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Scattering of an ensemble of photons taking their space – time localisation into account

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***Abstract*. A problem of scattering of an ensemble of photons by material particles is solved. The vector potential of each of the incident photons scattered by particles is described by a nonspreading wave packet. The expressions for cross sections for elastic and inelastic scattering of electromagnetic radiation are derived taking the space – time localisation of photons into account. The possible experiments for verifying these theoretical results are discussed.**

Keywords: *vector potential, scattering of photons, interaction of radiation with matter.*

1. Introduction

The quantities describing the interaction of electromagnetic radiation with matter, for example, the cross section for scattering of a photon by an atom (molecule) are commonly calculated using the perturbation theory in the parameter of interaction of the electromagnetic field with an electron, while the vector potential of the photon is described by a monochromatic wave [1, 2]. It was shown in Ref. [3] that there exist problems that can be solved only by taking the space – time localisation of photons into account, i.e., when the vector potential of photons is described by a wave packet propagating along the z-axis, which has the form

$$\boldsymbol{A}_{\rm ph}(\boldsymbol{r},t) = -[\gamma_0^3(1-g_0)^3\hbar/\pi c\Omega_0]^{1/2}\boldsymbol{u}_0 \mathrm{kei}(r\theta_r\gamma_0(1-g)/\sqrt{2}c)$$

$$\times \exp\left[-\frac{\gamma_0}{2}(1-g_0)(t-z/c)\right]\sin[\Omega_0(t-z/c)]\theta(t-z/c). \quad (1)$$

Here, $\gamma_0 = {}^2/_3 e^2\tilde{\Omega}^2/m_0c^3$ is the radiative decay constant of an oscillator emitting the wave packet (1); e is the electron charge; $\tilde{\Omega}_0$ and m_0 are the oscillator frequency and mass, respectively; c is the velocity of light in vacuum; $g_0 = (3\Omega_{10}/8\pi)\sin^2\theta_{v0}$; $\Omega_1 = (1-2\gamma_0\bar{\omega}_0/\pi\tilde{\Omega}_0)^{1/2}$; $\bar{\omega}_0 = \tilde{\omega}_{\max}/\tilde{\Omega}_0$; $\tilde{\omega}_{\max}$ is the cut-off frequency of the photon spectrum; $\sin\theta_{v0} = \boldsymbol{e}_{v0}\boldsymbol{u}_0$; $\boldsymbol{e}_{v0}$ and $\boldsymbol{u}_0$ are the unit vector of a straight line along which oscillations occur and the photon polarisation vector, respectively; $z = r\cos\theta_r \approx r - r\theta_r^2/2$; kei(...) is the Thomson function; and Ω_0 is the photon frequency. Hereafter, the parameters of a photon incident on a particle (an atom, a molecule) are labelled by '0'. The function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

The problem in which the space – time localisation of photons should be taken into account is the problem of scattering of an ensemble of parallel propagating photons by material particles. We will solve this problem using the following relation for the vector potential of a photon propagating along the z'-axis and appearing upon the interaction between the incident photon and a material particle [3]

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{3\Omega_1}{2\pi^2}\gamma\boldsymbol{u}\sin\theta_v\,\mathrm{kei}\left[r\theta_r'\gamma(1-g)/\sqrt{2}c\right]$$

$$\times\int_0^{t-z'/c}\mathrm{d}\tau\boldsymbol{e}_{\rm v}\boldsymbol{A}_{\rm ph}(\boldsymbol{r}_{\rm c}=\boldsymbol{r}(\boldsymbol{r}_{\rm c}'),\tau)\exp\left[-\frac{\gamma}{2}(t-z'/c-\tau)\right]$$

$$\times\cos[\Omega(t-z'/c-\tau)]\theta(t-z'/c). \quad (2)$$

Here, $\boldsymbol{A}_{\rm ph}(\boldsymbol{r}_{\rm c}=\boldsymbol{r}(\boldsymbol{r}_{\rm c}'),\tau)$ is described by expression (1): Ω_1, γ, $\sin\theta_{\rm v}$, $\boldsymbol{e}_{\rm v}$, $\boldsymbol{u}$, Ω are the same as in (1), however without the subscript '0'. The radius vector $\boldsymbol{r}_{\rm c} = \boldsymbol{r}(\boldsymbol{r}_{\rm c}')$ specifies the position of the oscillator in the primed coordinate system.

In this paper, we refine the expressions for cross sections for elastic and inelastic scattering of electromagnetic radiation taking the space – time localisation of photons into account. The exact criteria for classical scattering of an electromagnetic field by matter are obtained and possible experiments for verifying the results obtained are proposed.

2. Scattering by a harmonic ensemble of photons

Consider an ensemble of photons incident along the z-axis on a medium in the form of a cylinder of radius $R_{\rm s}$ and length $h_{\rm s}$. The vector potential of photons is described by the expression

$$\boldsymbol{A}_0(\boldsymbol{r},t) = \sum_{m=1}^{M}\sum_{l_m}^{L_m}\boldsymbol{A}_{ml_m}(t-z/c-t_{l_m},\boldsymbol{\rho}-\boldsymbol{\rho}_m). \quad (3)$$

Here, the vector $\boldsymbol{A}_{ml_m}(t-z/c-t_{l_m},\boldsymbol{\rho}-\boldsymbol{\rho}_m)$ is described by expression (1), in which the replacement $t \to t - t_{l_m}$, $r\theta_r \to \boldsymbol{\rho}-\boldsymbol{\rho}_m$; should be made, where $\boldsymbol{\rho},\boldsymbol{\rho}_m$ are radius vectors in the plane perpendicular to the z-axis; $\boldsymbol{r} = \boldsymbol{\rho}+\boldsymbol{e}_z z$; and $\boldsymbol{e}_z$ is the unit vector along the z-axis. Note that,

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depending on parameters entering (3) and on the properties of the distribution functions of random quantities in this expression, the wave packets $A_0(\boldsymbol{r},t)$ have coherent properties.

For simplicity, we assume that the parameters of all photons in expression (3) are identical and $\boldsymbol{u}_0=\boldsymbol{e}_x$, where $\boldsymbol{e}_x$ is the unit vector along the x-axis. The quantity t_{l_m} determines the instant of time of intersection of the xy plane by the l_mth photon at the point m with the radius vector $\boldsymbol{\rho}_m$. Integers M and L_m give a total number of the intersection points of photons with the xy plane to the instant of time t and a total number of photons that intersected this plane at the point with the radius vector $\boldsymbol{\rho}_m$, respectively. We assume that z and $\boldsymbol{\rho}$ are finite, the time $t\to\infty$ and, hence, $M, L_m\to\infty$.

