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Multiple reflection method for electromagnetic waves in layered dielectric structures

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Abstract. **Reflection and transmission of a plane electromagnetic wave propagating in a layered dielectric structure with an arbitrary number of layers of various thicknesses are investigated. For the general case of oblique incidence of the wave on this structure, the reflection and transmission coefficients are calculated for both TE and TM waves using a multiple reflection method. An algorithm to apply the obtained formulas for numerical and analytical calculations is suggested.**

Keywords: *layered dielectric structure, electromagnetic wave, oblique incidence, reflection coefficient.*

1. Introduction

The classical problem of electromagnetic wave propagation in a one-dimensional stack of dielectric layers [1] is still a hot topic due to the permanent interest to optic and electro-optic devices based on multilayered dielectric coatings. The development of laser physics constantly stimulates the search for new multilayered coatings for mirrors and filters of optical resonators. This search is often restricted by the use of different numerical methods where the material properties of layers can be easily varied in a computer program that calculates the change of the reflection and transmission properties of a layered structure.

However, in order to find the regions of parameters where new properties of these structures appear (in order to identify the class of real materials for the layers where these properties exist) and in order to understand physical processes occurring in them, it is still reasonable to use analytical methods. For example, some new possibilities for using the layered structures as ideal reflecting mirrors for all polarisations of incident electromagnetic waves, were obtained in principle in a quite recent, but already well-known article [2], from a more thorough analytical examination of the classical problem about the localisation of forbidden and allowed regions of frequencies in a two-layered periodic structure. Originally, this problem was considered as far back as [3]. In previous articles [4, 5], also using an analytical approach rather than numerical, we obtained some new results about the electromagnetic wave propagation in finite layered periodic structures for the case of normal incidence. In particular, it was shown the possibility of using these structures as an operating medium for light shutters [5].

As for methods which are applicable to an arbitrary layered (not necessary periodic) structure, usually one of many modifications of the so-called transfer matrix method which was initially proposed by Abeles for optic [6] and by Thomson and Haskell for acoustic waves [7, 8] is used. A detailed and modern analysis of matrix approaches was done in a review [9]. Some recursive methods are also widely used, the essence of which are to relate some physical parameters in successive layers. For example, it was shown in papers [10, 11] that recursive expressions relate the reflection coefficients in $(j+1)$ and j layers for the case of normal wave propagation. Using these recursive relations the necessary number of times, the reflection and transmission coefficients of a total layered structure are easily obtained.

The basic idea of matrix and recursive approaches is to derive the system of algebraic equations for the amplitudes of reflected and transmitted waves, using boundary conditions, that means, in the case of electromagnetic waves, the continuity of electric and magnetic fields across the interfaces, i.e. at the points of discontinuity of the refractive index.

The characteristic feature of both matrix and recursive methods is that they become essentially numerical if the number of layers in a stack is more than five (including the semi-infinite media on either side of the layered structure itself). This due to the fact that although analytical expressions for the reflection and transmission coefficients can be obtained in principle, they become so cumbersome that their use makes no practical sense. As a result, the matrix and recursive methods employ directly the numerical values of the layer parameters and wavelength of the wave incident on the stack.

The goal of the present paper is to develop an approximate analytical method for the calculation of the reflection coefficient for an electromagnetic wave incident at an arbitrary angle upon a layered structure (not necessary periodic) with an arbitrary number of layers. The basic idea of the suggested method is to represent the wave reflected (transmitted) from a total stack as a sum of waves multiply reflected from the interfaces between individual layers. For a single layer that is placed between two semi-infinite media the method is a well-known one. It is called the multiple reflection method. The application of this method to an elec-

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tromagnetic wave normally incident upon the layer can be found in Ref. [1] and the application to the propagation of electrons in one-dimensional potential barriers can be found in Ref. [12].

It was already noted that the multiple reflection formalism developed in this paper for electromagnetic waves includes the case of oblique incidence as well. Therefore, two possible polarisations of an incident wave, that are so-called TE and TM modes, are considered. Moreover, the expressions obtained for the reflection coefficients can be used for both wave polarisations. Besides, we suppose that the structure can have physical absorption or amplification, i.e. the dielectric permittivity of each layer is complex in general.

