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Green function for three-wave coupling problems

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Abstract. **The Green function is found for three-wave coupling problems. The function was used for analysis of parametric amplification in dissipative and active media. It is shown that the parametric increment in active media can become exponential. As an example, the nonstationary stimulated scattering of electromagnetic waves by sound and temperatures waves is considered.**

Keywords: *three-wave coupling, Green function, parametric increment, active medium.*

1. Introduction

Three-wave interactions encompass a broad scope of nonlinear phenomena: various types of stimulated scattering of electromagnetic and acoustic waves by waves of the different nature, the generation of sum- and difference-frequency waves, second harmonic generation, parametric amplification, etc. When the approximations of slowly varying amplitudes and infinite plane waves are used and the depletion of a pump wave is neglected, all these interactions can be described with the help of a system of two coupled first-order partial differential equations. Methods for deriving these equations and the substantiation of the approximations used are described in detail, for example, in Refs [1 – 5]. In this paper, we solve these equations using the Green function.

2. Specified field approximation

Under phase-matching conditions, a standard system of equations describing the three-wave coupling in the approximation of the specified field of a pump wave has the form [6 – 12]

$$A_0\frac{\partial\Phi}{\partial t}+A_1\frac{\partial\Phi}{\partial x}+A_2\Phi=A_3\Psi^*\Pi,$$
$$B_0\frac{\partial\Psi^*}{\partial t}+B_1\frac{\partial\Psi^*}{\partial x}+B_2\Psi^*=B_3\Phi\Pi^*, \tag{1}$$

where Φ, Ψ, Π are the complex amplitudes of the interacting quasi-plane, quasi-monochromatic waves with freqeuncies ω_1, ω_2, ω_0 and wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_0$, respectively $\omega_1+\omega_2=\omega_0$, $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_0$; the amplitude $\Pi=\Pi_0$ of the pump wave is independent of the time t and coordinate x; the x axis is directed along the propagation direction of the exciting wave; A_i and B_i are constants determined by the physical statement of the problem. In a number of physical problems, dissipative coefficients A_2, B_2, and nonlinear coupling coefficients A_3 and B_3 can be complex. The ratios of coefficients A_1/A_0 and B_1/B_0 (velocities of the corresponding waves along the x axis) are assumed real.

System of equations (1) should be supplemented with boundary conditions. For definiteness, we write them in the form

$$\Phi(0,t)=\Phi_0(t),\quad \Psi^*(0,t)=\Psi_0(t),$$
$$\Phi(x,0)=\Psi^*(x,0)=0. \tag{2}$$

Here, the boundary conditions are specified at the input to the nonlinear medium ($x=0$), which is convenient when the waves propagate along the x axis. This corresponds to the conditions $A_1/A_0>0$ and $B_1/B_0>0$. If one of the waves propagates in the opposite direction, it is convenient to specify the boundary conditions for this wave at the medium boundary $x=L$.

System (1) was solved in previous papers using various simplifying approximations. For example, in Ref. [10], this system was solved for $B_1=A_2=0$, in Refs [6, 7], for $B_1=0$ and using some assumptions about temporal and spatial limits, in Ref. [8], for $A_0=A_2=B_2=0$, and in Ref. [9], for $A_2=B_2=0$ and $\Phi_0(t)=0$. In Ref. [11], an exact integral solution of system (1) was also obtained for $A_2=B_2=0$, but for arbitrary functions $\Phi_0(t)$, $\Psi_0(t)$. This solution was written in the Riemann form using the average characteristic for the waves Ψ and Φ. In paper [11], the procedure is also described which takes the quantities A_2 and B_2 into account, but the final expressions containing $A_2, B_2\neq 0$ are not presented because they are cumbersome.

Below, we show that the exact solution of system (1) can be written in a convenient integral form in terms of the Green function $G(x,t)$, and find the expression for this function. A similar integral solution was obtained for the first time in Ref. [12] for a particular case of system (1) for $A_0=B_0=1$. The Green function obtained below is valid for any coefficients A_i, B_i, including their zero values. This allows one to use this function for a broader class of problems, for example, those considered in section 3.

