the result

$$w_0(n_0) = S^{-1}\varphi\left(\frac{n_0 - \bar{n}_0}{\beta N^{2/3}t}\right), \quad \varphi(y) = \mathrm{Re}\int_0^{\infty}\mathrm{e}^{\mathrm{i}yu + g(u)}\mathrm{d}u,$$

$$\bar{n}_0 = N\left(1 - t^{3/2}\right), \quad \beta = \pi^{-1}\zeta^{-2/3}(3/2) \approx 0.168. \qquad (19)$$

The integral in (19) is written taking into account that the condition $\pi/\alpha \gg 1$ admits the calculation of this integral with infinite limits and using an important property of the function $g(u)$, $g(u) = g^*(-u)$, which allows us to write the result as a real part of the integral over positive values of u.

Note that for the value $\mu = 0$ chosen by us, i.e., for the circle radius $|z| = \mathrm{e}^{\mu} = 1$ in the integral in (8), the value of $\bar{n}_0$ coincides with a mean number $\tilde{n}_0$ of particles in the condensate obtain from (4) and (5) only when $t = T/T_{\mathrm{c}} < 1$. For $t > 1$, the value of $\bar{n}_0$ becomes negative, which is admissible in our case. To avoid misunderstanding, we emphasise that all the mean values obtained from (19) are denoted hereafter by angle brackets.

The function $\varphi(y)$, which can be studied only numerically, is shown in Fig. 1. For $y \leqslant 10$, the function can be calculated quite accurately by keeping only two first terms in expansion (17):

$$g(u) \approx -8.25u^2 - \mathrm{i}2.8u^3, \qquad (20)$$

while for $y < 7$, both the exact function $\varphi(y)$ and its approximate value calculated by expression (20) differ less than by 10 % from the function $\varphi(y)$ calculated by retaining only the first term in expansion (17), i.e., from the Gaussian

$$\varphi(y) = \frac{1}{2}\left(\frac{\pi}{c_2}\right)^{1/2}\mathrm{e}^{-y^2/4c_2}. \qquad (21)$$

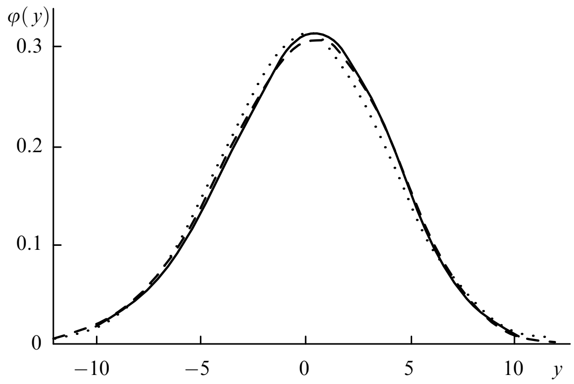

Figure 1. Function $\varphi(y)$ calculated by expression (19) with exact $g(u)$ calculated by expression (16) (solid curve) and by expression (20) (dashed curve), as well as the function $\varphi(y)$ calculated by expression (21) (dotted curve).

All the moments of the function $\varphi(y)$ exist and can be calculated analytically:

$$\langle y^m\rangle = \int_{-\infty}^{+\infty} y^m\varphi(y)\mathrm{d}y = \pi\left(\frac{\mathrm{d}^m}{\mathrm{d}u^m}\mathrm{e}^{g(\mathrm{i}u)}\right)_{u=0},$$

$$g(\mathrm{i}u) = \sum_{n=2}^{\infty}(-1)^n c_n u^n, \qquad (22)$$

which shows that $\varphi(y)$ decays exponentially at $|y| \gg 1$. For large positive values of y, using (18) and calculating the integral in (19) by the saddle point method, we find the value

$$\varphi(y) = \frac{\sqrt{y}}{2\pi\sqrt{\pi}}\mathrm{e}^{-y^3/12\pi^4}, \quad y \gg 1, \qquad (23)$$

which, however, can be reached only for very large $y > 100$.

The distribution $w_0(n_0)$ qualitatively changes at large positive, small and large negative values of $\bar{n}_0$. In a broad temperature range below the critical temperature

$$t \gg \beta^2 N^{-2/3}, \quad 1 - t \gg \frac{2}{3}\beta N^{-1/3} \qquad (24)$$

the distribution function is exponentially small at its two extreme points $n_0 = 0$ and $n_0 = N$. The second condition in (24), which ensures the smallness of $w_0(n_0)$ at the lower boundary, allows a very close approach to the critical temperature. In the calculation of the normalising factor

$$S = \sum_{n_0=0}^{N} w_0(n_0)$$

and all the mean values, summation can be replaced by integration with infinite limits to find

$$S = \pi\beta N^{2/3} t, \quad \langle n_0 \rangle = \bar{n}_0,$$
$$\big\langle (n_0 - \bar{n}_0)^m \big\rangle = \frac{1}{\pi} \big(\beta N^{2/3} t\big)^m \big\langle y^m \big\rangle. \tag{25}$$