Let us assume that the ends of radius vectors $\boldsymbol{\rho}_m$ in expression (3) are distributed over the area S in the xy plane with the probability density $1/S$, where $S\to\infty$. We also assume for simplicity that all material particles are identical and their positions are determined by the radius vectors $\boldsymbol{r}_n$, where the number of a particle is $1\leqslant n\leqslant N$. We assume that the particles differ only in the spatial orientation of the unit vectors along which oscillations occur. We assume also that photons are weakly absorbed by particles. Then, using (1) – (3) and making the replacement $\boldsymbol{r}\to|\boldsymbol{r}-\boldsymbol{r}_n|, \theta_r\to\theta_{nl_m}$, $\boldsymbol{e}_{\rm v}\to\boldsymbol{e}_{{\rm v}n}, \boldsymbol{u}\to\boldsymbol{u}_{nl_m}, \theta_{\rm v}\to\theta_{{\rm v}nl_m}, \gamma(1-g)/\sqrt{2}c\to\gamma\left(1-g_{nl_m}\right)/\sqrt{2}c$ in (2), we obtain the expression

$$\overline{\boldsymbol{A}(\boldsymbol{r},t)}=\sum_{n=1}^{N}\overline{\boldsymbol{A}_n(\boldsymbol{r},t)}. \tag{4}$$

for the vector potential of an ensemble of photons scattered by particles. Here,

$$\boldsymbol{A}_n(\boldsymbol{r},t)=\sum_{m=1}^{M}\sum_{l_m}^{L_m}\boldsymbol{A}_{nml_m}(\boldsymbol{r},t);$$

$$\boldsymbol{A}_{nml_m}(\boldsymbol{r},t)=-\frac{3}{2\pi^2}\Omega_1\gamma\,\boldsymbol{u}_{nl_m}\sin\theta_{{\rm v}nl_m}$$
$$\times\mathrm{kei}\Big(\frac{\gamma}{2c}\left(1-g_{nl_m}\right)|\boldsymbol{r}-\boldsymbol{r}_n|\theta_{nl_m}\Big)$$
$$\times\int_0^{\bar t_n}d\tau\,\boldsymbol{e}_{{\rm v}n}\boldsymbol{A}_{ml_m}(\tau,\boldsymbol{\rho}_n-\boldsymbol{\rho}_m)\exp\Big[-\frac{\gamma}{2}(\bar t_n-\tau)\Big]$$
$$\times\cos[\Omega(\bar t_n-\tau)]\theta(\bar t_n);$$

$\bar t_n=t-|\boldsymbol{r}-\boldsymbol{r}_n|/c+|\boldsymbol{r}-\boldsymbol{r}_n|\theta^2_{nl_m}/2c-z_n/c-t_{l_m}>0$; the bar means averaging over the instants of time t_{l_m} and positions of vectors $\boldsymbol{\rho}_m$, as well as over spatial variables $\boldsymbol{r}_n, \boldsymbol{e}_{{\rm v}n}, \theta_{{\rm v}nl_m}$, etc. We will perform averaging in (4) over the instants of time $\{t_{l_m}\}$ with the distribution density [3]

$$P(t_{l_m})=\frac{\alpha l_m C_{L_m}^{l_m}}{T}\left(\frac{t_{l_m}}{T}\right)^{\alpha l_m-1}\left[1-\left(\frac{t_{l_m}}{T}\right)^{\alpha}\right]^{L_m-l_m},$$

where $0<t_{l_m}<T;\ T\to\infty;\ 0\leqslant l_m\leqslant L_m;\ L_m\to\infty$;

$$C_{L_m}^{l_m}=\frac{L_m!}{(L_m-l_m)!l_m!};$$

$$\alpha=\alpha(n_{\rm ph})=-\frac{n_0}{2n_{\rm ph}}+\left[\left(\frac{n_0}{2n_{\rm ph}}\right)^2+\frac{n_0}{n_{\rm ph}}\right]^{1/2}\ (0<\alpha<1);$$

$$n_0=\frac{(\Omega_0/c)^3\gamma_0(1-g_0)}{128\pi^2\Omega_0};$$

$n_{\rm ph}=j/c$ is the mean volume photon density in the ensemble incident on matter; j is the photon flux density; and l_m is the photon number.

Let us calculate the mean value of the vector potential for an ensemble of photons scattered by particles and then calculate the quantity $\overline{\boldsymbol{E}^2}$, which gives the scattered radiation intensity ($\boldsymbol{E}=-(1/c)(\partial\boldsymbol{A}/\partial t)$).

Note that, because $r\to\infty$ and vectors $\boldsymbol{e}_{knl_m}$ and $\boldsymbol{e}_r$ are substantially different ($\boldsymbol{e}_r$ is the unit vector along the vector $\boldsymbol{r}$), expression (4) vanishes after averaging. The nonzero result is obtained only when $\boldsymbol{e}_{knl_m}\approx\boldsymbol{e}_r$. In this case, we have

$$\boldsymbol{u}_{nl_m}=\frac{\boldsymbol{e}_{{\rm v}n}-\boldsymbol{e}_{knl_m}(\boldsymbol{e}_{knl_m}\boldsymbol{e}_{{\rm v}n})}{\left[1-(\boldsymbol{e}_{knl_m}\boldsymbol{e}_{{\rm v}n})^2\right]^{1/2}}\approx\frac{\boldsymbol{e}_{{\rm v}n}-\boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_{{\rm v}n})}{\left[1-(\boldsymbol{e}_r\boldsymbol{e}_{{\rm v}n})^2\right]^{1/2}},$$
$$\sin\theta_{{\rm v}nl_m}=\boldsymbol{u}_{nl_m}\boldsymbol{e}_{{\rm v}n}\approx\left[1-(\boldsymbol{e}_r\boldsymbol{e}_{{\rm v}n})^2\right]^{1/2}\approx\sin\theta_{{\rm v}n}, \tag{5}$$
$$\sin\theta_{nl_m}\approx\theta_{nl_m}\approx\left[2(1-\boldsymbol{e}_{knl_m}\boldsymbol{e}_r)\right]^{1/2},$$

where $\boldsymbol{e}_{knl_m}$ is the unit vector along the wave vector $\boldsymbol{k}_{nl_m}$, which determines the propagation direction of the l_mth photon scattered by the nth particle.