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Using the method of generalised functions [13], system (1) can be transformed to the following equations with homogeneous boundary conditions:

$$A_1B_1\frac{\partial^2\Phi}{\partial x^2}+(A_1B_0+B_1A_0)\frac{\partial^2\Phi}{\partial x\partial t}+A_0B_0\frac{\partial^2\Phi}{\partial t^2}$$

$$+(A_2B_0+B_2A_0)\frac{\partial\phi}{\partial t}+(A_2B_1+A_1B_2)\frac{\partial\Phi}{\partial x}$$

$$+\left(A_2B_2-A_3B_3|\Pi_0|^2\right)\Phi=g(x,t),$$

$$\Psi^*=\frac{1}{A_3\Pi_0}\left(A_0\frac{\partial\Phi}{\partial t}+A_1\frac{\partial\Phi}{\partial x}+A_2\Phi\right). \tag{3}$$

Here,

$$g(x,t)=A_1B_1\Phi_0(t)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}+\Bigg[A_1B_2\Phi_0(t)+A_3B_1\Pi_0\Psi_0(t)$$

$$+A_1B_0\frac{\mathrm{d}\Phi_0(t)}{\mathrm{d}t}\Bigg]\delta(x);$$

$\delta(x)$ is the delta function.

The solution of inhomogeneous equation (3) is determined in terms of the Green function in the form

$$\Phi(x,t)=-\int_0^x\mathrm{d}x'\int_0^t G(x',t')g(x-x',t-t')\mathrm{d}t'$$

$$=-\int_0^t\Bigg\{A_1\Phi_0(t-t')\Bigg[B_1\frac{\partial G(x,t')}{\partial x}+B_0\frac{\partial G(x,t')}{\partial t'}$$

$$+B_2G(x,t')\Bigg]+A_3B_1\Pi_0\Psi_0(t-t')G(x,t')\Bigg\}\mathrm{d}t', \tag{4}$$

where the Green function satisfies the equation with homogeneous boundary conditions

$$A_1B_1\frac{\partial^2G}{\partial x^2}+(A_1B_0+B_1A_0)\frac{\partial^2G}{\partial x\,\partial t}+A_0B_0\frac{\partial^2G}{\partial t^2}$$

$$+(A_2B_0+B_2A_0)\frac{\partial G}{\partial t}+(A_2B_1+A_1B_2)\frac{\partial G}{\partial x}$$

$$+\left(A_2B_2-A_3B_3|\Pi_0|^2\right)G=-\delta(x)\delta(t). \tag{5}$$

To find the Green function, we will apply to (5) the two-dimensional Laplace transform ($t\to\omega$, $x\to p$). As a result, we obtain the Laplace transform of the Green function

$$\Gamma(\omega,p)=-\big[(A_1p+A_0\omega+A_2)$$

$$\times(B_1p+B_0\omega+B_2)-A_3B_3|\Pi_0|^2\big]^{-1}. \tag{6}$$

The inverse Laplace transform applied to expression (6) (using tables from [14]) gives the required Green function for system (1):

$$G(x,t)=\frac{\exp[Z_1(x,t)]}{A_1B_0-B_1A_0}J_0[Z_2(x,t)][\eta(\tau_2)-\eta(\tau_1)], \tag{7}$$

where $\tau_1=t-xA_0/A_1$; $\tau_2=t-xB_0/B_1$; $Z_1(x,t)=(A_2B_1\tau_2-A_1B_2\tau_1)/(A_1B_0-B_1A_0)$; $Z_2(x,t)=2(A_1B_1A_3B_3|\Pi_0|^2\tau_1\times\tau_2)^{1/2}(A_1B_0-B_1A_0)^{-1}$; J_0 is the zero-order Bessel function of the first kind; and η is the Heaviside function.

Expression (7) defines more accurately the Green function and generalises it to a broad class of three-wave coupling problems described by the system of equations (1). This function was first presented in Ref. [12], without any rigorous mathematical substantiation, for SBS (a particular case of such problems). Note that the form of the Green function (7) remains the same under other boundary conditions that differ from condition (4). Only the form of the 'source' $g(x,t)$ will change in inhomogeneous equation (3) and the integrand in (4).

