Quantum Electronics **31** (9) 753 – 759 (2001)

PHYSICAL FOUNDATIONS OF QUANTUM ELECTRONICS

PACS numbers: 42.50.Ct; 14.70.Bh
DOI: 10.1070/QE2001v031n09ABEH002040

Vector potential of the electromagnetic field of a photon

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Abstract. **A solution of D'Alembert's equation for the vector potential of an electromagnetic field is found in the form of a wave packet, which does not spread in time and space. The expression obtained for the vector potential of a photon is used for the solution of some problems.**

Keywords: *vector potential, interaction of radiation with matter.*

1. Introduction

In many experimental studies of the interaction of electromagnetic radiation with matter, the radiation fluxes can correspond to such a low photon density $n_{\rm ph} \sim w/Sc\hbar\Omega$ that the vector potential of this radiation, which is required for theoretical calculations of some experimentally measured quantities, cannot be considered classical [1]. Here, w is the power of electromagnetic radiation, S is the cross section of the photon flux, c is the velocity of light in vacuum, and Ω is the photon frequency.

The criterion for the field $\boldsymbol{E}$ to be classical at the characteristic time interval Δt has the form [1]

$$\left(\overline{\boldsymbol{E}^2}\right)^{1/2} \gg \frac{(\hbar c)^{1/2}}{(c\Delta t)^2} \sim \frac{(\hbar c)^{1/2}}{(c/\Omega)^2}.$$

For example, when a helium-neon laser is used ($\hbar\Omega \sim 1$ eV, $w \sim 10^{-3}$ W, $S \sim 0.01$ cm^{-2}), we have $n_{\rm ph} \sim 10^7$ cm^{-3}, which is obviously insufficient for the fulfilment of the above criterion because $(\hbar c)^{1/2}(c/\Omega)^{-2} \sim 0.01$ V m^{-1}. A similar situation takes place in experiments with X rays [2]: for $\hbar\Omega \sim 10^4$ eV and $w/S \sim 10^{-7}$ W cm^{-2}, the value $(\overline{\boldsymbol{E}^2})^{1/2} \sim (\hbar\Omega n_{\rm ph})^{1/2} \sim 10^{-6}$ V m^{-1} is significantly lower than $(\hbar c)^{1/2}(c/\Omega)^{-2} \sim 10^6$ V m^{-1}.

Thus, in a real situation we can deal with the interaction of low-intensity radiation with matter, when we should consider in fact the interaction of one material particle with one photon. In this case, one the one hand, the vector potential of a photon cannot be considered classical, and on the other, a photon cannot be treated as a plane wave localised over the entire space because this results in physically meaningless results for some problems, as shown below.

In this connection it is interesting to consider, at least using a simple model, the problem of calculating the vector potential of the electromagnetic field for one photon. The corresponding result for the photon ensemble will represent a sum of vectors of electromagnetic fields for individual photons.

In the model under study, a material particle is described by a nonrelativistic one-dimensional harmonic oscillator. To solve the formulated problem, we derived the expression for the vector potential $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}, t)$ of a single external photon. This vector potential satisfies D'Alembert's equation and describes the propagation of a localised one-photon wave packet along a straight line. It is natural that the volume integral of the energy density calculated over the entire space with the help of $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}, t)$ should be equal to the photon energy $\hbar\Omega$. It is important to note that $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}, t)$ takes into account the finiteness of the region of photon localisation, whose size cannot be less than the wavelength [3].

The results obtained in this paper can be used in some problems related to the coherence and statistics of photons [4] because they permit the consideration of effects caused by the space–time localisation of photons.

2. Vector potential of the electromagnetic field of one photon

To calculate the vector potential $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}, t)$ of the electromagnetic field of one photon, we find first the expression for the vector potential $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}, t)$ of the electromagnetic field that appears upon scattering of one photon by a material particle.

Consider the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \qquad (1)$$

for a system with the Hamiltonian

$$\hat{H} = \hat{H}_{\rm v} + \hat{H}_{\rm f} + \hat{V}_{\rm f} + \hat{V}_{\rm ph}.$$

Here,

$$\hat{H}_{\rm v} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\tilde{s}^2} + \frac{1}{2}m\tilde{\Omega}^2\tilde{s}^2$$

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Received 5 February 2001
Kvantovaya Elektronika **31** (9) 753 – 759 (2001)
Translated by M N Sapozhnikov

is the Hamiltonian of a one-dimensional nonrelativistic harmonic oscillator; $m, \tilde{s}, \tilde{\Omega}$ are the oscillator mass, coordinate, and frequency, respectively;

$$\hat{H}_{\rm f} = -\frac{\hbar}{2}\sum_{\alpha}\tilde{\omega}_{\alpha}\left(-\frac{\partial^2}{\partial\xi_{\alpha}^2}+\xi_{\alpha}^2\right)$$

is the Hamiltonian of a quantised electromagnetic field, where the variables ξ_{α} can be formally treated as normal coordinates of continuously distributed oscillators representing the quantised electromagnetic field;

$$\hat{V}_{\rm f} = \frac{e}{mc}\frac{\hbar}{\rm i}\,\boldsymbol{e}_{\rm v}\sum_{\alpha}\hat{\boldsymbol{A}}_{\alpha}(\tilde{s})\frac{\partial}{\partial\tilde{s}}$$

is the operator of interaction of the oscillator with the quantised electromagnetic field;

$$\hat{\boldsymbol{A}}_{\alpha}(\tilde{s}) = \left(\frac{2\pi\hbar}{V}\right)^{1/2} c\,\frac{\boldsymbol{e}_{\alpha}}{\tilde{\omega}_{\alpha}^{1/2}}$$

$$\times[\hat{a}_{\alpha}\exp({\rm i}q\boldsymbol{e}_q\boldsymbol{e}_{\rm v}\tilde{s}) + \hat{a}_{\alpha}^{+}\exp(-{\rm i}q\boldsymbol{e}_q\boldsymbol{e}_{\rm v}\tilde{s})];$$