2. Properties of Electromagnetic Waves in Layered Complex Dielectric Media

The geometry of the problem at hand is shown in Fig. 1. A layered dielectric structure of length L is surrounded by two semi-infinite media with a real dielectric permittivity ε_0 and with a complex dielectric permittivity ε_f. Such a structure can be represented as m plane homogeneous layers with complex dielectric permittivities ε_j and thicknesses $d_j = z_j - z_{j-1}$, where $j = 1, 2, \ldots, m$, $z_0 \equiv 0$, $z_m \equiv L$, $d_0 \equiv 0$:

$$\varepsilon(z) = \begin{cases} \varepsilon_0, & z < 0, \\ \varepsilon_1, & 0 < z < z_1, \\ \ldots, & \ldots, \\ \varepsilon_m, & z_{m-1} < z < z_m, \\ \varepsilon_\mathrm{f}, & z > L\,. \end{cases} \tag{1}$$

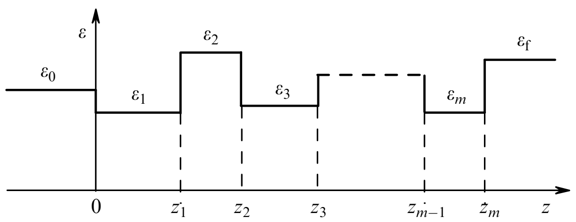

Figure 1. Layered dielectric structure.

We remind the reader that the complex dielectric permittivity means the presence of a real physical absorption in the medium. We suppose that, in each layer $\mu = 1$, i.e. $\boldsymbol{B} = \boldsymbol{H}$. Let a plane monochromatic wave with angular frequency ω and wave number in vacuum k is obliquely incident on a layered structure from the medium ε_0 under an arbitrary angle θ_0 to the interface normal.

According to the Maxwell equations, the coordinates x, y can be selected so that the electromagnetic field inside the dielectric structure (1) depends only on one of them, for example, on the coordinate x. This means that plane monochromatic waves propagating in this structure can have two independent polarisations. In the first case the field $\boldsymbol{E}$ is perpendicular to the plane of propagation xz, i.e. it is directed along the y axis, and the magnetic field $\boldsymbol{H}$ lies in this plane (the case of TE waves). In the second case the electric field $\boldsymbol{E}$ lies in the plane xz, and the field $\boldsymbol{H}$ is directed along the y axis (the case of TM waves). However, there are some general rules for both kinds of waves that follow from the Maxwell equations. For example, in each layer ε_j the dispersion relation between the wave vector $\boldsymbol{k}_j$, which is complex in general, and the wave frequency ω have the form,

$$\boldsymbol{k}_j^2 = \frac{\omega^2}{c^2}\,\varepsilon_j\,,$$

or in the more detailed notation

$$k_j'^2 - k_j''^2 + 2\mathrm{i}\,\boldsymbol{k}_j'\boldsymbol{k}_j'' = \frac{\omega^2}{c^2}\,(\varepsilon_j' + \mathrm{i}\varepsilon_j''), \tag{2}$$

where $\boldsymbol{k}_j'$ and $\boldsymbol{k}_j''$ are real vectors with directions perpendicular to equiphase and equiamplitude planes respectively in a wave; ε_j' and ε_j'' are the real and imaginary parts of the dielectric permittivity ε_j. Then, due to the uniformity of the problem in the plane xy, for each layer the relation

$$k_{0x} = k_{jx}\,, \quad j = 1, 2, \ldots, m\,, \tag{3}$$

is valid, where $\boldsymbol{k}_0$ is the wave vector of the incident wave in the medium ε_0. The relation between the electric and magnetic fields of a plane monochromatic wave, propagating in a nonmagnetic medium, has the form, see for example [13],

$$\boldsymbol{H}_j = \frac{c}{\omega}\,\boldsymbol{k}_j \times \boldsymbol{E}_j\,, \quad \boldsymbol{E}_j = -\frac{c}{\omega\varepsilon_j}\,\boldsymbol{k}_j \times \boldsymbol{H}_j\,, \tag{4}$$

where all three vectors are complex. Note that the relations (2)–(4) are obviously valid for semi-infinite media ε_0, ε_f as well.

One can see from the relation (2), that even for a transparent medium ε_0 $(\varepsilon_0'' = 0)$ $\boldsymbol{k}_0$ can be complex if $\boldsymbol{k}_0' \perp \boldsymbol{k}_0''$. However, such a situation occurs only in the case of a total internal reflection, i.e., it is logical to consider the wave vector of the incident wave $\boldsymbol{k}_0$ as a real vector by definition. Then its modulus and projections on the x- and y axis are determined by the formulas

$$k_0 = \frac{\omega}{c}\,\varepsilon_0^{1/2}, \; k_{0x} = \frac{\omega}{c}\,\varepsilon_0^{1/2}\sin\theta_0, \; k_{0z} = \frac{\omega}{c}\,\varepsilon_0^{1/2}\cos\theta_0\,, \tag{5}$$

where θ_0 is the angle between the wave vector $\boldsymbol{k}_0$ of the incident wave and the z axis. The relation (3) means that the x components of the wave vectors are real in any layer ε_j and in the medium ε_f as well. As a result, $\boldsymbol{k}_j''$ is directed along the z axis in any layer, i.e. absorption in any layer occurs perpendicular to its interfaces. Therefore, the angle θ_j between vectors $\boldsymbol{k}_j'$ and $\boldsymbol{k}_j''$ in any layer j is the angle between the vector $\boldsymbol{k}_j'$ of a refracted (reflected) wave and the z axis, i.e. the ordinary angle of the refraction (reflection). We emphasise that such an angle θ_j is always real. In the case of the total internal reflection at the interface between any two layers it equals $\pi/2$ in the second layer. We should note that for the description of total internal reflection and for the description of absorptive media it is sometimes reasonable to use complex angles of reflection and refraction [14].