As an example of the application of the Green function, consider the solutions of the system of equations describing nonstationary SBS. This system, in the approximation of the specified pump-wave field E_0, is similar to system (1) [7]:

$$\frac{\partial E_1}{\partial t}+c\varphi_1\frac{\partial E_1}{\partial x}+c\alpha_1E_1=\mathrm{i}AP^*E_0,$$

$$\frac{\partial P^*}{\partial t}+u_\mathrm{s}\varphi_2\frac{\partial P^*}{\partial x}+u_\mathrm{s}\alpha_2P^*=-\mathrm{i}BE_1E_0^*, \tag{8}$$

where $\varphi_1=\boldsymbol{k}_1\boldsymbol{k}_0/(k_1k_0)$; $\varphi_2=\boldsymbol{k}_2\boldsymbol{k}_0/(k_2k_0)$; E_1 and P^* are the complex amplitudes of a scattered electromagnetic wave (with frequency ω_1 and wave vector $\boldsymbol{k}_1$) and of an acoustic wave ($\omega_2=\omega_0-\omega_1$, $\boldsymbol{k}_2=\boldsymbol{k}_0-\boldsymbol{k}_1$); α_1, α_2, c, u_s are the absorption coefficients and velocities of the corresponding waves; $A=0.25Y\beta_\mathrm{s}\omega_1/n^2$; $B=\omega_2Y/16\pi$; Y is the nonlinearity parameter of the electrostriction coupling; β_s is the adiabatic compressibility coefficient; and n is the refractive index.

Let us supplement system (8) with boundary conditions $E_1(0,t)=E_{10}(t)$, $P^*(0,t)=P_0(t)$, $E_1(x,0)=P^*(x,0)=0$. Expressions (4) and (7) allows one to write immediately the dependences $E_1(x,t)$ and $P^*(x,t)$ in the form

$$E_1(x,t)=\frac{1}{2}\int_0^t E_{10}(t-t')\exp(Z_1')I_1(Z_2')\frac{Z_2'}{\tau_1'}$$

$$\times\left[\eta(\tau_1')-\eta(\tau_2')\right]\mathrm{d}t'+E_{10}(\tau_1)\exp\left(-\frac{\alpha_1x}{\varphi_1}\right)\eta(\tau_1) \tag{9a}$$

$$+\frac{\mathrm{i}AE_0u_\mathrm{s}\varphi_2}{c\varphi_1-u_\mathrm{s}\varphi_2}\int_0^t P_0(t-t')\exp(Z_1')I_0(Z_2')\left[\eta(\tau_1')-\eta(\tau_2')\right]\mathrm{d}t',$$

$$P^*(x,t)=-\frac{\mathrm{i}BE_0^*c\varphi_1}{c\varphi_1-u_\mathrm{s}\varphi_2}\int_0^t E_{10}(t-t')\exp(Z_1')I_0(Z_2')$$

$$\times\left[\eta(\tau_1')-\eta(\tau_2')\right]\mathrm{d}t'+P_0(\tau_2)\exp\left(-\frac{\alpha_2x}{\varphi_2}\right)\eta(\tau_2) \tag{9b}$$

$$+\frac{1}{2}\int_0^t P_0(t-t')\exp(Z_1')I_1(Z_2')\frac{Z_2'}{\tau_2'}\left[\eta(\tau_1')-\eta(\tau_2')\right]\mathrm{d}t',$$

where I_ν is the modified ν-order Bessel function of the first kind;

$$Z_1(x,t)=-\frac{\alpha_2u_\mathrm{s}c\varphi_1\tau_1}{c\varphi_1-u_\mathrm{s}\varphi_2}+\frac{\alpha_1u_\mathrm{s}c\varphi_2\tau_2}{c\varphi_1-u_\mathrm{s}\varphi_2};$$

$$Z_2(x,t)=\frac{2(-AB|E_0|^2u_{\rm s}c\varphi_1\varphi_2\tau_1\tau_2)^{1/2}}{c\varphi_1-u_{\rm s}\varphi_2};$$

$$\tau_1=t-\frac{x}{c\varphi_1};\ \tau_2=t-\frac{x}{u_{\rm s}\varphi_2};\ Z'_{1,2}(x,t)=Z_{1,2}(x,t');$$

$$\tau'_{1,2}(x,t)=\tau_{1,2}(x,t').$$

The misprints committed in expressions for amplitudes $E_1(x,t)$ and $P^*(x,t)$ in paper [12] are eliminated in expressions (9).