We assume that a scattering layer of matter represents a solid solution of identical particles, which are randomly distributed in space. The latter circumstance permits the averaging of expression (4) over the directions of the wave vectors of scattered photons with the uniform probability density

$$p(\varphi_{nl_m},\ \theta_{nl_m})=1/4\pi. \tag{6}$$

Let us calculate $\overline{\boldsymbol{A}(\boldsymbol{r},t)}$ assuming that $|\boldsymbol{r}-\boldsymbol{r}_n|\approx r-\boldsymbol{e}_r\boldsymbol{r}_n$. Using the above expression for the distribution density $P(t_{l_m})$ and expressions (5) and (6) and taking into account that the result of averaging over angles θ_{nl_m} in (4) is mainly determined by the angles for which the inequalities $\gamma r\theta^2_{nl_m},\ \gamma r_n\theta^2_{nl_m}\ll 1$ are satisfied, we obtain

$$\overline{\boldsymbol{A}(r,t)}=-\frac{3}{4}|\boldsymbol{A}_0|\frac{\gamma\Omega_0 cn_{\rm s}}{r}\int_0^T{\rm d}t_{l_m}\int_0^{\pi}{\rm d}\theta_{{\rm v}n}\sin\theta_{{\rm v}n}$$
$$\times[\boldsymbol{e}_{{\rm v}n}-\boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_{{\rm v}n})](\boldsymbol{e}_{{\rm v}n}\boldsymbol{e}_x)\int_{V_{\rm s}}{\rm d}^3r_nJ(t_{l_m},r_n,\theta_{{\rm v}n}), \tag{7}$$

where the explicit expression for $J(t_{l_m},r_n,\theta_{{\rm v}n})$ is presented in Appendix 1.

Let us analyse the result obtained. Note first of all that only those values of r are of interest for which

$$r/r_{\rm c},\ r/r_{\rm c0}\ll 1, \tag{8}$$

where $r_{\rm c}=8c\Omega/\gamma^2\approx r_{{\rm c}n}(\omega=\Omega)$ and $r_{\rm c0}=8c\Omega_0/\gamma^2\approx r_{{\rm c}n}(\omega=\Omega_0)$. It is in this range of values, as follows from (7), that the vector $\overline{\boldsymbol{A}(\boldsymbol{r},t)}$ is inversely proportional to r. Note, for example, that for the values $\Omega,\ \Omega_0\sim 10^{15}$ s^{-1}, and $\gamma,\ \gamma_0\sim 10^8$ s^{-1}, parameters $r_{\rm c}$ and $r_{\rm c0}$ are so large that inequalities (8) are valid for all values of r of interest.

The conditions imposed on the radius r, under which the vector $\overline{\boldsymbol{A}(\boldsymbol{r},t)}$ is inversely proportional to r, will be more exact if we take into account the results [3] and the behaviour of the asymptotics of functions $\mathrm{ci}(x) \approx \ln x$ and $\mathrm{si}(x) - \mathrm{si}(-x) \approx 2x$ for $x = r/r_{cn} \ll 1$. These conditions have the form

$$\frac{c}{\gamma} \ll r \ll \frac{r_{\mathrm{c}}}{\ln(r_{\mathrm{c}}/r)}, \qquad \frac{c}{\gamma_0} \ll r \ll \frac{r_{\mathrm{c}0}}{\ln(r_{\mathrm{c}0}/r)}. \tag{9}$$

Note that, within the framework of our model, under the conditions that are reverse to conditions (8), (9), the vector $\overline{\boldsymbol{A}(\boldsymbol{r},t)}$ is inversely proportional to r^2, as follows from (7).

Further, we are interested only in the situation corresponding to conditions (9). Assuming for simplicity that $R_{\mathrm{s}}, h_{\mathrm{s}} \ll c/\Omega$ and $R_{\mathrm{s}}, h_{\mathrm{s}} \ll c/\Omega_0$ and using (7), we obtain

$$\overline{\boldsymbol{E}(r,t)} = \frac{1}{r}\gamma\, cN|E_0|[\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)]\alpha\left(\frac{n_{\mathrm{ph}}}{2n_0}\right)^{1/2}$$
$$\times\left\{\frac{1}{\Omega}\exp\left(-\frac{\gamma}{2}\,\bar t\right)\left[\left[\frac{\Omega_0-\Omega}{(\Omega_0-\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4} + \frac{\Omega_0+\Omega}{(\Omega_0+\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4}\right]\sin\Omega\bar t\right.\right.$$
$$+\left[\frac{[\gamma-\gamma_0(1-g_0)]/2}{(\Omega_0-\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4} - \frac{[\gamma-\gamma_0(1-g_0)]/2}{(\Omega_0+\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4}\right]\cos\Omega\bar t\Bigg] \tag{10}$$
$$-\frac{1}{\Omega_0}\exp\left[-\frac{\gamma_0}{2}(1-g_0)\bar t\right]\left[\left[\frac{\Omega_0-\Omega}{(\Omega_0-\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4} + \frac{\Omega_0+\Omega}{(\Omega_0+\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4}\right]\sin\Omega_0\bar t\right.$$
$$\left.\left.+\left[\frac{[\gamma-\gamma_0(1-g_0)]/2}{(\Omega_0-\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4} + \frac{[\gamma-\gamma_0(1-g_0)]/2}{(\Omega_0+\Omega)^2+[\gamma-\gamma_0(1-g_0)]^2/4}\right]\cos\Omega\bar t\right]\right\},$$

where $|\boldsymbol{E}_0| = (8\pi\hbar\,\Omega_0 n_{\mathrm{ph}})^{1/2}$; $\bar t = t - r/c - T$; $T = (1/v_0)E\times[v_0(t\;-r/c)]$; E is the integer part of a number; and $1/v_0 = {}^1/_4\pi c n_{\mathrm{ph}} s_0\alpha$ is the mean time interval between two nearest instants of intersection by photons of a part of the xy plane of area

$$s_0 = S\Bigg/\lim_{\{L_m\},T\to\infty}\sum_{m=1}^{M}(L_m/T)\Bigg/\lim_{\{L_m\},T\to\infty}(L_m/T).$$

The parameter $1/s_0$ determines the order of the surface density of incident photons in the xy plane.

To obtain the expressions for scattering amplitudes of an electromagnetic wave, we consider expression (10) in a classical situation, i.e., when $\alpha \approx (n_0/n_{\mathrm{ph}})^{1/2} \ll 1$ [3]. In addition, we will assume that the condition

$$\gamma_0 \ll v_0 \ll \Omega_0, \quad \gamma \ll v_0 \ll \Omega \tag{11}$$

is fulfilled.