From expressions (2), (3), and (5), we can easily obtain the expressions for the projections of a complex vector $\boldsymbol{k}_j$ on the x and z axis

$$\begin{aligned} k_{jx} &= \frac{\omega}{c}\,\varepsilon_0^{1/2}\sin\theta_0, \\ k_{jz} &= \pm\,\frac{\omega}{c}\left[(\varepsilon_j' + \mathrm{i}\varepsilon_j'') - \varepsilon_0\sin^2\theta_0\right]^{1/2}, \end{aligned} \tag{6}$$

where the sign «+» corresponds to the wave propagating under an acute angle (the direction of a vector $\mathbf{k}_j'$) with the z axis. It was already noted that the projection k_{jx} is always real and does not depend on the layer number j. As for the projection k_{jz}, it is complex. Therefore, its expression in (6) is a bit formal as it does not show explicitly the real and imaginary parts. Moreover, in order to pick out these parts we need to find the vectors $\mathbf{k}_j'$ and $\mathbf{k}_j''$ themselves, and the angle θ_j between them as well, a very cumbersome problem [15].

According to the Maxwell equations the electromagnetic fields of TE and TM waves in each layer ε_j ($j = 1, 2, \ldots, m$) can be expressed as

$$\begin{aligned} E_{jy}^{\mathrm{TE}}(x,z,t) &= \{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad + B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)], \\ H_{jx}^{\mathrm{TE}}(x,z,t) &= -\frac{k_{jz}}{k}\{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad - B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)], \\ H_{jz}^{\mathrm{TE}}(x,z,t) &= \frac{k_{jx}}{k}\{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad + B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)] \end{aligned} \tag{7}$$

and

$$\begin{aligned} H_{jy}^{\mathrm{TM}}(x,z,t) &= \{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad + B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)], \\ E_{jx}^{\mathrm{TM}}(x,z,t) &= \frac{k_{jz}}{k\varepsilon_j}\{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad - B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)], \\ E_{jz}^{\mathrm{TM}}(x,z,t) &= -\frac{k_{jx}}{k\varepsilon_j}\{A_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad + B_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)]. \end{aligned} \tag{8}$$

For semi-infinite media ε_0 and ε_{f}, expressions (7) and (8) remain valid if we suppose that $z_{-1} \equiv 0$ and instead of index f, use $m+1$. In addition, as the initial incoming wave goes from medium ε_0, we have $B_{\mathrm{f}} = 0$. After that, without loss of generality we can put $A_0 = 1$. Then, the reflection and transmission coefficients R and T are written as $R = B_0$ and $T = A_{\mathrm{f}}$.

Let us introduce the Fresnel reflection and transmission coefficients for a single inhomogeneous plane wave at the interface between arbitrary layers ε_j and ε_{j+1}. Let us consider first the TE wave. The electric field of this wave in layers ε_j and ε_{j+1} takes the form

$$\begin{aligned} E_{jy}^{\mathrm{TE}}(x,z,t) &= \{a_j \exp[\mathrm{i}k_{jz}(z-z_{j-1})] \\ &\quad + b_j \exp[-\mathrm{i}k_{jz}(z-z_{j-1})]\} \exp[\mathrm{i}(k_{jx}x-\omega t)], \\ E_{j+1,y}^{\mathrm{TE}}(x,z,t) &= \{a_{j+1} \exp[\mathrm{i}k_{j+1,z}(z-z_j)] \exp[\mathrm{i}(k_{jx}x-\omega t)]. \end{aligned} \tag{9}$$

The magnetic field of the wave in layers ε_j and ε_{j+1} can be expressed from (9) using the Maxwell equations. Then, the expressions for the Fresnel reflection and transmission coefficients immediately follow from the boundary conditions:

$$r_{j,j+1}^{\mathrm{TE}} \equiv \frac{E_j^-(z_j-0)}{E_j^+(z_j-0)} = \frac{b_j \exp(-\mathrm{i}k_{jz}d_j)}{a_j \exp(\mathrm{i}k_{jz}d_j)} = \frac{k_{jz}-k_{j+1,z}}{k_{jz}+k_{j+1,z}}, \tag{10}$$

$$t_{j,j+1}^{\mathrm{TE}} \equiv \frac{E_{j+1}^+(z_j+0)}{E_j^+(z_j-0)} = \frac{a_{j+1}}{a_j \exp(\mathrm{i}k_{jz}d_j)} = \frac{2k_{jz}}{k_{jz}+k_{j+1,z}}. \tag{11}$$