Using (9), we can determine the time $t_{\rm s}$ of the establishment of stationary SBS in the form

$$t_{\rm s}=\frac{x}{c\varphi_1}+\frac{x(c\varphi_1-u_{\rm s}\varphi_2)}{2c\varphi_1u_{\rm s}\varphi_2}\left\{1-\left(\frac{\alpha_2}{\varphi_2}-\frac{\alpha_1}{\varphi_1}\right)(c\varphi_1u_{\rm s}\varphi_2)^{1/2}\right.$$
$$\left.\times\left[c\varphi_1u_{\rm s}\varphi_2\left(\frac{\alpha_2}{\varphi_2}-\frac{\alpha_1}{\varphi_1}\right)^2+4AB|E_0|^2\right]^{-1/2}\right\}.$$

Here, $t_{\rm s}$ corresponds to the maximum of the exponent of exponentials in integrals in (9); note that $I_\nu(z)\sim\exp z(2\pi z)^{-1/2}$ for $z\gg 1$. When the duration of the pump electromagnetic radiation pulse $t<t_{\rm s}$, the amplification coefficient υ of the scattered wave in the nonstationary regime is twice the exponent in integrals in (9), i.e., for $Z_2\gg 1$, we have $\upsilon=2[Z_1(x,t)+Z_2(x,t)]$. For $t>t_{\rm s}$, the stationary SBS regime is established with the known amplification coefficient [7]

$$\upsilon_{\rm s}=\upsilon(t_{\rm s})=-\left(\frac{\alpha_2}{\varphi_2}+\frac{\alpha_1}{\varphi_1}\right)x$$
$$+\left[\left(\frac{\alpha_2}{\varphi_2}-\frac{\alpha_1}{\varphi_1}\right)^2+\frac{4AB|E_0|^2}{u_{\rm s}c\varphi_1\varphi_2}\right]^{1/2}x.$$

Any three-wave interactions described by system (1) can be considered similarly using the Green function. Of course, exact integral solutions written in terms of the Green function and in the Riemann form should coincide with each other. Expression (9a) can be compared, for example, with expression (2.7) from paper [11], which was obtained for $\alpha_1=\alpha_2=0$. In Ref. [11], the parametric interaction of wave packets in a specified field of a plane monochromatic wave was considered. The amplitudes of the waves are written in the Riemann form using the average characteristic $\eta_{12}=(\eta_1+\eta_2)/2$, where η, η_2 are the characteristics of the interacting waves. The integration was performed over the variable η_{12}/v_{12}, where v_{12} is the mismatch of reverse group velocities of the wave packets.

Expression (9a) (for $\alpha_1=\alpha_2=0$) and relation (2.7) from Ref. [11] can be reduced to the same form accurate to notation. The difference is as follows. First, the second term in the integrand in (9a) contains the factor i and, second, division by τ'_1 rather than by τ'_2 is performed in the first integral in (9a). These differences are not important and are possibly caused by some inaccuracies or misprints in [11]. The use of the Green function in such problems is more convenient than the Riemann method because there is no need to pass from variables x, t to the characteristics of the interacting waves, to redefine the boundary conditions for these characteristics, and to eliminate dissipative terms using special substitutions. The integral solutions written in terms of the Green function can be readily generalised to any boundary conditions.

3. Three-wave coupling in the specified field of a dissipating pump wave

In dissipative media with large decrements α_0 at the pump-wave frequency, the specified field approximation $\Pi=\Pi_0$ proves to be inadequate to the three-wave coupling conditions even for small amplitudes of the scattered wave (Φ, $\Psi^*\ll\Pi_0$). In this case, the specified amplitude approximation for a dissipating pump wave is less crude [15–19]:

$$\Pi=\Pi_0\exp(-\alpha_0x). \tag{10}$$

Below, we will find, using the Green function (7), the solution of system (1) for the coefficient $B_1=0$ and the pump-wave amplitude (10). The approximation $B_1=0$ is used, for example, for the description of scattering of electromagnetic (STS) and sound (STSS) waves by temperature waves [15–20], scattering of acoustic waves by vortex perturbations [21], SRS [22], as well as SBS and stimulated enthalpy scattering at large acoustic decrements [10, 23].