$V(V\to\infty)$ is the volume of the quantised electromagnetic field; e is the electron charge; $\boldsymbol{e}_{\alpha}$ is the unit polarisation vector of the electromagnetic wave, which corresponds to the subscript $\alpha = \{\boldsymbol{q}, \sigma\}$ representing a set of the wave vector $\boldsymbol{q} = \boldsymbol{e}_q q$ and two polarisations $\sigma = 1, 2$ of the transverse electromagnetic field; $\boldsymbol{e}_q$ is the unit vector in the direction of the vector $\boldsymbol{q}$; $\tilde{\omega}_{\alpha}$ is the frequency of the mode α of the quantised electromagnetic field;

$$0 \leqslant \tilde{\omega}_{\alpha} = \tilde{\omega}_q = cq \leqslant cq_{\max} = \tilde{\omega}_{\max}\,;$$

$$\hat{a}_{\alpha} = \frac{1}{\sqrt{2}}\left(\xi_{\alpha}+\frac{\partial}{\partial\xi_{\alpha}}\right); \quad \hat{a}_{\alpha}^{+} = \frac{1}{\sqrt{2}}\left(\xi_{\alpha}-\frac{\partial}{\partial\xi_{\alpha}}\right);$$

and $\boldsymbol{e}_{\rm v}$ is the unit vector along the straight line on which the oscillator moves.

Note that in the problem under study, the frequency $\tilde{\omega}_{\alpha}$ is unlimited [1], which requires the renormalisation of the mass and charge in quantum electrodynamics. In our nonrelativistic problem, it is sufficient to renormalise only the oscillator mass, which is equivalent to the restriction of the frequency spectrum by some frequency $\tilde{\omega}_{\max}$. The operator

$$\hat{V}_{\rm ph} = \frac{e}{mc}\frac{\hbar}{\rm i}\,\boldsymbol{e}_{\rm v}\,\hat{\boldsymbol{A}}_{\rm ph}(t,\tilde{s})\,\frac{\partial}{\partial\tilde{s}}$$

describes the interaction of the oscillator with the electromagnetic field of one external photon. We assume that the operator $\hat{\boldsymbol{A}}_{\rm ph}(t,\tilde{s})$ parametrically depends on the variables characterising some material system in the infinitely removed region of space where the external photon has been produced. Let us average equations (1) over the variables of this system assuming that

$$\overline{\hat{\boldsymbol{A}}_{\rm ph}\boldsymbol{\Psi}} = \overline{\hat{\boldsymbol{A}}_{\rm ph}}\,\overline{\boldsymbol{\Psi}} = \hat{\boldsymbol{A}}_{\rm ph}(t,\tilde{s})\overline{\boldsymbol{\Psi}}.$$

Let us introduce the dimensionless variables

$$s = \frac{\tilde{s}}{(\hbar/m\tilde{\Omega})^{1/2}} \quad \text{and} \quad \omega_{\alpha} = \omega_q = \frac{\tilde{\omega}_{\alpha}}{\tilde{\Omega}},$$

where $0 \leqslant \omega_{\alpha} \leqslant \omega_{\max}/\tilde{\Omega} = \overline{\omega}$. Then, by passing to new variables $\xi_{\alpha} = \xi_{\alpha}\omega_{\alpha}^{-1/2}$ and taking into account that only the region $\omega_{\alpha} \sim 1$ and $s \sim 1$ is important for the following and that the inequality $\hbar\tilde{\Omega}/mc^2 \ll 1$ is satisfied (which corresponds to the long-wavelength approximation), we obtain in the momentum representation for the averaged wave function (by omitting in it the upper bar for simplicity)

$$\Psi_{\{n_{\lambda}\}}(t,\{Q_{\lambda}\}) = \prod_{\lambda}\Phi_{n_{\lambda}}\left[s_{\lambda}^{1/2}(Q_{\lambda}-\eta_{\lambda})\right]$$

$$\times\exp\left\{{\rm i}\left[\frac{1}{\tilde{\Omega}}\frac{\partial\eta_{\lambda}}{\partial t}(Q_{\lambda}-\eta_{\lambda}) - \tilde{\Omega}\int_{-\infty}^{t}L_{\lambda}(\tau){\rm d}\tau\right]\right\}, \qquad (2)$$

where $\Psi_{\{n_{\lambda}\}}[s_{\lambda}^{1/2}(Q_{\lambda}-\eta_{\lambda})]$ is the wave function of the harmonic oscillator

$$\eta_{\lambda}(t) = \frac{\tilde{Q}}{s_{\lambda}}\int_{-\infty}^{t}{\rm d}\tau X_{0\lambda}f(\tau)\sin\left[\tilde{Q}s_{\lambda}(t-\tau)\right]$$

and is determined by the equation [5]

$$\frac{1}{\tilde{\Omega}^2}\frac{\partial^2\eta_{\lambda}}{\partial t^2} + s_{\lambda}^2\eta_{\lambda} = X_{0\lambda}f(t)\,;$$

$$f(t) = \frac{e}{c\left(m\hbar\tilde{\Omega}\right)^{1/2}}\,\boldsymbol{e}_{\rm v}\boldsymbol{A}_{\rm ph}(t);$$

$$L_{\lambda}(t) = \frac{1}{2\tilde{\Omega}^2}\left(\frac{\partial\eta_{\lambda}}{\partial t}\right)^2 - \frac{1}{2}s_{\lambda}^2{\eta_{\lambda}}^2 + X_{0\lambda}f(t)\eta_{\lambda}(t);$$

$\{n_{\lambda}\}$ and $\{Q_{\lambda}\}$ are the sets of quantum numbers and normal coordinates, and [6]

$$Q_{\lambda} = \sum_{\alpha}X_{0\lambda}\zeta_{\alpha} + X_{0\lambda}p,\ \ p = \sum_{\lambda}X_{0\lambda}Q_{\lambda},\ \ \zeta_{\alpha} = \sum_{\lambda}X_{\alpha\lambda}Q_{\lambda}\,;$$

$$X_{0\lambda}^2 = 1\bigg/\frac{{\rm d}G}{{\rm d}z}\bigg|_{z=z_{\lambda}},\quad X_{\alpha\lambda} = \frac{\varepsilon_{\alpha}X_{0\lambda}}{z_{\lambda}-\omega_{\alpha}^2},$$

where $z_{\lambda} = s_{\lambda}^2$ are the roots of the equation

$$G(z) = z - 1 - \frac{\sum_{\alpha}\varepsilon_{\alpha}^2}{z_{\lambda}-\omega_{\alpha}^2} = 0;\ \varepsilon_{\alpha} = 2\left(\frac{\pi}{Vm}\right)^{1/2}\frac{\boldsymbol{e}_{\rm v}\boldsymbol{e}_{\alpha}}{\tilde{\Omega}}.$$