The term containing the factor $\exp(-\gamma/2\bar t)$ in expression (10) represents a diverging electromagnetic wave caused by inelastic scattering of the incident electromagnetic wave. The term containing the factor $\exp[(-\gamma_0/2)(1-g_0)\bar t]$ represents a diverging electromagnetic wave caused by elastic scattering of the initial electromagnetic wave.

Let the frequency range be determined by inequalities

$$|\gamma - \gamma_0(1-g_0)|\Omega_0 \ll \Omega_0^2 \ll \Omega^2. \tag{12}$$

In this case, taking into account the condition $\alpha \approx (n_0/n_{\mathrm{ph}})^{1/2} \ll 1$ and inequalities (11), we obtain the amplitude of inelastic scattering

$$A_{\mathrm{u}} = \frac{2\sqrt{2}}{3} r_{\mathrm{e}} N(\Omega_0/\Omega)|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|, \tag{13}$$

where $r_{\mathrm{e}} = e^2/mc^2$ is the Thomson radius; m is the electron mass; and $\Omega, \Omega_0 \neq 0$.

For the frequency range under study, the amplitude of elastic scattering is

$$A_{\mathrm{e}} = \frac{2\sqrt{2}}{3} r_{\mathrm{e}} N|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|. \tag{14}$$

When conditions (12) are fulfilled for the model under study, expression (14) is independent of the photon frequency Ω_0 and the oscillator frequency Ω [1, 4]. For the frequency range

$$\Omega_0 \gg \Omega \gg \gamma, \gamma_0, \tag{15}$$

we have

$$A_{\mathrm{u}} = \frac{2\sqrt{2}}{3} r_{\mathrm{e}} N(\Omega/\Omega_0)|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|, \qquad A_{\mathrm{e}} = \frac{2\sqrt{2}}{3} r_{\mathrm{e}} N(\Omega/\Omega_0)^2|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|. \tag{16}$$

Note that scattering amplitudes (16) in the frequency range (15) decrease with increasing photon energy $\hbar\Omega_0$, which correlates to some extent with the behaviour of scattering amplitudes of fast electrons and atoms [5].

Finally, consider the resonance case, when the corresponding results differ from those obtained in Refs [1, 2] assuming that the electromagnetic field of a photon represents a plane infinite wave. The difference consists, in particular, in the fact that the 'resonance' denominators in expressions (10) contain the difference $[\gamma - \gamma_0(1-g_0)]/2$, whereas denominators in the corresponding expressions in Refs [1, 2] contain $[\gamma - \gamma_0(1-g_0)]/2$.

The amplitude (cross section) of resonance scattering, which is determined by the condition

$$|\Omega_0 - \Omega| \ll v_0. \tag{17}$$

differs from the known results [1, 2] more substantially.

Taking into account inequalities (11), from which it follows that the quantities

$$|\Omega_0 - \Omega|\bar t = \{v_0(t-r/c) - E[v_0(t-r/c)]\}|\Omega_0 - \Omega|/v_0,$$

$$\gamma\bar{t} = \{v_0(t - r/c) - E[v_0(t - r/c)]\}\gamma/v_0,$$

$$\gamma_0\bar{t} = \{v_0(t - r/c) - E[v_0(t - r/c)]\}\gamma_0/v_0$$

in (10) are small, and performing in (10) the expansions in these quantities, we obtain

$$\overline{\boldsymbol{E}(\boldsymbol{r}, t)} = \frac{\sqrt{2}}{3}|\boldsymbol{E}_0|\frac{r_e}{r}N[\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)]\Omega\bar{t}\cos\Omega\,\bar{t}. \qquad (18)$$

Because it follows from (16) that the resonance scattering amplitude $A_{res}(t) = (\sqrt{2}/3)r_e N\Omega\,\bar{t}\,[\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)]$ is a periodic function with the period $1/v_0$, we will define A_{res} as

$$A_{res} = \frac{\sqrt{2}}{3}r_e N\Omega\,|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|\langle\bar{t}^2\rangle^{1/2}, \qquad (19)$$

where

$$\langle\bar{t}^2\rangle = v_0\int_0^{1/v_0} t^2\mathrm{d}t = 1/3v_0^2.$$

Finally, using the expressions for α and v_0 presented above and taking into account that $(n_0/n_{ph})^{1/2} \ll 1$, we obtain from (19)

$$A_{res} = \frac{\sqrt{2}}{288}\frac{c}{\Omega}\frac{1}{s_0}\left[\frac{r_e}{(1 - g_0)n_{ph}}\right]^{1/2}|\boldsymbol{e}_x - \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{e}_x)|.$$

Note that the condition reverse to inequality (17) is a criterion for the possibility of frequency separation of electromagnetic radiation into inelastic and elastic scattering.

3. Scattering cross section for a photon ensemble

By using the expression under the averaging bar in (4), we calculate the quantity $\overline{\boldsymbol{E}^2(\boldsymbol{r}, t)}$, which is directly related to the experimentally measured intensity of scattered electromagnetic radiation

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{n,m,l_m}\boldsymbol{E}_{nml_m}(\boldsymbol{r}, t), \quad \boldsymbol{E}_{nml_m}(\boldsymbol{r}, t) = -\frac{1}{c}\frac{\partial\boldsymbol{A}_{nml_m}}{\partial t}.$$

Under the assumptions used in deriving (10) and assuming that $g_n \ll 1$ in (7), we obtain

$$\overline{\boldsymbol{E}^2(\boldsymbol{r}, t)} = I_1(r, t) + I_2(r, t) + I_3(r, t). \qquad (20)$$

Here,

$$I_1(r, t) = \lim_{\{L_m\}, T\to\infty}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{l_m=1}^{L_m}\overline{\boldsymbol{E}_{nml_m}(r, t)}\right)^2 = \left(\overline{\boldsymbol{E}(r, t)}\right)^2,$$

$\overline{\boldsymbol{E}(r, t)}$ is described by the expression (10)

$$I_2(r, t) = \lim_{\{L_m\}, T\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{l_m=1}^{L_m}\overline{\boldsymbol{E}^2_{nml_m}}(r, t);$$

$$I_3(r, t) = \lim_{\{L_m\}, T\to\infty}\sum_{m=1}^{M}\sum_{l_m=1}^{L_m}\left(\left\langle\left\langle\sum_{n=1}^{N}\boldsymbol{E}_{nml_m}\right\rangle_n^2\right\rangle_{ml_m} - \left\langle\sum_{n=1}^{N}\left\langle\boldsymbol{E}_{nml_m}\right\rangle_n^2\right\rangle_{ml_m}\right);$$

angle brackets $\langle\ldots\rangle_n$ mean averaging over all random variables containing the subscript n; angle brackets $\langle\ldots\rangle_{ml_m}$ mean averaging over all random quantities entering the expression for the vector $\boldsymbol{E}_{nml_m}$, which contain only subscripts m, l_m; the quantities I_1, I_2, I_3 are assumed to be averaged over periods of high-frequency oscillations 2Ω, $2\Omega_0$, $\Omega_0 + \Omega$. Because the expressions for I_1, I_2, I_3 are cumbersome they are presented in Appendix 2.