For the TM wave the expressions (9) are valid for the magnetic fields ε_j and ε_{j+1} in layers H_{jy} and $H_{j+1,y}$ if we change E for H. The electric fields can be obtained from these expressions using the Maxwell equations. Then, the Fresnel coefficients for the TM wave take the form

$$r_{j,j+1}^{\mathrm{TM}} \equiv \frac{H_j^-(z_j-0)}{H_j^+(z_j-0)} = \frac{b_j \exp(-\mathrm{i}k_{jz}d_j)}{a_j \exp(\mathrm{i}k_{jz}d_j)} = \frac{\varepsilon_{j+1}k_{jz}-\varepsilon_j k_{j+1,z}}{\varepsilon_{j+1}k_{jz}+\varepsilon_j k_{j+1,z}}, \tag{12}$$

$$t_{j,j+1}^{\mathrm{TM}} \equiv \frac{H_{j+1}^+(z_j+0)}{H_j^+(z_j-0)} = \frac{a_{j+1}}{a_j \exp(\mathrm{i}k_{jz}d_j)} = \frac{2\varepsilon_{j+1}k_{jz}}{\varepsilon_{j+1}k_{jz}+\varepsilon_j k_{j+1,z}}. \tag{13}$$

After that, all the theory we have developed is equally applied to both polarisations if, for the Fresnel coefficients, we use the corresponding formula either from (10), (12) or from (11), (13).

For example, the so-called Stokes relations have the same form for both TE and TM waves

$$r_{j,j+1} = -r_{j+1,j}, \quad r_{j,j+1}^2 + t_{j,j+1}t_{j+1,j} = 1. \tag{14}$$

Now, let us derive the reflection coefficients R for TE and TM waves incident on the stack (1) using the multiple reflection method. The case of an arbitrary polarisation of an incident wave can be then considered with the aid of its decomposition into TE and TM modes.

3. Exact calculation of the reflection coefficient for one layer

The application of the multiple reflection method to a single layer is well-known in the literature. It is usually given as the theory of the Fabry–Perot interferometer. Let us briefly describe the method for this case.

We can represent the wave reflected from a stack of layers (the semi-infinite media ε_0 and ε_{f}, and the layer ε_1) as a superposition of waves: a) the wave reflecting from the interface $\varepsilon_0/\varepsilon_1$; b) the wave transmitting through the interface $\varepsilon_0/\varepsilon_1$, then passing through the layer, reflecting from the interface $\varepsilon_1/\varepsilon_{\mathrm{f}}$, passing through the layer again, and leaving the layer through the interface $\varepsilon_0/\varepsilon_1$; c) the wave penetrating in the layer, reflecting from the interface $\varepsilon_1/\varepsilon_{\mathrm{f}}$ twice and reflecting from the interface $\varepsilon_0/\varepsilon_1$ ones, passing the layer forward and back twice, and leaving the layer through the interface $\varepsilon_0/\varepsilon_1$, etc (see Fig. 2). Summing the contribution of all these partial waves to the total reflection coefficient R_1, we obtain

$$R_1 = r_{01} + t_{01}\exp(\mathrm{i}k_{1z}d_1) r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) t_{10} + t_{01}\exp(\mathrm{i}k_{1z}d_1)$$

$$\times\, r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) r_{10} \exp(\mathrm{i}k_{1z}d_1) r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) t_{10} + \dots .$$

Beginning with the second term, we have an infinite geometric series. Summing it up and taking into account (14), we obtain

$$R_1 = r_{01} + \frac{t_{01}t_{10}r_{1\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)}{1 + r_{01}r_{1\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)} = \frac{r_{01} + r_{1\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)}{1 + r_{01}r_{1\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)}, \tag{15}$$

where the Fresnel reflection and transmission coefficients for TE and TM waves are given by formulas (10)–(13).

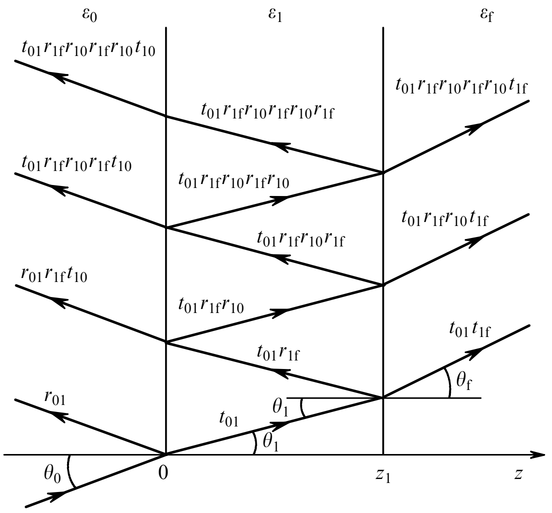

Figure 2. Application of a multiple reflection method to one layer.