First, we calculate the average value of the space–time operator $\hat{\boldsymbol{A}}_{\alpha}(\boldsymbol{r},t)$ of the vector potential of the electromagnetic field over the quantum states determined by expression (2):

$$\boldsymbol{A}_{\alpha}(\boldsymbol{r},t) = \frac{{\rm Sp}\left[\exp\left(-\hat{H}(t\to-\infty)/T_r\right)\hat{\boldsymbol{A}}_{\alpha}(\boldsymbol{r},t)\right]}{{\rm Sp}\left[\exp\left(-\hat{H}(t\to-\infty)/T_r\right)\right]}. \qquad (3)$$

Here, T_r is the temperature of the system of which the final result is independent due to the features of the model; and

$$\hat{\boldsymbol{A}}_{\alpha}(\boldsymbol{r},t) = \exp\left[{\rm i}\int_{-\infty}^{t}H(\tau){\rm d}\tau\right]\hat{\boldsymbol{A}}_{\alpha}(\boldsymbol{r})\exp\left[-{\rm i}\int_{-\infty}^{t}H(\tau){\rm d}\tau\right],$$

$$\hat{\boldsymbol{A}}_{\alpha}(\boldsymbol{r}) = \left(\frac{2\pi\hbar}{V}\right)^{1/2}c\left[\frac{\boldsymbol{e}_{\alpha}}{\left(\tilde{\Omega}\omega_{\alpha}\right)^{1/2}}\right]$$

$$\times\left[\hat{a}_{\alpha}\exp\left({\rm i}\frac{\tilde{\Omega}}{c}\omega_{\alpha}\boldsymbol{e}_q\boldsymbol{r}\right) + \hat{a}_{\alpha}^{+}\exp\left(-{\rm i}\frac{\tilde{\Omega}}{c}\omega_{\alpha}\boldsymbol{e}_q\boldsymbol{r}\right)\right].$$

The Hamiltonian $\hat{H}(t)$ in expressions for $A_\alpha(\boldsymbol{r}, t)$ and $\hat{A}_\alpha(\boldsymbol{r}, t)$ is written in the occupation number representation:

$$\hat{H}(t) = \tilde{\Omega}\sum_\lambda \left[\hat{a}_\lambda^+\hat{a}_\lambda + \frac{1}{2}s_\lambda + \frac{1}{\sqrt{2}s_\lambda}X_{0\lambda}(\hat{a}_\lambda + \hat{a}_\lambda^+)f(t)\right],$$

$$\hat{a}_\alpha = \sum_\lambda X_{0\lambda}\hat{a}_\lambda\,,$$

where

$$\hat{a}_\alpha = \frac{1}{\sqrt{2}}\left(Q_\lambda + \frac{\partial}{\partial Q_\lambda}\right).$$

We will seek the expression for the vector potential of a one-photon wave packet $\boldsymbol{A}(\boldsymbol{r}, t)$ propagating along the z-axis for t, $|\boldsymbol{r}| = r \to \infty$. The wave packet $\boldsymbol{A}(\boldsymbol{r}, t)$ detected by an observer located on the z-axis at the distance $r_0 \to \infty$ represents a superposition of expressions (3) over all the modes of planes waves α with the weight function, i.e., with the probability density that provides the fulfilment of all the requirements to a one-photon wave packet discussed in Introduction.

As shown in Appendix 1, this weight function has the form

$$P_{\omega_\alpha}(\theta_q, \varphi_q, \lambda; \nu) = \frac{C}{a^2+\lambda^2}\,\mathrm{e}^{-u}u^{\lambda\nu}[\Gamma(1+\lambda\nu)]^{-1}.$$

Here, $\Gamma(\ldots)$ is the gamma function; $C = 2a^2\omega_\alpha\tilde{\Omega}^2\nu\Delta R^2\times(\pi c^2)^{-1}$; a is a parameter describing the variance of the integration variable λ $(0<\lambda<\infty)$; $\nu\,(\nu\to\infty)$ is a parameter; θ_q and φ_q are the polar and azimuthal angles determining the direction of the vector $\boldsymbol{q}$; $u = (\omega_\alpha\tilde{\Omega}/c)^2\times(1-\cos\theta_q)\nu a\Delta R^2$; ΔR is a parameter characterising the size of the localisation region of the one-photon wave packet. The weight function provides the summation of only those modes α of the vector potential (3) for which the directions of the wave vectors $\boldsymbol{q}$ are sufficiently close to the direction of the z-axis along which a photon propagates.

Assuming that $\varpi \gg 1$ and $\boldsymbol{A}(\boldsymbol{r}, t) = 0$ in the region $t - z/c < 0$ to which the electromagnetic field $\boldsymbol{A}(\boldsymbol{r}, t)$ did not come yet, we obtain for the scattered one-photon packet $\boldsymbol{A}(\boldsymbol{r}, t)$

$$\boldsymbol{A}(\boldsymbol{r}, t) = -\frac{3}{\pi^3}\gamma\boldsymbol{u}\sin\theta_{\mathrm{v}}\,\mathrm{kei}\left(\frac{r\theta_r\sqrt{2}}{\Delta R}\right)\left\{\int_0^\infty \mathrm{d}\tau\,\boldsymbol{e}_{\mathrm{v}}\boldsymbol{A}_{\mathrm{ph}}(\tau)\right.$$
$$\times\left\{1-\cos\left[\varpi\tilde{\Omega}\left(t-\frac{z}{c}-\tau\right)\right]\right\}\left[\tilde{\Omega}\left(t-\frac{z}{c}-\tau\right)\right]^{-1}$$
$$+\frac{\pi}{2}\Omega_1\int_0^{t-z/c}\mathrm{d}\tau\,\boldsymbol{e}_{\mathrm{v}}\boldsymbol{A}_{\mathrm{ph}}(\tau)\exp\left[-\frac{\gamma}{2}\left(t-\frac{z}{c}-\tau\right)\right]$$
$$\left.\times\cos\left[\Omega\left(t-\frac{z}{c}-\tau\right)\right]\theta\left(t-\frac{z}{c}\right)\right\}. \quad (4)$$