After averaging expressions $\boldsymbol{E}^2/4\pi = \hbar\Omega_0 n_{ph}\exp[-\gamma_0(1 - g_0)(t - z/c - T)]$ [3] and (18) over the oscillation period $1/v_0$, we obtain the intensity of radiation scattered by matter within a unit solid angle ω_s:

$$\overline{\boldsymbol{E}^2}r^2 = \boldsymbol{E}_0^2\sigma'\frac{\{1 - \exp[-\gamma_0(1 - g_0)/v_0]\}}{\gamma_0(1 - g_0)/v_0}. \qquad (21)$$

Here, $\sigma' = \sigma'_1 + \sigma'_2 + \sigma'_3 = \mathrm{d}\sigma/\mathrm{d}\omega_s$ is the cross section for scattering of the photon flux per unit solid angle, and σ'_1, σ'_2, σ'_3 correspond to I_1, I_2, I_3.

For simplicity, we present here expressions for $\sigma'_{1,2,3}$ only for two typical cases determined by the inequalities

$$\gamma, \gamma_0 \ll v_0, \qquad (22)$$

$$\gamma, \gamma_0 \gg v_0. \qquad (23)$$

We assume that the inequality

$$\gamma, \gamma_0, v_0 \ll \Omega_0, \Omega$$

is fulfilled.

Consider first scattering far from the resonance, i.e., we assume that the inequality reverse to (17) is valid. Then,

$$\sigma' = \sigma'_{1u} + \sigma'_{2u} + \sigma'_{3u} + \sigma'_{1e} + \sigma'_{2e} + \sigma'_{3e}, \qquad (24)$$

where $\sigma'_{1u,2u,3u}$ are components of the cross section for inelastic scattering (secondary emission) per unit solid angle and $\sigma'_{1e,2e,3e,}$ are components of the cross section for elastic scattering per unit solid angle.

Let conditions (12) and (22) be fulfilled and

$$|\Omega - \Omega_0| \gg \gamma, \gamma_0, v_0. \qquad (25)$$

In this case, we have for terms in (24)

$$\sigma'_{1u} = \frac{4}{9}r_e^2 N^2(\Omega_0/\Omega)^2[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2]\alpha^2 n_{ph}/n_0,$$

$$\sigma'_{2u} = \frac{2}{5}N\left(c\Omega_0/\Omega^2\right)^2\left[1 - \frac{2}{3}(\boldsymbol{e}_r\boldsymbol{e}_x)^2\right]\alpha m(1 - g_0)/m_0,$$

$$\sigma'_{3u} = \frac{4}{9}N(N - 1)r_e(\Omega_0/\Omega)^2[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2]\alpha m(1 - g_0)/m_0, \qquad (26)$$

$$\sigma'_{1e} = \frac{4}{9}N^2 r_e[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2]\alpha^2 n_{ph}/n_0,$$

$$\sigma'_{2e} = \frac{8}{5}N(c\Omega_0/\Omega^2)^2\left[1 - \frac{2}{3}(\boldsymbol{e}_r\boldsymbol{e}_x)^2\right]\alpha,$$

$$\sigma'_{3e} = \frac{4}{9}N(N - 1)r_e^2[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2]\,\alpha.$$

Because $\alpha^2 n_{\rm ph}/n_0 + \alpha = 1$ [3], the quantity $\sigma'_{1\rm e} + \sigma'_{3\rm e} = {}^4/_9 N(N-\alpha) r_{\rm e}^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2]$ does not depend on frequencies Ω, Ω_0 and the photon flux density $j = c n_{\rm ph}$ in the quasi-classical region where $\alpha \ll 1$. If conditions (15), (22), and (25) are fulfilled, then

$$\sigma'_{1\rm u} = \frac{4}{9} N^2 r_{\rm e} (\Omega/\Omega_0)^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha^2 n_{\rm ph}/n_0,$$

$$\sigma'_{2\rm u} = \frac{8}{5} N (c/\Omega_0)^2 \left[1 - \frac{2}{3} (\boldsymbol{e}_r \boldsymbol{e}_x)^2\right] \alpha m (1 - g_0)/m_0,$$

$$\sigma'_{3\rm u} = \frac{4}{9} N(N-1) r_{\rm e}^2 (\Omega/\Omega_0)^4 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha, \tag{27}$$

$$\sigma'_{1\rm e} = \frac{4}{9} N^2 r_{\rm e}^2 (\Omega/\Omega_0)^4 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha^2 n_{\rm ph}/n_0,$$

$$\sigma'_{2\rm e} = \frac{8}{5} N (c/\Omega_0)^2 \left[1 - \frac{2}{3} (\boldsymbol{e}_r \boldsymbol{e}_x)^2\right] \alpha,$$

$$\sigma'_{3\rm e} = \frac{4}{9} N(N-1) r_{\rm e}^2 (\Omega/\Omega_0)^4 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha.$$

Note that $\sigma'_{1\rm e} + \sigma'_{3\rm e} = {}^4/_9 N(N-\alpha) r_{\rm e}^2 (\Omega/\Omega_0)^4 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2]$. Provided conditions (12), (23), and (25) are fulfilled, we should make the substitution

$$\sigma'_{1\rm u} \to \frac{\gamma_0(1-g_0)}{\gamma} \sigma'_{1\rm u},\ \sigma'_{2\rm u} \to \frac{\gamma_0(1-g_0)}{\gamma} \sigma'_{2\rm u},$$
$$\sigma'_{3\rm u} \to \frac{\gamma_0(1-g_0)}{\gamma} \sigma'_{3\rm u},\ \sigma'_{1\rm e} \to \sigma'_{1\rm e},\ \sigma'_{2\rm e} \to \sigma'_{2\rm e},\ \sigma'_{3\rm e} \to \sigma'_{3\rm e} \tag{28}$$

in expression (24). In this case, $\sigma'_{1\rm u,2u,3u}$ and $\sigma'_{1\rm e,2e,3e}$ are described by expressions (26). When conditions (15), (23), and (25) are fulfilled, the substitution in (24) should be analogous to (28), however, $\sigma'_{1\rm u,2u,3u}$ and $\sigma'_{1\rm e,2e,3e}$ are now described by expressions (27).