It follows from this, in particular, that a mean number of particles in the condensate coincides with that obtained from (1), while the mean-square fluctuation

$$\big\langle \Delta n_0^2 \big\rangle = \big\langle (n_0 - \bar{n}_0)^2 \big\rangle = \big\langle n_0^2 \big\rangle - \langle n_0 \rangle^2 = 2c_2 \big(\beta N^{2/3} t\big)^2 \tag{26}$$

decreases with decreasing temperature proportionally to T^2. The distribution shape is close to a Gaussian, however, it is slightly asymmetric and the distribution maximum is located at $n_0^{\max} \approx \langle n_0 \rangle + 0.5\beta N^{2/3} t$, which somewhat exceeds the mean value.

In the narrow vicinity of temperatures above the critical temperature

$$\frac{2}{3}\beta N^{-1/3} \ll t - 1 \ll 1 \tag{27}$$

the value of $\bar{n}_0$ becomes negative and the modulus $|\bar{n}_0| \approx (3/2)N(t-1) \gg N^{2/3}\beta$ becomes very large. In this case, the integral in (19) contains contributions from large values $u \approx (\bar{n}_0 \alpha)^2$. However, the corresponding values $x = \alpha u \approx \bar{n}_0^2 \alpha^3 \ll 1$ are still small, i.e., the replacement of summation over p in (13) by integration in (15) is still justified and expression (19) is still valid. By using the asymptotic value (23) for $\varphi(y)$, we find

$$w_0(n_0) = S^{-1} \sqrt{|\bar{n}_0| + n_0} \exp\Big[-\frac{\gamma}{N^2 t^3} \big(|\bar{n}_0| + n_0\big)^3 \Big],$$
$$\gamma = \frac{\zeta^2(3/2)}{12\pi}.$$

One can easily verify that in the temperature range (27), only the linear term over n_0 may be retained in the exponent in the above expression, while n_0 under the root sign can be neglected. As a result, we obtain

$$w_0(n_0) = \frac{27}{4}\gamma(t-1)^2 \exp\Big[-\frac{27}{4}\gamma(t-1)^2 n_0 \Big].$$

This distribution coincides with (1) if the parameter $\mu = -(27/4)\gamma(t-1)^2$ is used in (1). One can easily verify that the same μ is obtained in this case from condition (5).

As the temperature further increases and the condition $t - 1 > 1$ is satisfied, very large values of u corresponding to $x \gg 1$ contribute to the integral in (19). In this case, the replacement of summation over p in (13) by integration becomes invalid, so that we introduce, similarly to the procedure used in [1], the parameter $\mu < 0$ and require the fulfilment of the condition

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \mathrm{i} \sum_{\boldsymbol{k} \neq 0} \tilde{n}_{\boldsymbol{k}} = \mathrm{i} \sum_{p=1}^{\infty} \mathrm{e}^{\mu p} \sum_{\boldsymbol{k} \neq 0} \mathrm{e}^{-\alpha p k^2} = \mathrm{i}(N - \tilde{n}_0) \approx \mathrm{i}N$$

for mean values, which coincides with (5).

By keeping in this relation only the term with $p = 1$ and replacing summation over $\boldsymbol{k}$ by integration, we obtain $\mathrm{e}^{\mu} = t^{-3/2}$ and, respectively,

$$F(x) = F(0) + \mathrm{i}Nx - \frac{1}{2}Nx^2. \tag{28}$$

By substituting (28) into (12), we again obtain distribution (1). In this case, $\mathrm{e}^{\mu} \ll 1$ and in fact we deal with the Boltzmann distribution.

It follows from the above discussion that in a narrow vicinity of T_c between regions (24) and (27), i.e., when the condition $|t - 1| \leqslant \beta N^{-1/3}$ is satisfied, the distribution function of the number of particles in the condensate is abruptly (virtually jumpwise) transformed, by changing its shape from almost Gaussian to the usual shape (1) (see Fig. 2).

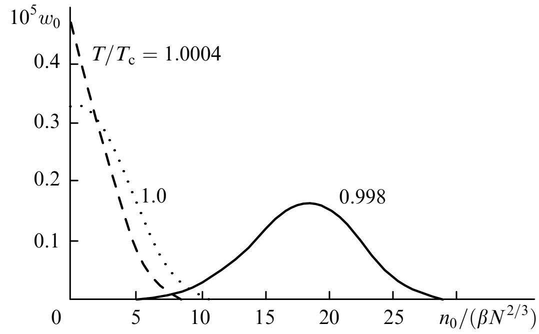

Figure 2. Distributions $w_0(n_0)$ at different temperatures in the vicinity of the critical temperature for $N = 10^9$.