Unfortunately, already for two layers the picture of partial reflected and transmitted waves is getting so complicated (Fig. 3), that there is no direct and exact method to sum all partial waves contributing to the total reflection coefficient R_2.

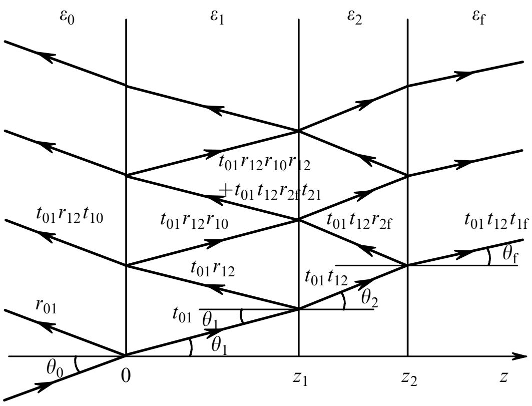

Figure 3. Application of a multiple reflection method to two layers.

4. Recursive calculation of the reflection coefficient for a multilayered structure

Let r_1 be the total reflection coefficient from the interface $\varepsilon_1/\varepsilon_2$ between the first layer and the remaining stack of $m-1$ layers plus the medium ε_f. As a result, repeating the calculation of the previous section for one layer, we obtain

$$R_m = r_{01} + \frac{t_{01}t_{10}r_1\exp(2\mathrm{i}k_{1z}d_1)}{1 + r_{01}r_1\exp(2\mathrm{i}k_{1z}d_1)}. \tag{16}$$

Then, representing the wave reflected from the interface $\varepsilon_1/\varepsilon_2$ as a sum of partial waves reflected from the interface $\varepsilon_2/\varepsilon_3$, we obtain

$$r_1 = r_{12} + \frac{t_{12}t_{21}r_2\exp(2\mathrm{i}k_{2z}d_2)}{1 + r_{12}r_2\exp(2\mathrm{i}k_{2z}d_2)}, \tag{17}$$

where r_2 is the total reflection coefficient from the interface $\varepsilon_2/\varepsilon_3$ between the second layer and the remaining stack of $m-2$ layers plus the medium ε_f. For an arbitrary interface $\varepsilon_j/\varepsilon_{j+1}$, we obtain

$$r_j = r_{j,j+1} + \frac{t_{j,j+1}t_{j+1,j}\,r_{j+1}\exp(2\mathrm{i}k_{j+1,z}d_j)}{1 + r_{j,j+1}\,r_{j+1}\exp(2\mathrm{i}k_{j+1,z}d_j)}, \tag{18}$$

where r_{j+1} is the total reflection coefficient from the interface $\varepsilon_{j+1}/\varepsilon_{j+2}$ between the $j+1$ layer and the remaining stack of $m-j-1$ layers plus the medium ε_f. For the last interface, i.e. for the interface $\varepsilon_m/\varepsilon_\mathrm{f}$, $j = m$ and $r_m = r_{m\mathrm{f}}$, where $r_{m\mathrm{f}}$ is the Fresnel reflection coefficient at the interface between the last layer and the medium ε_f.

Thus, the problem of the reflection of the TE or TM wave from a stack of an arbitrary number of plane layers can be solved using recursively the formula (18) m times and taking into account $r_m = r_{m\mathrm{f}}$. The procedure gives us the exact analytical results. Taking into account Stokes relations, we obtain, for example, the reflection coefficient for two layers ($m = 2$) in the form

$$R_2 = \frac{r_{01} + r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)}{1 + r_{01}r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{12}r_{2\mathrm{f}}\exp(2\mathrm{i}k_{2z}d_2)} \to$$

$$\to \frac{+\,r_{01}r_{12}r_{2\mathrm{f}}\exp(2\mathrm{i}k_{2z}d_2)}{+\,r_{01}r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)}. \tag{19}$$