Finally, consider the resonance case and present the expressions for terms entering the formula for $\sigma' = \sigma'_{\rm res}$ in (21) where

$$\sigma'_{\rm res} = \sigma'_{1\rm res} + \sigma'_{2\rm res} + \sigma'_{3\rm res} \tag{29}$$

and where, obviously, there are no longer the separation into inelastic and elastic scattering and each term in (29) corresponds to the appropriate term in (20).

If the condition (22) is fulfilled and $\Omega_0 \to \Omega$, then

$$\sigma'_{1\rm res} = \frac{1}{6} N^2 (c\gamma/v_0\Omega)^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha^2 n_{\rm ph}/n_0,$$

$$\sigma'_{2\rm res} = \frac{16}{5} N (c/\gamma)^2 \left[1 - \frac{2}{3} (\boldsymbol{e}_r \boldsymbol{e}_x)^2\right] \alpha \Big/ \left[1 + \frac{\gamma_0}{\gamma}(1 - g_0)\right], \tag{30}$$

$$\sigma'_{3\rm res} = 2N(N-1)(c/\Omega)^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha \Big/ \left[1 + \frac{\gamma_0}{\gamma}(1 - g_0)\right].$$

When the condition (23) is fulfilled and $\Omega_0 \to \Omega$, then

$$\sigma'_{1\rm res} = 2N^2 (c/\Omega)^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha^2 (n_{\rm ph}/n_0) \gamma/[\gamma + \gamma_0(1 - g_0)],$$

$$\sigma'_{2\rm res} = \frac{16}{5} N (c/\gamma)^2 \left[1 - \frac{2}{3} (\boldsymbol{e}_r \boldsymbol{e}_x)^2\right] \alpha \big[3 + 25\gamma_0(1 - g_0)/\gamma$$
$$-11(\gamma_0/\gamma)^2 (1-g_0)^2 + (\gamma_0/\gamma)^3 (1-g_0)^3\big]$$
$$\times \gamma_0(1-g_0)/\gamma[1 + \gamma_0(1-g_0)/\gamma]^4, \tag{31}$$

$$\sigma'_{3\rm res} = 2N(N-1)(c/\Omega)^2 [1 - (\boldsymbol{e}_r \boldsymbol{e}_x)^2] \alpha \big[3 + 25\gamma_0(1 - g_0)/\gamma$$
$$-11(\gamma_0/\gamma)^2 (1-g_0)^2 + (\gamma_0/\gamma)^3 (1-g_0)^3\big]$$
$$\times \gamma_0(1-g_0)/\gamma[1 + \gamma_0(1-g_0)/\gamma]^4.$$

Let us find out first of all in what of the cases from (26)–(31) a classical situation can be realised for the scattered electromagnetic field, which corresponds, as was noted in Ref. [3], to the condition $\overline{\boldsymbol{E}^2} - \overline{\boldsymbol{E}}^2 \ll \overline{\boldsymbol{E}}^2$, or, taking (19) into account, to the inequalities $\sigma'_1 \gg \sigma'_2, \sigma'_3$.

Taking into account the conditions of deriving expression (10) and expressions for n_0 and γ [see (1) and (13)], we can represent the inequalities providing classical scattering of the electromagnetic field when expressions (26) and (28) are valid in the form

$$n_{\rm s}(c/\Omega_0)^3, n_{\rm s}(c/\Omega)^3 \gg N \gg (\Omega_0/\Omega)^2 m/8\sqrt{2}\pi m_0 (r_{\rm e}^3 n_{\rm ph})^{1/2},$$
$$n_{\rm s}(c/\Omega_0)^3, n_{\rm s}(c/\Omega)^3 \gg N \gg (\Omega_0/\Omega)^4/8\sqrt{2}\pi (r_{\rm e}^3 n_{\rm ph})^{1/2}, \tag{32}$$

where, according to (12), $\Omega \gg \Omega_0$. The first and second inequalities in (32) provide quasi-classical inelastic and elastic scattering of the electromagnetic field, respectively. It is obvious that, despite the smallness of $r_{\rm e} \sim 10^{-13}$ cm, inequalities (32) can be fulfilled for physically reasonable values of parameters contained in them. For example, assuming that $n_{\rm s} \sim 10^{21} - 10^{22}$ cm^{-3}, $\Omega_0 \sim 10^{15}$ s^{-1}, $\Omega \sim 10^{16}$ s^{-1}, and $m/m_0 \sim 10^{-3}$, we see that inequalities (32) are fulfilled for physically reasonable values $n_{\rm ph} \sim 10^{16} - 10^{18}$ cm^{-3}. The conditions for the electromagnetic field of an ensemble of parallel propagating photons to be classical are substantially less stringent than that for an ensemble of scattered photons [3].

In the cases when formulas (26) and (28) are valid, expression (19) is in general a nonlinear function of the parameter α, which in turn depends nonlinearly on the photon flux density $j = c n_{\rm ph}$. Therefore, the intensity of scattered radiation is a nonlinear function. It is possible that this nonlinear dependence of (21) on j can be observed experimentally. Note that corrections in scattering cross sections concerning the intensity of the incident radiation reflect the influence of statistical properties of an ensemble of incident photons on the scattering process and are not related to collective scattering from oscillating particles.

Under conditions (27), the electromagnetic field is scattered classically when the formal conditions

$$n_{\rm s}(c/\Omega_0)^3, n_{\rm s}(c/\Omega)^3 \gg N \gg (\Omega_0/\Omega)^2 m/8\sqrt{2}\pi m_0 (r_{\rm e}^3 n_{\rm ph})^{1/2},$$
$$n_{\rm s}(c/\Omega_0)^3, n_{\rm s}(c/\Omega)^3 \gg N \gg (\Omega_0/\Omega)^4/8\sqrt{2}\pi (r_{\rm e}^3 n_{\rm ph})^{1/2}, \tag{33}$$

are valid, where, according to (15), $\Omega_0 \gg \Omega$. Because $\Omega_0 \gg \Omega$ and $r_{\rm e} \sim 10^{-13}$, conditions (13) can be satisfied only for a very high volume photon density $n_{\rm ph}$, which cannot be physically attained. This means that the inequality $\sigma'_2 = \sigma'_{2\rm u} + \sigma'_{2\rm e} \gg \sigma'_1 = \sigma'_{1\rm u} + \sigma'_{1\rm e}$, $\sigma'_3 = \sigma'_{3\rm u} + \sigma'_{3\rm e}$ takes place instead of the inequality $\sigma_1' \gg \sigma_2', \sigma_3'$, i.e., classical scattering of the electromagnetic field cannot occur under conditions (27), and when conditions (22) are fulfilled, expression (21) takes the form

$$\overline{\boldsymbol{E}^2} \approx \boldsymbol{E}_0^2(\sigma_{2\mathrm{u}}' + \sigma_{2\mathrm{e}}'). \tag{34}$$

When conditions (23) are fulfilled, the expression of the type (34) takes place.