Here, $\boldsymbol{u} = \boldsymbol{e}_x\cos\varphi_{\mathrm{v}} + \boldsymbol{e}_y\sin\varphi_{\mathrm{v}}$ is the unit polarisation vector of the wave packet $\boldsymbol{A}(\boldsymbol{r}, t)$; $\sin\theta_{\mathrm{v}} = \boldsymbol{e}_{\mathrm{v}}\boldsymbol{u}$; $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are the unit vectors along the axes of the Cartesian coordinate system; θ_{v} and φ_{v} are the polar and azimuthal angles of the vector $\boldsymbol{e}_{\mathrm{v}}$;

$$\Omega_1^2 = 1 - \frac{2\gamma\varpi}{\pi\tilde{\Omega}};\quad \gamma = \frac{2}{3}\frac{e^2\tilde{\Omega}^2}{mc^3};\quad \Omega = \Omega_1\tilde{\Omega};$$

$$\mathrm{kei}(z) = \frac{K_0[z\exp(\mathrm{i}\pi/4)] + K_0[z\exp(-\mathrm{i}\pi/4)]}{2\mathrm{i}};$$

and K_0 is the Basset function. In expression (4), the coordinate $z = r\cos\theta_r \approx r - {}^1/_2 r\theta_r^2$ and $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

Note that expression (4) satisfies all the requirements formulated in Introduction. Indeed, it satisfies D'Alembert's equation with accuracy to the terms of the order

$$\left(\frac{\gamma}{\tilde{\Omega}}\right)^2 \ll 1\,, \quad (5)$$

which are typical for the optical range ($\gamma \sim 10^7 - 10^9$ s^{-1}, $\tilde{\Omega} \sim 10^{15}$ s^{-1}). Expression (4) does not spread in time. Finally, the volume integral of the energy density over the entire space calculated with the help of (4) is equal to the photon energy $\hbar\Omega$:

$$\frac{1}{8\pi}\int_V \mathrm{d}^3r\left[\mathrm{rot}^2\boldsymbol{A}(\boldsymbol{r}, t) + \frac{1}{c^2}\left(\frac{\partial\boldsymbol{A}(\boldsymbol{r}, t)}{\partial t}\right)^2\right] = \hbar\Omega. \quad (6)$$

Let us find the vector potential $\boldsymbol{A}_{\mathrm{ph}}(t)$ using expression (4). We will assume that the same oscillator as in the case considered above is located at the point in space with the radius vector $\boldsymbol{r} = -\boldsymbol{r}_0(r_0 = |\boldsymbol{r}_0| \to \infty)$. We assume that this oscillator emits a photon along the z-axis, which excited the oscillator considered by us, and the latter oscillator emitted the same photon along the z-axis, which corresponds to $r = r_0$ and $\theta_r = 0$.

By replacing $t - z/c$ by the variable t and taking into account that

$$\boldsymbol{A}(\boldsymbol{r}_0, t) = \boldsymbol{A}_{\mathrm{ph}}(t),\quad -\mathrm{kei}(0) = \frac{\pi}{4},$$

we obtain for $A_{\mathrm{ph}}(t) = \boldsymbol{e}_{\mathrm{v}}\boldsymbol{A}_{\mathrm{ph}}(t)$ the equation

$$A_{\mathrm{ph}}(t) = g\gamma\left[\int_0^t \mathrm{d}\tau K(t-\tau)A_{\mathrm{ph}}(t)\right.$$
$$\left.+\int_0^\infty K_1(t-\tau)A_{\mathrm{ph}}(\tau)\mathrm{d}\tau\right],\quad t\geqslant 0\,, \quad (7)$$

where

$$K(t) = \mathrm{e}^{\gamma/2}\cos\Omega t\,;\ K_1(t) = \frac{2}{\pi}\frac{1-\cos(\varpi\tilde{\Omega}t)}{\Omega t};$$
$$g = \frac{3\Omega_1}{8\pi}\sin^2\theta_{\mathrm{v}}\,. \quad (8)$$

It is shown in Appendix 2 that, taking into account the solution of equation (7) and (8), we have

$$\boldsymbol{A}_{\mathrm{ph}}(\boldsymbol{r}, t) = -\left[\frac{\gamma^3(1-g)^3\hbar}{\pi c\Omega}\right]^{1/2}\boldsymbol{u}\,\mathrm{kei}\left(\frac{r\theta_r\sqrt{2}}{\Delta R}\right)$$
$$\times\exp\left[-\frac{\gamma}{2}(1-g)\left(t-\frac{z}{c}\right)\right]\sin\left[\Omega\left(t-\frac{z}{c}-\tau\right)\right], \quad (9)$$

where $t - z/c > 0$; and $z = r\cos\theta_r \approx r - r\theta_r^2/2$. It follows from (9) and the condition that the parameter ΔR in (4) is the same for all three axes of the coordinate system that $\Delta R = 2c/\gamma(1-g)$.

Note that the model considered here and our calculations, including the renormalisation of the initial values of the oscillator mass m' and its frequency Ω' ($m', \Omega' \to \infty$), which is required for the formally infinite spectrum of frequencies ω_α, are intrinsically consistent because we use the nonrelativistic approximation. Indeed, the oscillator mass m' is proportional to its frequency Ω' because the energy quantum of the oscillator is $\hbar\Omega' \sim m'e^4/\hbar^2$ [8, 9] and the coefficient of rigidity is $m'\Omega'^2 \sim e^2/a_0^3$, where $a_0 = \hbar^2/m'e^2$. In this case, the condition of nonrelativity $\hbar\Omega' \sim m'e^4/\hbar^2 \ll m'c^2$ is valid.

Consider the problem where the consideration of the space–time localisation of a photon is important. This is the problem on the calculation of the average occupation number $\overline{\hat{n}(t)}$ for the energy levels of the oscillator in equation (1).