For a stack consisting of more layers the final results are exact but too cumbersome. For example, already for three layers ($m = 3$) the reflection coefficient R_3 takes the form

$$R_3 = \frac{r_{01} + r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{23}\exp[2\mathrm{i}(k_{1z}d_1 + k_{2z}d_2)]}{1 + r_{01}r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{12}r_{23}\exp(2\mathrm{i}k_{2z}d_2)} \to$$

$$\to \frac{+\,r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{1z}d_1 + k_{2z}d_2 + k_{3z}d_3)] + r_{01}r_{12}r_{23}\exp(2\mathrm{i}k_{2z}d_2)}{+\,r_{23}r_{3\mathrm{f}}\exp(2\mathrm{i}k_{3z}d_3) + r_{12}r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{2z}d_2 + k_{3z}d_3)]} \to$$

$$\to \frac{+r_{01}r_{23}r_{3\mathrm{f}}\exp(2\mathrm{i}k_{3z}d_3) + r_{01}r_{12}r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{2z}d_2 + k_{3z}d_3)]}{+\,r_{01}r_{23}\exp[2\mathrm{i}(k_{1z}d_1 + k_{2z}d_2)] + r_{01}r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{1z}d_1} \to$$

$$\to \frac{+\,r_{12}r_{23}r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{1z}d_1 + k_{3z}d_3)]}{+\,k_{2z}d_2 + k_{3z}d_3)] + r_{01}r_{12}r_{23}r_{3\mathrm{f}}\exp[2\mathrm{i}(k_{1z}d_1 + k_{3z}d_3)]}. \tag{20}$$

As a result, it is reasonable to use the algorithm (18) directly in a numerical form for given parameters (dielectric permittivities and thicknesses) of layers. However, as we will see in the next section, the multiple reflection method still allows us to obtain analytical results, which are approximate but in good agreement with the exact ones for realistic optic materials, in physically intelligible and relatively simple form.

5. Approximate calculation of the reflection coefficient

The core physical idea of the suggested scheme is that the waves having the least number of reflections inside the stack contribute more to the total reflection coefficient. This follows from the fact that the Fresnel coefficients at any interface satisfy the condition $|r_{j,j+1}| < 1$. It is obvious that any partial wave has odd number of reflections inside the stack. Therefore, in the first approximation we should take into account only terms which are proportional to $r_{j,j+1}$, where $j = 0, 1, 2, \ldots, m$, and $m + 1 = \mathrm{f}$. In the second approximation we should also take into account terms which are proportional to $r_{j,j+1} r_{p,p+1} r_{q,q+1}$ etc. As a result, only partial waves having a single reflection in the stack contribute to the first approximation, partial waves having one and three reflections in the stack contribute to the second approximation etc.

Let us illustrate the aforementioned method by concrete examples. Consider again the case of a single layer in the stack $m = 1$. The reflection coefficient in the first approximation takes into account the contribution of the wave reflected from the interface $\varepsilon_0/\varepsilon_1$ and the contribution of the wave penetrating the interface $\varepsilon_0/\varepsilon_1$, passing through the layer, reflecting from the interface $\varepsilon_1/\varepsilon_\mathrm{f}$, passing through the layer again and leaving through the interface $\varepsilon_0/\varepsilon_1$. Then,

$$R_1 \approx r_{01} + t_{01} \exp(\mathrm{i}k_{1z}d_1) r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) t_{10}$$

$$= r_{01} + \left(1 - r_{01}^2\right) r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1) \approx r_{01} + r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1),$$

where we neglect the term $-r_{01}^2 r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1)$, which is the contribution to the second approximation. As a result the reflection coefficient from one layer in the first approximation of the multiple reflection theory is

$$R_1^{(1)} = r_{01} + r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1). \tag{21}$$

In the second approximation there is also a contribution from the wave passing the layer twice and leaving it after two reflections at the interface $\varepsilon_1/\varepsilon_\mathrm{f}$ and one reflection at the interface $\varepsilon_1/\varepsilon_0$. Taking into account this wave, we have

$$R_1 \approx r_{01} + t_{01} \exp(\mathrm{i}k_{1z}d_1) r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) t_{10} + t_{01} \exp(\mathrm{i}k_{1z}d_1)$$

$$\times r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) r_{10} \exp(\mathrm{i}k_{1z}d_1) r_{1\mathrm{f}} \exp(\mathrm{i}k_{1z}d_1) t_{10}$$

$$= r_{01} + \left(1 - r_{01}^2\right) r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1) + \left(1 - r_{01}^2\right) r_{10} r_{1\mathrm{f}}^2$$

$$\times \exp(4\mathrm{i}k_{1z}d_1) \approx r_{01} + r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1)$$

$$- r_{01}^2 r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1) - r_{01} r_{1\mathrm{f}}^2 \exp(4\mathrm{i}k_{1z}d_1),$$

where we neglect the term $r_{01}^3 r_{1\mathrm{f}}^2 \exp(4\mathrm{i}k_{1z}d_1)$, which is the contribution to the third approximation. Therefore, the reflection coefficient from one layer in the second approximation of the multiple reflection theory is

$$R_1^{(2)} = r_{01} + r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1) - r_{01}^2 r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1)$$

$$- [r_{1\mathrm{f}} \exp(2\mathrm{i}k_{1z}d_1)]^2 r_{01}. \tag{22}$$

The comparison of the results for the reflection coefficient from one layer in the first and second approximation with the exact formula (15) is shown in Fig. 4.