Now, the initial system of equations has the form

$$\begin{aligned}&A_0\frac{\partial\Phi}{\partial t}+A_1\frac{\partial\Phi}{\partial x}+A_2\Phi=A_3\Psi^*\Pi_0\exp(-\alpha_0x),\\&B_0\frac{\partial\Psi^*}{\partial t}+B_2\Psi^*=B_3\Phi\Pi_0^*\exp(-\alpha_0x).\end{aligned}\tag{11}$$

Using boundary conditions (2), we introduce new functions

$$\bar\Phi=\Phi\exp\left(\frac{A_2x}{A_1}\right),\quad\bar\Psi=\Psi^*\exp\left(\alpha_0x+\frac{A_2x}{A_1}\right)\tag{12}$$

and new variables

$$y=t-\frac{A_0x}{A_1},\quad\xi=\frac{1-\exp(-2\alpha_0x)}{2\alpha_0}.\tag{13}$$

Using expressions (12) and (13), system (11) can be written in the form

$$\begin{aligned}&A_1\frac{\partial\bar\Phi}{\partial\xi}=A_3\bar\Psi\Pi_0,\\&B_0\frac{\partial\bar\Psi}{\partial y}+B_2\bar\Psi=B_3\bar\Phi\Pi_0^*.\end{aligned}\tag{14}$$

Solutions (14) follow from expressions (4) and (7) (in which coefficients A_0, A_2 and B_1 should be set to zero):

$$\bar\Phi(\xi,y)=-A_1\int_0^y\Phi_0(y-y')$$
$$\times\left[B_0\frac{\partial G(\xi,y')}{\partial y'}+B_2G(\xi,y')\right]{\rm d}y',\tag{15}$$

$$\bar\Psi(\xi,y)=\frac{A_1}{A_3\Pi_0}\frac{\partial\bar\Phi(\xi,y)}{\partial\xi}.\tag{16}$$

Here,

$$G(\xi, y) = -\frac{\exp(-B_2 y/B_0)}{A_1 B_0} J_0[Z(\xi, y)]\eta(y),$$
$$Z(\xi, y) = 2\left(-\frac{A_3 B_3 |\Pi_0|^2 y\xi}{A_1 B_0}\right)^{1/2}. \quad (17)$$

Returning to variables x, t in expressions (15)–(17), we obtain the required expressions for the amplitudes of the interacting waves:

$$\Phi(x,t) = -A_1 \exp\left(-\frac{A_2 x}{A_1}\right)\int_0^t \Phi_0(t-t')$$
$$\times\left[B_0 \frac{\partial G(x,t')}{\partial t'} + B_2 G(x,t')\right]\mathrm{d}t',$$
$$\Psi(x,t) = \frac{\exp(\alpha_0 x)}{A_3 \Pi_0}\left(A_0 \frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x} + A_2\right)\Phi(x,t), \quad (18)$$

where

$$G(x,t) = -\frac{\exp[-B_2(t - xA_0/A_1)]}{A_1 B_0} J_0[Z(x,t)]\eta\left(t - \frac{xA_0}{A_1}\right);$$

$$Z(x,t) = \left[-2A_3 B_3 |\Pi_0|^2 (t - xA_0/A_1)\frac{1-\exp(-2\alpha_0 x)}{A_1 B_0 \alpha_0}\right]^{1/2}.$$

These results can be readily generalised to any boundary conditions. For example, if the boundary condition for the wave Φ is specified at the boundary $x = L$, then x in expressions (12), (13) should be simply replaced by $x - L$. In this case, system (14) retains its form, but coefficients A_3 and B_3 are multiplied by $\exp(-\alpha_0 L)$.

Let us use the expressions obtained above for describing, for example, nonstationary STS. The system of equations describing STS (by neglecting electrocaloric effect) in the approximation of the specified dissipating pump amplitude has the form [20]

$$\frac{\partial E_1}{\partial t} + c\varphi_1 \frac{\partial E_1}{\partial x} + c\alpha_1 E_1 = A_T E_0 \exp(-\alpha_0 x) T_1^*,$$
$$\frac{\partial T_1^*}{\partial t} + \alpha_2 T_1^* = B_T E_0^* \exp(-\alpha_0 x) E_1, \quad (19)$$

where E_0 and E_1 are the amplitudes of the pump and scattered electromagnetic waves with frequencies ω_0, ω_1 and wave vectors $\boldsymbol{k}_0, \boldsymbol{k}_1$; T_1 is the amplitude of the temperature wave with the frequency $\omega_2 = \omega_0 - \omega_1$ and the wave vector $\boldsymbol{k}_2 = \boldsymbol{k}_0 - \boldsymbol{k}_1$ (for Stokes scattering $\omega_2 > 0$ and $\omega_2 < 0$ for anti-Stokes scattering); α_0, α_1 are decrements of the electromagnetic waves; $\alpha_2 = \mathrm{i}\omega_2 + \chi k_2^2$; χ is the thermal diffusivity; $A_T = \mathrm{i}\omega_1(\partial\varepsilon/\partial T)_p/(4n^2)$; $B_T = \alpha_1 n^2 (2\pi\rho_0 c_p)^{-1}$; ε is the dielectric constant; c_p is the specific heat capacity at constant pressure p; and ρ and T are the density and temperature of the medium, respectively.