Let the vector potential $\boldsymbol{A}_{\rm ph}(t)$ of an external photon in (1) be determined by expression (9), in which all the parameters of this photon have the subscript zero. Upon substitution of (9) into (1), we assume that $r\theta_r \simeq z \simeq 0$, which corresponds to the placement of the oscillator to the coordinate origin. We will take into account that the operator

$$\hat{n} = \frac{1}{2}\left(-\frac{\partial^2}{\partial s^2} + s^2\right) - \frac{1}{2}$$

and $\overline{\hat{n}(t)}$ is determined by expression (3), in which the operator $\hat{A}_\alpha(r)$ should be replaced by the operator $\hat{n}$. Then, taking into account the wave function (2), we obtain

$$\overline{\hat{n}(t)} = \frac{1}{8}\tilde{\Omega}\left\{\left[\int_0^t \exp\left[-\frac{\gamma}{2}(t-t')\right]\cos\left[\Omega(t-t')\right]f(t')\,\mathrm{d}t'\right]^2 + \left[\int_0^t \exp\left[-\frac{\gamma}{2}(t-t')\right]\sin\left[\Omega(t-t')\right]f(t')\,\mathrm{d}t'\right]^2\right\} + \left[\exp\left(\frac{\hbar\Omega}{T_r}\right) - 1\right]^{-1}.$$

Here, as in (2),

$$f(t) = \frac{e}{c\left(m\hbar\tilde{\Omega}\right)^{1/2}}\,\boldsymbol{e}_{\rm v}\boldsymbol{A}_{\rm ph}(t).$$

After the calculation of integrals over the variable t, we have

$$\overline{\hat{n}(t)} = \frac{3\pi^2}{512}\frac{\gamma\gamma_0^3}{\tilde{\Omega}\Omega_0}\sin^2\theta_{\rm v0}\left\{\left[\exp\left(-\frac{\gamma}{2}t\right)[a(\omega=-\Delta\Omega)\cos\Omega t + b(\omega=-\Delta\gamma)\sin\Omega t] + \exp\left(-\frac{\gamma_0}{2}(1-g_0)t\right)\times[-a(\omega=-\Delta\Omega)\cos\Omega_0 t + b(\omega=\Delta\gamma)\sin\Omega_0 t]\right]^2 + \left[\exp\left(-\frac{\gamma}{2}t\right)[a(\omega=\Delta\Omega)\sin\Omega t - b(\omega=\Delta\gamma)\cos\Omega t] + \exp\left(-\frac{\gamma_0}{2}(1-g_0)t\right)[a(\omega=-\Delta\Omega)\sin\Omega_0 t + b(\omega=\Delta\gamma)\cos\Omega_0 t]\right]^2\right\} + \left[\exp\left(\frac{\hbar\Omega}{T_r}\right) - 1\right]^{-1}.$$

Here,

$$\sin\theta_{\rm v0} = \boldsymbol{e}_{\rm v}\boldsymbol{u}_0;\quad a(\omega) = \frac{\omega}{\Delta\Omega^2 + \Delta\gamma^2} + \frac{\Omega + \Omega_0}{(\Omega+\Omega_0)^2 + \Delta\gamma^2};$$

$$\Delta\Omega = \Omega - \Omega_0';\ \Delta\gamma = \frac{\gamma}{2} - \frac{\gamma_0(1-g_0)}{2};$$

$$b(\omega) = \frac{\omega}{\Delta\Omega^2 + \Delta\gamma^2} + \frac{\Delta\gamma}{(\Omega+\Omega_0)^2 + \Delta\gamma^2}.$$

One can see from the above expression that the function $\overline{\hat{n}(t)}$ depends on time according to the physical process of interaction of the oscillator with the photon, i.e., it first increases by oscillating and then decreases to zero for $t \to \infty$. Note, in particular, that under the resonance conditions, i.e., when $|\Delta\Omega|t$, $|\Delta\gamma t| \ll 1$, the above expression gives

$$\overline{\hat{n}(t)} = \frac{3\pi^2}{512}\sin^2\theta_{\rm v0}\,\frac{\gamma^4}{\tilde{\Omega}\Omega}\,t^2\exp(-\gamma t) + \left[\exp\left(\frac{\hbar\Omega}{T_r}\right) - 1\right]^{-1}.$$

Note that the result obtained for $\overline{\hat{n}(t)}$ cannot be obtained without considering the space–time localisation of a photon. Indeed, the expression for $\overline{\hat{n}(t)}$ will have no physical sense if we substitute into the expression for the function $f(t)$ instead of the vector $\boldsymbol{A}_{\rm ph}(t)$ the expression $\boldsymbol{A}_{\rm ph}(t) = 2c\,(2\pi\hbar/\Omega_0 V)^{1/2} \times\boldsymbol{u}_0\sin\Omega_0 t$, which represents a monochromatic wave localised in the space of volume $V \to \infty$.

3. Vector potential of the electromagnetic field and the energy density for an ensemble of parallel moving photons

Consider the conditions under which the average vector potential of an ensemble of photons propagating along the positive direction of the z-axis of the Cartesian coordinate system transforms into a classical vector potential. For this purpose, we should calculate an observable, namely, the radiation intensity (the energy density) of the photon ensemble, which is proportional to $\boldsymbol{E}^2$.

Taking into account the result (9), the vector potential of the photon ensemble has the form

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{m=1}^{M}\sum_{l_m=1}^{L_m}\boldsymbol{A}_{m\,l_m}\left(t - \frac{z}{c} - t_{l_m}, \boldsymbol{\rho} - \boldsymbol{\rho}_m\right), \tag{10}$$

where

$$\boldsymbol{A}_{m\,l_m}\left(t - \frac{z}{c} - t_{l_m}, \boldsymbol{\rho} - \boldsymbol{\rho}_m\right) = -\boldsymbol{B}_0\,\mathrm{kei}\left(\frac{|\boldsymbol{\rho} - \boldsymbol{\rho}_m|(1-g_0)}{\sqrt{2}c}\right)$$

$$\exp\left[-\left(t - \frac{z}{c} - t_{l_m}\right)\frac{\gamma_0(1-g_0)}{2}\right]\sin\left[\Omega_0\left(t - \frac{z}{c} - t_{l_m}\right)\right];$$

$$t - \frac{z}{c} - t_{l_m} > 0;\ \boldsymbol{B}_0 = \left[\frac{\gamma_0^3(1-g_0)^3\hbar}{\pi c\Omega_0}\right]^{1/2}\boldsymbol{u}_0;$$

$$g_0 = \frac{3}{8\pi}\Omega_{10}\sin^2\theta_{\rm v0};$$

$\boldsymbol{\rho}$ and $\boldsymbol{\rho}_m$ are the radius vectors in the plane perpendicular to the z-axis. For simplicity, we assume that the values of $\gamma_0, g_0, u_0, \Omega_0$ in (10) are the same for all photons in the ensemble. The parameter t_{l_m} determined the instant of the intersection of the xy plane by the l_mth photon at the point m with the radius vector $\boldsymbol{\rho}_m$. The integers M and L_m determine the total number of the points of intersection of the xy plane by photons by the time t and the total number of photons intersecting this plane at the point with the radius vector $\boldsymbol{\rho}_m$, respectively. We assume that z and $\boldsymbol{\rho}$ in (9) are finite and t, M and $L_m \to \infty$.