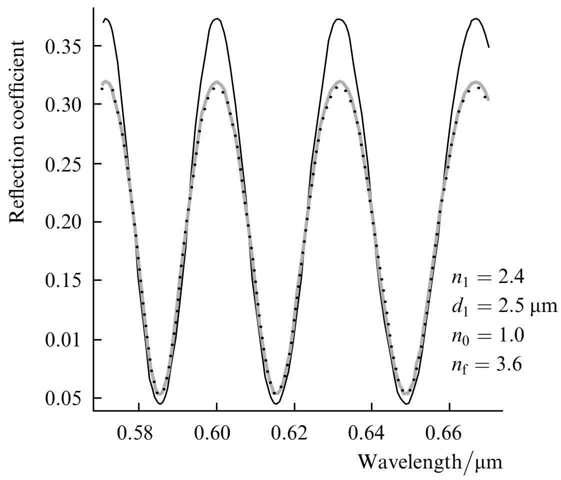

Figure 4. Dependence of the reflection coefficient on the wavelength of an incident wave for a stack consisting of a single layer; solid line – first approximation, dashed line – second approximation, grey line – exact result.

As an example, we consider the dependence of the reflection coefficient on the wavelength of incident light in the optical range for the case of normal incidence in the absence of absorption. In this case, the Fresnel reflection and transmission coefficients (10)–(13) are real and identical for both TE and TM waves:

$$r_{j,j+1} = \frac{n_j - n_{j+1}}{n_j + n_{j+1}}, \quad t_{j,j+1} = \frac{2n_j}{n_j + n_{j+1}}, \quad n_j = \varepsilon_j^{1/2}. \tag{23}$$

The following selection of refractive indexes $n_0 \simeq 1$ (air), $n_1 \simeq 2.4$ (chalcogenide glass [16]), and $n_\mathrm{f} \simeq 3.6$ (GaAs) realises a rather extreme situation among realistic cases from the point of view of the suggested method (the typical range of the refractive indexes of optic materials is $1 < n_j < 4$). The reason for this is the relatively big difference in the refractive indexes of these materials that results to the respectively high r_{01} and $r_{1\mathrm{f}}$ ($|r_{01}| \simeq 0.41$, $|r_{1\mathrm{f}}| \simeq 0.2$), according to formulas (23). If, for example $n_1 = 1.3$ (fluorinated ethylene propylene) and $n_\mathrm{f} = 1.5$ (silicon oxide), then $|r_{01}| \simeq 0.13$, $|r_{1\mathrm{f}}| \simeq 0.07$, and the higher order terms contribute much less to the total reflection coefficient R_1.

Consider now the case of two layers ($m = 2$). The reflection coefficient in the first approximation takes into account the contribution of partial waves which have a single internal reflection, i.e. (a) the contribution of the wave reflected from interface $\varepsilon_0/\varepsilon_1$, its contribution is r_{01}; (b) the contribution of the wave penetrating interface $\varepsilon_0/\varepsilon_1$, passing through layer ε_1, reflecting from interface $\varepsilon_1/\varepsilon_2$, passing through layer ε_1 again, and leaving the stack through interface $\varepsilon_1/\varepsilon_0$; its contribution to the total reflection coefficient is $t_{01}\exp(\mathrm{i}k_{1z}d_1)r_{12}\exp(\mathrm{i}k_{1z}d_1)t_{10}$, and its contribution to the reflection coefficient in the first approximation is $r_{12}\exp(2\mathrm{i}k_{1z}d_1)$; (c) the contribution of the wave penetrating interface $\varepsilon_0/\varepsilon_1$, passing through layer ε_1 and interface $\varepsilon_1/\varepsilon_2$, passing through layer ε_2, reflecting from interface $\varepsilon_2/\varepsilon_\mathrm{f}$, passing layer ε_2, interface $\varepsilon_2/\varepsilon_1$, layer ε_1, and finally leaving the stack through interface $\varepsilon_1/\varepsilon_0$; its contribution to the total reflection coefficient is $t_{01}\exp(\mathrm{i}k_{1z}d_1)t_{12}\exp(\mathrm{i}k_{2z}d_2)\times r_{2\mathrm{f}}\exp(\mathrm{i}k_{2z}d_2)t_{21}\exp(\mathrm{i}k_{1z}d_1)t_{10}$, and its contribution to the reflection coefficient in the first approximation is $r_{2\mathrm{f}}\times\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)$. As a result,

$$R_2^{(1)} = r_{01} + r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)\,. \tag{24}$$