Similarly to (12), we can write the joint distribution [1]:

$$w_{0,i\neq 0}(n_0, n_i) =$$
$$= S^{-1} \mathrm{e}^{\mu(n_0+n_i)} \mathrm{e}^{-\varepsilon_i n_i} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(N-n_0)x + F(x)} \big(1 - \mathrm{e}^{\mu + \mathrm{i}x - \varepsilon_i}\big) \mathrm{d}x.$$

It follows from this that in the temperature ranges (24) and (27) we have

$$w_{0,i\neq 0}(n_0, n_i) = S^{-1} \mathrm{e}^{-\varepsilon_i n_i} \Big[\varphi\Big(\frac{n_0 + n_i - \bar{n}_0}{\beta N^{2/3} t} \Big) - \mathrm{e}^{-\varepsilon_i} \varphi\Big(\frac{n_0 + n_i + 1 - \bar{n}_0}{\beta N^{2/3} t} \Big) \Big].$$

By summing (integrating) this distribution over n_0, we find that the distribution function

$$w_{i\neq 0} = \sum_{n_0=0}^{N} w_{0,i\neq 0}(n_0, n_i)$$

of the number of particles in excited states coincides with (4). By using (28) at higher temperatures $t - 1 > 1$, we again obtain (4).

Note now that we can easily pass from a cubic volume of quantisation to a rectangular parallelepiped. To do this, the change of variables

$$\alpha \to (\alpha_x \alpha_y \alpha_z)^{1/3}, \quad \alpha_s = \frac{(2\pi\hbar)^2}{2mTL_s^2}, \quad \Omega_s = \frac{\alpha_s}{(\alpha_x \alpha_y \alpha_z)^{1/3}},$$

$$s = x, y, z,$$

should be made in the above expressions. Then, we have

$$g(u) = -\sum_{k \neq 0} \left[\ln \left(1 - \mathrm{i} \frac{u}{\Omega_x k_x^2 + \Omega_y k_y^2 + \Omega_z k_z^2} \right) + \mathrm{i} \frac{u}{\Omega_x k_x^2 + \Omega_y k_y^2 + \Omega_z k_z^2} \right].$$

In this case, the definition of coefficients c_n changes in an obvious way. For example, when $L_x = L_y = l$, $L_z = L$, and $L/l \gg 1$, these coefficients take the form $c_n = (2/n)\zeta(2n) \times (L/l)^{4n/3}$. Therefore, according to (26), at temperatures below the critical temperature (in the region (24)), the width $\langle \Delta n_0^2 \rangle^{1/2}$ of the distribution at a constant volume increases proportionally to the parameter $(L/l)^{4/3}$, while the shape of this distribution differs more and more from a Gaussian, i.e., the shape of the distribution function in this region depends on the volume shape. At higher temperatures (already in the region (27)), this dependence vanishes because the asymptotic value (23) is independent of the volume shape, and only in this case the distribution is no longer related to the discreteness of the energy spectrum, which is determined by the shape of the quantisation volume.

Therefore, the summation of the Gibbs distribution showed that in the case of a canonical ensemble, i.e., when condition (6) is fulfilled rather than condition (5) for mean values, the distribution function of the number of particles in the ground state (in the condensate) drastically changes at temperatures below the critical temperature ($T < T_c$), whereas the distribution function of the number of particles in excited states is described at any temperature by expression (4) obtained by Einstein [2], regardless of whether condition (5) for mean values (grand canonical ensemble) or condition (6) (canonical ensemble) is satisfied upon summation of the Gibbs distribution.

In particular, this means that the correct distribution $w_0(n_0)$ can be obtained by using the equality $w_0(n_0) = w^*(N^* = N - n_0)$, which is obvious for a canonical ensemble, where $w^*(N^* = N - n_0)$ is the distribution function for a total number $N^* = n_1 + n_2 + ...$ of particles in excited states, if we assume (which is naturally has no *a priori* foundation) that the distribution function of the number of particles in excited states is described by expression (4). It is this approach that was used in papers [5, 6] (see also references therein) for calculating mean-square fluctuations, which coincide with (26). The value of $\langle \Delta n_0^2 \rangle$ calculated in Ref. [7] by a qualitatively different method exceeds the value (26) by a factor of sixteen.

In Ref. [8], the distribution $w_0(n_0)$ was obtained as a stationary solution of the model kinetic equation for the number of particles in the condensate assuming the validity of (4). The study of this distribution shows that it coincides with (2) for large N in the case of a parabolic trap, and differs from (19) in the case of the free boson gas, although gives the mean-square fluctuation coincident with (26).

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