The problem is reduced to the averaging of the vector potential (10) and

$$\boldsymbol{E}^2 = \left(-\frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t}\right)^2.$$

over the instants $\{t_{l_m}\}$ and radius vectors $\{\boldsymbol{\rho}_m\}$.

We will perform averaging over the instants $\{t_{l_m}\}$ with the Bernoulli distribution density

$$P(t_{l_m}) = \frac{\alpha l_m C_{L_m}^{l_m}}{T}\left(\frac{t_{l_m}}{T}\right)^{\alpha l_m - 1}\left[1 - \left(\frac{t_{l_m}}{T}\right)^{\alpha}\right]^{L_m - l_m}, \qquad (11)$$

where

$$0 < t_{l_m} < T\,;\; T \to \infty\,;\; 0 \leqslant l_m \leqslant L_m\,;\; L_m \to \infty\,;$$

$$C_{L_m}^{l_m} = \frac{L_m!}{(L_m - l_m)!\,l_m!}\,;\quad 0 < \alpha \leqslant 1\,.$$

As will be shown below, the parameter α in (11) determines the degree to which an electromagnetic field is quasi-classical:

$$\alpha = \alpha(n_{\rm ph}) = -\frac{n_0}{2n_{\rm ph}} + \left[\left(\frac{n_0}{2n_{\rm ph}}\right)^2 + \left(\frac{n_0}{n_{\rm ph}}\right)\right]^{1/2}. \qquad (12)$$

Here,

$$n_0 = \left(\frac{\Omega_0}{c}\right)^3\frac{\gamma_0(1 - g_0)}{128\pi^2\Omega_0};$$

$n_{\rm ph} = j/c$ is the photon density in the flux;

$$j = \frac{1}{4\pi}\lim_{L_m, T\to\infty}\frac{L_m}{Ts_0} = \frac{1}{4\pi}\lim_{\{L_m\}, T\to\infty}\sum_{m=1}^{M}\frac{L_m}{TS}$$

is the photon flux density; S is the area through which the photon flux propagates in (10); and

$$s_0 = S\Bigg/\lim_{\{L_m\}, T\to\infty}\sum_{m=1}^{M}\frac{L_m}{T}\Bigg/\lim_{L_m, T\to\infty}\frac{L_m}{T}.$$

Note that the quasi-classical condition for the electromagnetic field corresponds to the condition $\alpha \ll 1$, i.e., $(n_0/n_{\rm ph})^{1/2} \ll 1$. It is obvious that if we set $\Delta t = \sqrt{\pi}\Omega_0^{-1}\times\{128\Omega_0[\gamma_0(1 - g_0)]^{-1/4}\}$ and take into account that $\boldsymbol{E}^2 = \hbar\Omega_0 n_{\rm ph}$, then this condition will coincide with the quasi-classical condition [1].

Using expression (11), we can see that the relation

$$\frac{1}{\nu_0} = \frac{1}{\bar{\nu}\alpha}, \qquad (13)$$

is valid for the average value of the time interval $1/\nu_0$ between two nearest moments of the intersection of the given area s_0 by photons in the xy plane. Here, $\bar{\nu} = \lim_{L_m, T\to\infty} L_m/T = 4\pi c n_{\rm ph} s_0$ and α is determined by expression (12).

Taking into account that the variable $t_{l_m} < t - z/c \to \infty$ in (10) and that, according to (11), $\bar{t}_{L_m} = L_m/\nu_0$ $(L_m \to \infty)$, we can assume for simplicity that $L_m = E[\nu_0(t - z/c)]$ in (10) is independent of m, where $E[x]$ is the integer of the number x. Then, we can assume in (11) that

$$T = \frac{1}{\nu_0}E\left[\nu_0\left(t - \frac{z}{c}\right)\right], \qquad (14)$$

because it is reasonable to set $T = L_m/\nu_0$.

Let us perform averaging of expression (10) over the radius vectors $\boldsymbol{\rho}_m$ in the xy plane with the probability density $1/S$, where $S \to \infty$ is the area through which the total photon flux propagates. Then, we average the obtained expression over the instants t_{l_m} with the probability density (11). Taking into account that $T \to \infty$ and, according to (5), $\Omega_0 \gg \gamma_0$, we obtain

$$\overline{\boldsymbol{A}(\boldsymbol{r}, t)} = \boldsymbol{A}_0 \exp\left[-\frac{\gamma_0}{2}(1 - g_0)\left(t - \frac{z}{c} - T\right)\right]$$
$$\times \cos\left[\Omega_0\left(t - \frac{z}{c} - T\right)\right]\theta\left(t - \frac{z}{c} - T\right), \qquad (15)$$

where

$$T = T\left(t - \frac{z}{c}\right) = \frac{1}{\nu_0}E\left[\nu_0\left(t - \frac{z}{c}\right)\right];\; t - \frac{z}{c} - T > 0;$$

$$\boldsymbol{A}_0 = \frac{32\pi^2\boldsymbol{B}_0 c^3\alpha n_{\rm ph}}{\gamma_0^2(1 - g_0)^2\Omega_0}\,;\quad \alpha = \alpha(n_{\rm ph})\,.$$

We will see below that expression (15) directly determines an observable (the radiation intensity) only in the classical case when

$$\alpha \approx \left(\frac{n_0}{n_{\rm ph}}\right)^{1/2} \ll 1\,. \qquad (16)$$

We consider expression (15) for this case and estimate the parameters entering this expression. We assume that

$$\left(\frac{n_0}{n_{\rm ph}}\right)^{1/2} \sim 0.1,\; \Omega_0 \sim 10^{15}\ {\rm s}^{-1},\; \gamma_0 \sim 10^8\ {\rm s}^{-1},$$

$$n_0 = \left(\frac{\Omega_0}{c}\right)^3\frac{\gamma_0(1 - g_0)}{128\pi^2\Omega_0} \sim 10^4\ {\rm cm}^{-3}.$$