In the resonance case, when $\Omega_0 \to \Omega$, expressions (30) and (31) are valid, and the formal criterion for classical scattering of the electromagnetic field has the form

$$\begin{aligned} &n_{\mathrm{s}}(c/\Omega_0)^3, n_{\mathrm{s}}(c/\Omega)^3 \gg N \gg (v_0 c/\Omega^2 r_{\mathrm{e}})^2/8\sqrt{2}\pi \\ &\qquad \times[1 + m(1-g_0)/m_0](r_{\mathrm{e}}^3 n_{\mathrm{ph}})^{1/2} \text{ for } v_0 \gg \gamma, \gamma_0, \\ &n_{\mathrm{s}}(c/\Omega_0)^3, n_{\mathrm{s}}(c/\Omega)^3 \gg N \gg m/8\sqrt{2}\pi \\ &\qquad \times m_0(1 + m/m_0)^3(r_{\mathrm{e}}^3 n_{\mathrm{ph}})^{1/2} \text{ for } v_0 \ll \gamma, \gamma_0. \end{aligned} \tag{35}$$

It is obvious that inequalities (35) cannot be fulfilled for physically possible values of n_{ph}. Therefore, the scattered electromagnetic field cannot be classical, i.e., the inequality $\sigma_{2\mathrm{res}}' \gg \sigma_{1\mathrm{res}}', \sigma_{3\mathrm{res}}'$ takes place instead of the inequality $\sigma_1' \gg \sigma_2', \sigma_3'$, which corresponds to the relation

$$\overline{\boldsymbol{E}^2} \approx \boldsymbol{E}_0^2 \sigma_{2\mathrm{res}}'. \tag{36}$$

Here, $\sigma_{2\mathrm{res}}'$ is defined in (30) if inequalities (22) are valid. If inequalities (23) are valid, then $\sigma_{2\mathrm{res}}'$ from (36) is defined in (31). Note that, because of the dependence of α and v_0 on n_{ph}, expressions (35) and (36) are nonlinear functions of $j = cn_{\mathrm{ph}}$.

Therefore, within the framework of our model, the conditions of classical scattering of the electromagnetic field by matter are determined by inequalities $(n_0/n_{\mathrm{ph}})^{1/2} \ll 1$, (12), and (32). As follows from expressions for scattering cross sections (26), (27), (30), and (31), these conditions depend substantially on the parameters $m, m_0, \gamma, \gamma_0, g_0$ determining the spatial localisation of electromagnetic fields of photons.

Note in conclusion that, although material particles were described here by harmonic oscillators, nevertheless our result that the space – time localisation of photons should be taken into account in some problems on the interaction of the electromagnetic radiation with matter is quite general. In this connection, it would be interesting to verify these theoretical results experimentally. Obviously, these should be laser studies of the spectral characteristics of resonantly scattered radiation, which are described by expressions (19), (30), and (31), as well as experiments aimed at the discovery of corrections in the intensity of the incident photon flux in scattering cross sections, which are not related to the nonlinear properties of matter [see expressions (23), (27), (30), and (31)].

Appendix 1

The parameter $J(t_{l_m}, r_n, \theta_{vn}) =$

$$\begin{aligned} &\Big[\Big[I_{\mathrm{c}}\Big(\omega = \Omega, \Gamma = \frac{\gamma}{2}, t_{nl_m}\Big) - I_{\mathrm{c}}\Big(\omega = \Omega_0, \Gamma = \frac{\gamma_0}{2}(1-g_0), t_{nl_m}\Big)\Big] \\ &\qquad \times(\Omega_0 - \Omega) - \Big[I_{\mathrm{s}}\Big(\omega = \Omega, \Gamma = \frac{\gamma}{2}, t_{nl_m}\Big) \\ &\qquad -I_{\mathrm{s}}\Big(\omega = \Omega_0, \Gamma = \frac{\gamma_0}{2}(1-g_0), t_{nl_m}\Big)\Big] \\ &\times\Big[\frac{\gamma}{2} - \frac{\gamma_0}{2}(1-g_0)\Big]\Big] \Big/ \Big[(\Omega_0 - \Omega)^2 + \Big[\frac{\gamma}{2} - \frac{\gamma_0}{2}(1-g_0)\Big]^2\Big] \\ &+\Big[\Big[I_{\mathrm{c}}\Big(\omega = \Omega, \Gamma = \frac{\gamma}{2}, t_{nl_m}\Big) - I_{\mathrm{c}}\Big(\omega = \Omega_0, \Gamma = \frac{\gamma_0}{2}(1-g_0) t_{nl_m}\Big)\Big] \\ &\qquad \times(\Omega_0 + \Omega) + \Big[I_{\mathrm{s}}\Big(\omega = \Omega, \Gamma = \frac{\gamma}{2}, t_{nl_m}\Big) \\ &\qquad + I_{\mathrm{s}}\Big(\omega = \Omega_0, \Gamma = \frac{\gamma_0}{2}(1-g_0), t_{nl_m}\Big)\Big] \\ &\qquad \times\Big[\frac{\gamma}{2} - \frac{\gamma_0}{2}(1-g_0)\Big]\Big] \Big/ \Big[(\Omega_0 - \Omega)^2 - \Big[\frac{\gamma}{2} - \frac{\gamma_0}{2}(1-g_0)\Big]^2\Big]. \end{aligned}$$