For the boundary conditions (2), which are valid for $\theta < \pi/2$ (θ is the scattering angle), we obtain from (18)

$$E_1(x,t) = -\frac{1}{2}\int_0^t E_{10}(t-t')\frac{Z_T'}{\tau'} J_1(Z_T')\exp\left(-\alpha_2\tau' - \frac{\alpha_1 x}{\varphi_1}\right)$$
$$\times\eta(\tau')\mathrm{d}t' + E_{10}(\tau)\exp\left(-\frac{\alpha_1 x}{\varphi_1}\right)\eta(\tau), \quad (20)$$

where J_1 is the first-order Bessel function of the first kind;

$$Z_T = \left[-2A_T B_T |E_0|^2 \tau \frac{1-\exp(-2\alpha_0 x)}{\alpha_0 c}\right]^{1/2}; \quad \tau = t - \frac{x}{c\varphi_1};$$

$$\tau' = t' - \frac{x}{c\varphi_1}; \; Z_T' = Z_T(x, t'); \; E_1(0,t) = E_{10}(t).$$

This dependence coincides with that presented, for example, in Ref. [20] for $\alpha_0 = 0$, $E_{10} = \mathrm{const}$, and by neglecting the time derivative and the dissipative term $\sim\alpha_1$ in the first equation in (19).

Expression (20) allows one to determine the amplification coefficient of the scattered wave in the nonstationary regime (for $Z_T \gg 1$):

$$\upsilon_T = 2\mathrm{Re}\left\{-\alpha_2\tau - \frac{\alpha_1 x}{\varphi_1} + \mathrm{i}Z_T\,\mathrm{sign}\left[B_T\left(\frac{\partial\varepsilon}{\partial T}\right)_p\right]\right\}, \quad (21)$$

where $\mathrm{sing}\, a = +1$ or -1 for $a > 0$, or $a < 0$ respectively.

The time of the establishment of stationary STS corresponds to the maximum of the integrand in expression (20) and is equal (for $\alpha_2 > 0$) to the modulus of the quantity

$$t_{\mathrm{st}} = \frac{x}{c\varphi_1} + \frac{A_T B_T |E_0|^2 [1 - \exp(-2\alpha_0 x)]}{2c\alpha_2^2\alpha_0}. \quad (22)$$

Taking expression (22) into account, the amplification coefficient in the stationary regime ($t > |t_{\mathrm{st}}|$) can be written in the form

$$\upsilon_{\mathrm{st}} = \upsilon_T(t_{\mathrm{st}}) = -2\alpha_1 \frac{x}{\varphi_1}$$
$$+ \frac{\omega_2[1-\exp(-2\alpha_0 x)]|A_T B_T||E_0|^2 \mathrm{sign}[B_T(\partial\varepsilon/\partial T)_p]}{\alpha_0 c(\omega_2^2 + \chi^2 k_2^4)}. \quad (23)$$

In papers [15–19], the system of equations (11) at $A_0 = 0$ and $\Phi_0 = \mathrm{const}$ was transformed, using a special self-similar substitution, to the Bessel equation. Due to this procedure applied to the boundary value problem, the solutions of the system of equations (11) lost their integral form. The wave amplitude $\Phi(x,t)$ in [15–19] virtually coincided with the first term in (18) (after integration at $A_0 = 0$ and $\Phi_0 = \mathrm{const}$). Such a self-similar solution does not allow one to observe the establishment of the stationary regime of scattering. In papers [15–19], the quantity t_{M} (coinciding in fact with $|t_{\mathrm{st}}|$), at which the wave amplitude becomes maximal, is treated as a time during which the nonstationary parametric increment achieves its maximum. Based on this (not integral) solution, the conclusion was made that the nonstationary increment is a nonmonotonic function and tends to zero at $t \gg t_{\mathrm{M}}$. The integral form of the solution of systems (11) and (19) leads to a qualitatively different conclusion: for $t < |t_{\mathrm{st}}|$, the parametric increment increases, and for $t > |t_{\mathrm{st}}|$, the stationary regime is established with the increment υ_{st}. For $\alpha_0 = 0$, expression (22) for the time of the stationary-regime establishment coincides with that presented in Ref. [20].