Then, setting $s_0 \sim n_{\rm ph}^{-2/3} \sim 10^{-4}\ {\rm cm}^2$ for the estimate, we obtain $\nu_0 \approx 4\pi s_0(n_0 n_{\rm ph})^{1/2}c \sim 10^{12}\ {\rm s}^{-1}$. Therefore, when the condition (16) is valid, we have

$$\gamma_0 \ll \nu_0 \ll \Omega_0, \qquad (17)$$

In this case, the vector (15) represents the plane wave

$$\overline{\boldsymbol{A}(\boldsymbol{r}, t)} = \boldsymbol{A}_0 \cos\left[\Omega_0\left(t - \frac{z}{c} - T\right)\right], \qquad (18)$$

when $\Delta(t - z/c) \lesssim 1/\nu_0$, where

$$A_0 = \frac{(8\pi\hbar\Omega_0 n_{ph})^{1/2} \boldsymbol{u}_0 c}{\Omega_0}\,; \quad t - \frac{z}{c} - T > 0\,.$$

Assuming that the radius vectors $\boldsymbol{\rho}_m$ in the xy plane are distributed with the probability density $1/S_0$, where $S \to \infty$, we calculate the observable $\overline{\boldsymbol{E}^2}$. Taking into account that

$$\boldsymbol{E} = \sum_{m=1}^{M} \sum_{l_m=1}^{L_m} \boldsymbol{E}_{ml_m}\,,$$

where $\boldsymbol{E}$ and $\boldsymbol{E}_{ml_m}$ are determined using (10), i.e.,

$$\boldsymbol{E}_{ml_m} = -\frac{1}{c}\frac{\partial \boldsymbol{A}_{ml_m}}{\partial t}\,,$$

we obtain

$$\overline{\boldsymbol{E}^2}(t, n_{ph}) = I_1(t, n_{ph}) + I_2(t, n_{ph}) - I_3(t, n_{ph})\,. \qquad (19)$$

Here,

$$I_1(t, n_{ph}) = \lim_{\{L_m\}, T\to\infty} \left(\sum_{m=1}^{M} \sum_{l_m=1}^{L_m} \overline{E}_{ml_m} \right)^2;$$

$$I_2(t, n_{ph}) = \lim_{\{L_m\}, T\to\infty} \sum_{m=1}^{M} \sum_{l_m=1}^{L_m} \overline{E^2_{ml_m}}\,;$$

$$I_3(t, n_{ph}) = \lim_{\{L_m\}, T\to\infty} \sum_{m=1}^{M} \sum_{l_m=1}^{L_m} \overline{E}^2_{ml_m}\,.$$

Taking into account inequalities (5) and expressions (10)–(15), we obtain for I_1, I_2, I_3 averaged over the period $2\pi/\Omega_0$ of a high-frequency vibration

$$I_1(t, n_{ph}) = 2\pi I_0(t, n_{ph}) \alpha n_{ph} a_1\,,$$

$$I_2(t, n_{ph}) = I_0(t, n_{ph}) \left[\frac{\Omega_0^2}{c\gamma_0(1-g_0)} \right] a_2\,,$$

$$I_3(t, n_{ph}) = 0\,,$$

where

$$I_0(t, n_{ph}) = 4\pi \boldsymbol{B}_0^2 \alpha n_{ph} \exp\left[-\gamma_0(1-g_0)\left(t - \frac{z}{c} - T \right) \right];$$

$$a_1 = \left[\frac{8\pi c^2}{\gamma_0^2(1-g_0)^2} \right]^2; \quad a_2 = \frac{\pi c^2}{[\gamma_0(1-g_0)]^2}\,.$$

By substituting the above expressions for I_1, I_2 and I_3 into (19) and taking into account expression (12), we obtain

$$\overline{\boldsymbol{E}^2} = 4\pi\hbar\Omega_0 n_{ph} \left(\frac{\alpha^2 n_{ph}}{n_0} + \alpha \right) \times \exp\left[-\gamma_0(1-g_0)\left(t - \frac{z}{c} - T \right) \right]. \qquad (20)$$

Because it is clear from the physical point of view that the energy density $\overline{\boldsymbol{E}}^2/4\pi$ of the photon field of a plane electromagnetic wave should linearly depend on the concentration n_{ph} of photons both for a weak photon flux ($\alpha \approx 1$) and in a classical case ($\alpha \ll 1$), we should set

$$\frac{\alpha^2 n_{ph}}{n_0} + \alpha = 1 \qquad (21)$$

in (20). From equation (20), we obtain [see (12)]

$$\alpha = -\frac{n_0}{2n_{ph}} + \left[\left(\frac{n_0}{2n_{ph}} \right)^2 + \frac{n_0}{n_{ph}} \right]^{1/2}.$$

Note that $\alpha \approx 1$ for $n_{ph} \ll n_0$ and $\alpha \approx (n_0/n_{ph})^{1/2}$ for $n_{ph} \gg n_0$.

Therefore, the average energy density in the flux of parallel photons is

$$\frac{\overline{\boldsymbol{E}}^2}{4\pi} = \hbar\Omega_0 n_{ph} \exp\left[-\gamma_0(1-g_0)\left(t - \frac{z}{c} - T \right) \right]$$

and it does not depend on time for $\gamma_0 \ll \nu_0$.

Note that the classical electromagnetic field condition (16) means that the term $I_1(t, n_{ph})$ makes the main contribution to expression (20), which is the square of the electric field calculated with the help of the vector potential (15) and averaged over the period of the field oscillation with frequency Ω_0. In essence, this means that the classical electromagnetic field condition corresponds to the smallness of the variance $\overline{(E-\bar{E})^2} \ll \overline{E^2}$.