Here, $A_0 = 32\pi^2 B_0 c^3 \alpha n_{\mathrm{ph}}/\gamma_0^2(1-g_0)^2\Omega_0$; $B_0 = e_x[\gamma_0^3(1-g_0)^3 \times\hbar/\pi c\Omega_0]^{1/2}$; $n_{\mathrm{s}} = N/V_{\mathrm{s}}$ is the density of material particles in the volume V_s scattering photons; T is determined after all integrations in (7), as in Ref. [3]; and

$$\begin{aligned} I_{\mathrm{c}}(\omega, \Gamma, t_{nl_m}) = &-\frac{1}{\omega}\mathrm{e}^{-\Gamma t_{nl_m}}\Big\{\mathrm{e}^{-r/r_{\mathrm{c}n}}\sin(\omega t_{nl_m}) - \frac{1}{\pi}\cos(\omega t_{nl_m}) \\ &\times[-\sinh(r/r_{\mathrm{c}n})[\mathrm{ci}(-\mathrm{i}r/r_{\mathrm{c}n}) + \mathrm{ci}(\mathrm{i}r/r_{\mathrm{c}n})] + \mathrm{i}\cosh(r/r_{\mathrm{c}n}) \\ &\times[\mathrm{si}(-\mathrm{i}r/r_{\mathrm{c}n}) - \mathrm{si}(\mathrm{i}r/r_{\mathrm{c}n})]]\Big\}; \end{aligned}$$

$$\begin{aligned} I_{\mathrm{s}}(\omega, \Gamma, t_{nl_m}) = &\frac{1}{\omega}\mathrm{e}^{-\Gamma t_{nl_m}}\Big\{\mathrm{e}^{-r/r_{\mathrm{c}n}}\cos(\omega t_{nl_m}) + \frac{1}{\pi}\sin(\omega t_{nl_m}) \\ &\times[-\sinh(r/r_{\mathrm{c}n})[\mathrm{ci}(-\mathrm{i}r/r_{\mathrm{c}n}) + \mathrm{ci}(\mathrm{i}r/r_{\mathrm{c}n})] + \mathrm{i}\cosh(r/r_{\mathrm{c}n}) \\ &\times[\mathrm{si}(-\mathrm{i}r/r_{\mathrm{c}n}) - \mathrm{si}(\mathrm{i}r/r_{\mathrm{c}n})]]\Big\}; \end{aligned}$$

$$t_{nl_m} = t - r/c + \boldsymbol{e}_r\boldsymbol{r}_n/c - \boldsymbol{e}_z\boldsymbol{r}_n - t_{l_m};$$

$$r_{\mathrm{c}n} = r_{\mathrm{c}n}(\omega) = 8c\omega/\gamma^2(1-g_n)^2; \; g_n = \frac{3}{8\pi}\Omega_1 \sin^2\theta_{\mathrm{v}n}.$$

Appendix 2

The parameter

$$\begin{aligned} &I_1(r,t) = N^2\boldsymbol{E}_0^2\gamma^2 c^2\Big[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2\Big](\alpha^2 n_{\mathrm{ph}}/4n_0\Omega_0^2 r^2) \\ &\times\Big[(\Omega_0/\Omega)^2\mathrm{e}^{\gamma\bar{t}}i_{\mathrm{u}}\Big(\omega = -\Delta\Omega, \Gamma\Big) + \mathrm{e}^{-(\gamma-2\Gamma)\bar{t}}i_{\mathrm{e}}(\omega = \Delta\Omega, \Gamma) \\ &\quad -2\big(\Omega_0/\Omega\big)\mathrm{e}^{-(\gamma-\Gamma)\bar{t}}\,i_r(\omega = -\Delta\Omega, \Gamma)\cos(\Delta\Omega\bar{t})\Big]. \end{aligned}$$

Here,

$$i_{\mathrm{u}}(\omega, \Gamma) = \Big[\frac{-\omega}{\omega^2 + \Gamma^2} + \frac{\Omega + \Omega_0}{(\Omega + \Omega_0) + \Gamma^2}\Big]^2 + \frac{\Gamma^2}{(\omega^2 + \Gamma^2)^2};$$

$$\Delta\Omega = \Omega - \Omega_0; \;\; \Gamma = [\gamma - \gamma_0(1-g_0)]/2; \;\; \bar{t} = t - r/c - T;$$

$$i_{\rm e}(\omega, \Gamma) = i_{\rm u}(\omega, \Gamma); \quad i_r(\omega, \Gamma) = 1/(\omega^2 + \Gamma^2).$$

The parameter

$$I_2(r,t) = \frac{2}{5}\boldsymbol{E}_0^2 N\left[1 - \frac{2}{3}(\boldsymbol{e}_r\boldsymbol{e}_x)^2\right](\alpha c^2/r^2)$$
$$\times\Big\{(\Omega/\Omega_0)^2 \mathrm{e}^{-\gamma\bar{t}} i_{\rm u}(\omega = -\Delta\Omega,\ \Gamma)[(\gamma - 2\Gamma)/\gamma]$$
$$+\mathrm{e}^{-(\gamma-2\Gamma)\bar{t}}\, i_{\rm e}(\omega = -\Delta\Omega, \Gamma)$$
$$- 4(\Omega/\Omega_0)^2 \mathrm{e}^{-(\gamma-\Gamma)\bar{t}} i_{\rm r}(\omega = -\Delta\Omega,\ \Gamma)$$
$$\times\left[(\gamma - \Gamma)(\gamma - 2\Gamma)/\left(\Delta\Omega^2 + (\gamma - \Gamma)^2\right)\right](\pi\Omega r/\Delta\Omega r_{\rm c})^{1/2}$$
$$\times\left[\cos(2\Omega r/\Delta\Omega r_{\rm c})\left[\frac{1}{2} - S(2\Omega r/\Delta\Omega r_{\rm c})^{1/2}\right]\right.$$
$$\left.- \sin(2\Omega r/\Delta\Omega r_{\rm c})\left[\frac{1}{2} - C(2\Omega r/\Delta\Omega r_{\rm c})^{1/2}\right]\cos(\Delta\Omega\bar{t})\right]\Big\},$$

where

$$S(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \sin t^2 \mathrm{d}t, \quad C(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \cos t^2 \mathrm{d}t$$

are Fresnel integrals; other designations are as above [see also (7) and (10)].

The parameter

$$I_3(r,t) = N(N-1)\boldsymbol{E}_0^2\gamma^2 c^2\left[1 - (\boldsymbol{e}_r\boldsymbol{e}_x)^2\right](\alpha/4\Omega_0^2 r^2)$$
$$\times\Big[\mathrm{e}^{-\gamma\bar{t}} i_{\rm u}(\omega = -\Delta\Omega, \Gamma) + e^{-(\gamma-2\Gamma)\bar{t}} i_{\rm e}(\omega = -\Delta\Omega,\ \Gamma)$$
$$-\mathrm{e}^{-(\gamma-\Gamma)\bar{t}} i_r(\omega = -\Delta\Omega, \Gamma)$$
$$\times\left[(\gamma - \Gamma)(\gamma - 2\Gamma)/\left(\Delta\Omega^2 + (\gamma - \Gamma)^2\right)\right]\cos(\Delta\Omega\,\bar{t})\Big].$$

The expressions for $I_1(r,t), I_2(r,t), I_3(r,t)$ were obtained by neglecting the terms that are small according to inequalities (11).

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