In the case of backscattering ($\theta > \pi/2$), the replacements $x \to x - L$, A_T, $B_T \to A_T \exp(-\alpha_0 L)$, $B_T \exp(-\alpha_0 L)$ should be performed in expressions (20)–(23). As a result, upon forward scattering of electromagnetic waves ($\theta < \pi/2$), the amplification will take place at $B_T\omega_2(\partial\varepsilon/\partial T)_p > 0$, while upon backscattering, the amplification will occur at $B_T\omega_2(\partial\varepsilon/\partial T)_p < 0$.

Expressions (20)–(23) are valid for $\alpha_2 > 0$ and for any signs of α_0 and α_1 (the case of $\alpha_0 = 0$, $\alpha_2 < 0$ was studied in Ref. [24]), but the coefficient B_T changes in the active (inverted) medium. For example, the authors of Ref. [25] used the effective absorption coefficient $\alpha_{\text{eff}} = |\alpha_1|\omega_0/\Omega$ (where Ω is the transition frequency from the lower laser level to the ground level) instead of α_1 in the expression for the coefficient B_T in the case of fast relaxation of the lower laser level in the inverted carbon dioxide. Theoretically, the situations are possible in nonequilibrium media when $B_T < 0$.

Consider separately the cases of dissipative ($\alpha_0, \alpha_1 > 0$) and active ($\alpha_0, \alpha_1 < 0$) media. In the dissipative medium, $(\partial\varepsilon/\partial T)_p < 0$ for most substances [10, 20], so that the amplification in the case of forward scattering can only occur in the anti-Stokes region ($\omega_2 < 0$). Parametric increments (21) and (23) are linear at $\alpha_0 x \ll 1$. The presence of dissipation with the decrement α_0 restricts the nonlinear interaction length $x \sim 1/\alpha_0$. For $\alpha_1 > 0$, STS is a threshold process. The threshold pump intensity increases, according to (23), by a factor of $2\alpha_0 x/[1 - \exp(-2\alpha_0 x)]$ compared to the case of $\alpha_0 = 0$.

In the active medium at $\alpha_0, \alpha_1 < 0$, STS has no threshold, and along with electromagnetic waves the temperature wave will be also amplified even in the case of weak parametric coupling (when the second term in (23) is small compared to the first one). The amplitude $T_1 \sim \exp(-\alpha_0 x - \alpha_1 x/\varphi_1)$ of the temperature wave will rise as the Stokes and anti-Stokes components of the scattered light. A similar transfer of the increment of an unstable wave to a weak signal wave is sometimes called in the literature the superheterodyne amplification [26, 27].

For large $|E_0|^2$ or large x, the second term in (23) dominates and forward scattering will be at $B_T(\partial\varepsilon/\partial T)_p < 0$, as in a dissipative medium, anti-Stokes, while backscattering will be Stokes. The consideration of the pump-wave amplification ($\alpha_0 < 0$) results in an exponential increase in the increment along the x axis instead of a linear one and, hence, in a faster increase in the amplitudes E_1 and T_1 than in the dissipative medium. This conclusion was made for the first time in Ref. [28] where the stationary parametric amplification of ultrasonic waves in piezoelectric semiconductors was studied. The establishment time (22) of the stationary scattering regime will also increase exponentially with increasing x.

Similar variations in the properties of the parametric interaction in active media should be also expected for the interacting waves of a different nature, for example, upon the interaction of acoustic waves with temperature and vortex waves in media with a negative second (volume) viscosity [29].

4. Conclusions

The Green function and the solutions of systems of equations of the nonstationary three-wave coupling obtained in this paper can be used for investigating many problems, in particular, in the studies of SBS, STS, STSS and similar processes. The consideration of the pump-wave amplification in the active medium leads to an exponential increase in the parametric increment instead of a linear one and to an exponential increase in the time of establishment of the stationary regime.

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