Appendix 1

Taking into account the uncertainty relation for the momentum and coordinate, we assume that the weight function depends on

$$u = \frac{1}{\hbar^2}\left(\Delta p_x^2 R_x^2 + \Delta p_y^2 R_y^2 + \Delta p_z^2 R_z^2 \right),$$

where

$$\Delta p_x = \frac{1}{c}\,\hbar\omega_\alpha\tilde{\Omega}\sin\theta_q\cos\varphi_q, \;\; \Delta p_y = \frac{1}{c}\,\hbar\omega_\alpha\tilde{\Omega}\sin\theta_q\sin\varphi_q,$$

$$\Delta p_z = \frac{1}{c}\,\hbar\omega_\alpha\tilde{\Omega}(1-\cos\varphi_q)$$

are the projections of the difference $c^{-1}\hbar\omega_\alpha\tilde{\Omega}(\boldsymbol{e}_z - \boldsymbol{e}_q)$ of momenta on the coordinate axes and R_x, R_y, R_z are the projections of the characteristic sizes of the localisation region of the plane wave (3) on the coordinate axes. We assume that $R_x = R_y = R_z \to \infty$. Thus,

$$u = \left(\frac{\omega_\alpha\tilde{\Omega}}{c} \right)^2 (1-\cos\theta_q)\nu a \Delta R^2,$$

where $\nu a\Delta R^2 = R^2$ and the parameter $\nu \to \infty$ because a and ΔR are finite.

The quantity u is in fact the number of states for the mode α, i.e., the number of elementary cells in the phase space whose geometrical part is connected with the part of the spherical sector surface with the angle θ_q. The total number of cells on the sphere surface is

$$N = u(\theta_q = \pi) = 2\left(\frac{\omega_\alpha\tilde{\Omega}}{c} \right)^2 R^2 \to \infty.$$

The probability of finding of $\mu = \lambda\nu + 1$ $(0 < \lambda < \infty)$ cells from N cells on this spherical sector surface is determined by the expression

$$P\left(\xi < \frac{1-\cos\theta_q}{2}\right) = \frac{N!}{(N-\lambda\nu-1)!(\lambda\nu)!} \times \int_0^{(1-\cos\theta_q)/2} \mathrm{d}z z^{\lambda\nu}(1-z)^{N-\lambda\nu-1},$$

where $\xi < (1-\cos\theta_q)/2$ is a random quantity.

Let us differentiate this expression with respect to the angle θ_q and take into account that only the angles $\theta_q \to 0$ are of interest to us and that $N \to \infty$. Note that the number of cells μ (or, which is the same, the number of modes α forming a one-photon wave packet $\boldsymbol{A}(\boldsymbol{r}, t)$) tends to infinity. Let us assume that μ and ν are random quantities whose distribution is described by the normal law. Then, the distribution function of the ratio $\lambda = \mu/\nu$ is determined by the Cauchy formula

$$P(\lambda) = \frac{2a}{\pi}\left(\lambda^2 + a^2\right)^{-1}.$$

The product of the probability densities is $P(\lambda)$, $\mathrm{d}P(\xi < \frac{1}{2}(1-\cos\theta_q))/\mathrm{d}\theta_q$ and determines the weight function $P_{\omega_\alpha}(\theta_q, \varphi_q, \lambda, \nu)$.

Expression (4) can be obtained in the following way. Expression (3) is summed over polarisations $\sigma = 1, 2$. The obtained expression is integrated with the found weight function over angles φ_q and θ_q taking into account the dependence of vectors $\boldsymbol{e}_\alpha$ on angles φ_q and θ_q. These calculations give the expression in which parameters $\mu = \lambda\nu$ and ν enter only in the degenerate hypergeometric function, which has the form

$$_1F_1\left(1+\lambda\nu,\ 1;\ -\frac{r^2\sin^2\theta_r}{2\Delta R^2 a\nu}\right) \approx J_0\left[\left(\frac{\lambda}{a}\right)^{1/2} r\sin\theta_r \frac{1}{\Delta R/\sqrt{2}}\right] \approx J_0\left[\left(\frac{\lambda}{a}\right)^{1/2}\frac{r\theta_r}{\Delta R/\sqrt{2}}\right],$$

because $\nu \to \infty$, where J_0 is the Bessel function. Then, the integration over the parameter λ is performed, summation over the spectrum of frequencies s_λ and, finally, integration over frequencies ω_α.

Appendix 2

Equation (7) can be solved by the method of Wiener–Hopf [9]. However, its approximate solution can be obtained as follows. Let us represent equation (7) in the form

$$\frac{\mathrm{d}^2 A_{\mathrm{ph}}}{\mathrm{d}t^2} + \gamma(1-g)\frac{\mathrm{d}A_{\mathrm{ph}}}{\mathrm{d}t} + \left[\Omega^2 + \frac{\gamma^2}{2}\left(\frac{1}{2} - g\right)\right]A_{\mathrm{ph}}$$

$$= \frac{\mathrm{d}K_1}{\mathrm{d}t}A_{\mathrm{ph}}(+0) + K_1(t)\frac{\mathrm{d}A_{\mathrm{ph}}}{\mathrm{d}t}\bigg|_{t=+0} + \int_0^\infty \mathrm{d}\tau K_1(t-\tau)$$

$$\times\left(\frac{\mathrm{d}^2 A_{\mathrm{ph}}(\tau)}{\mathrm{d}\tau^2} + \gamma\frac{\mathrm{d}A_{\mathrm{ph}}(\tau)}{\mathrm{d}\tau^2} + \Omega^2 A_{\mathrm{ph}}(\tau)\right), \quad t > 0,$$

$$A_{\mathrm{ph}}(+0) = g\gamma\int_0^\infty K_1(-\tau+0)A_{\mathrm{ph}}(\tau)\mathrm{d}\tau,$$

where $A_{\mathrm{ph}}(+0) = A_{\mathrm{ph}}(t = +0)$.

The solution of the obtained differential equation, taking into account the inequality (5) and the initial condition, has the form

$$A_{\mathrm{ph}}(t) = A_{\mathrm{ph}}(+0)\exp\left[-\frac{\gamma}{2}(1-g)t\right] \times\left(\cos\Omega t + \frac{\Omega}{g\gamma}\sin\Omega t\right), \quad t > 0.$$

If the condition $t \gg \gamma/\Omega^2$ is fulfilled, then

$$A_{\mathrm{ph}}(t) = C\exp\left[-\frac{\gamma}{2}(1-g)t\right]\sin\Omega t,$$

where $C = A_{\mathrm{ph}}(+0)\Omega/g\gamma$. Finally, taking into account this expression, we obtain expression (9) for a one-photon wave packet from relation (4) using (5) and (6).

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