Notice that the wave (b) contributes also term $-r_{01}^2 r_{12}\times\exp(2\mathrm{i}k_{1z}d_1)$ to the second approximation, and the wave c) contributes term $-r_{01}^2 r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2) - r_{12}^2 r_{2\mathrm{f}}\times\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)$ to the second approximation and term $r_{01}^2 r_{12}^2 r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)$ to the third approximation, respectively. In the second approximation we should also take into account the contributions from the partial waves which have three internal reflections inside the stack of two layers (there are five such waves). As a result,

$$\begin{aligned}R_2^{(2)} &= r_{01} + r_{12}\exp(2\mathrm{i}k_{1z}d_1) + r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)\\ &- r_{01}^2 r_{12}\exp(2\mathrm{i}k_{1z}d_1) - r_{01}^2 r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)\\ &- r_{12}^2 r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2) - [r_{12}\exp(2\mathrm{i}k_{1z}d_1)\\ &+ r_{2\mathrm{f}}\exp(2\mathrm{i}k_{1z}d_1)\exp(2\mathrm{i}k_{2z}d_2)]^2 r_{01} - [r_{2\mathrm{f}}\exp(2\mathrm{i}k_{2z}d_2)]^2\\ &\times r_{12}\exp(2\mathrm{i}k_{1z}d_1).\end{aligned} \tag{25}$$

The comparison of the results for the reflection coefficient from the stack of two layers in the first and second approximation with the exact formula (19) is shown in Fig. 5 for the case of normal incidence of an optic wave. Again, we used the typical values of the refractive indexes of optic materials: $n_0 \simeq 1.0$ (air), $n_1 \simeq 2.4$ (chalcogenide glass), $n_2 \simeq 1.5$ (silicon oxide) and $n_\mathrm{f} \simeq 3.6$ (GaAs). Such a selection of materials is again testing an unfavourable situation with the point of view of the suggested method, as the difference in refractive indexes of these materials are high enough. However, as we can see in Fig. 5, the second approximation is already in good agreement with the exact result (19).

In the case of an arbitrary number of layers m, summing up all corresponding partial waves we obtain in the first approximation of the suggested multiple reflection method

$$R_m^{(1)} = \sum_{j=0}^{m}\left\{r_{j,j+1}\prod_{t=0}^{j}\exp(2\mathrm{i}k_{tz}d_t)\right\}, \tag{26}$$

and in the second approximation

$$\begin{aligned}R_m^{(2)} &= \sum_{j=0}^{m}\left\{r_{j,j+1}\prod_{t=0}^{j}\exp(2\mathrm{i}k_{tz}d_t)\right\} - \sum_{j=1}^{m}\left\{\left(\sum_{p=0}^{j-1}r_{p,p+1}^2\right)\right.\\ &\left.\times r_{j,j+1}\prod_{t=0}^{j}\exp(2\mathrm{i}k_{tz}d_t)\right\} - \sum_{j=0}^{m-1}\left\{\left[\sum_{p=j+1}^{m}r_{p,p+1}\right.\right.\\ &\left.\left.\times\prod_{t=1}^{p}\exp(2\mathrm{i}k_{tz}d_t)\right]^2 r_{j,j+1}\prod_{t=0}^{j}\exp(2\mathrm{i}k_{tz}d_t)\right\}.\end{aligned} \tag{27}$$

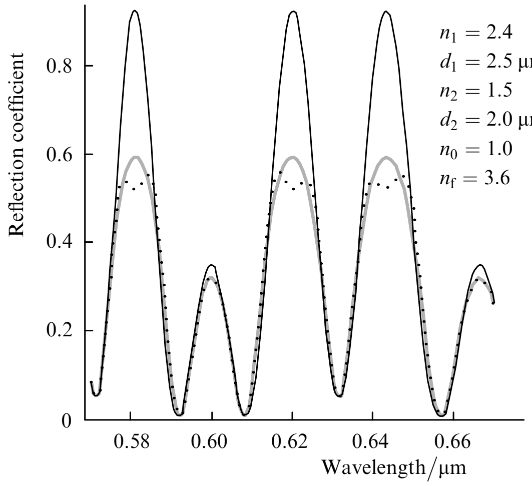

Figure 5. Dependence of the reflection coefficient on the wavelength of an incident wave for a stack consisting of two layers; solid line – first approximation, dashed line - second approximation, grey line – exact result.

We remind the reader that in the above formulas $d_0 \equiv 0$. For the case of normal incidence of an electromagnetic wave from vacuum on a stack of $m = 5$ layers with arbitrary selected refractive indexes in the range $1.5 < n_j < 3$, Fig. 6 illustrates the comparison between the exact reflection coefficient that was obtained from the numerical recursive use of algorithm (18) and the approximate reflection coefficients that were obtained from the analytical formulas (26) and (27). The widths of layers are within few microns.

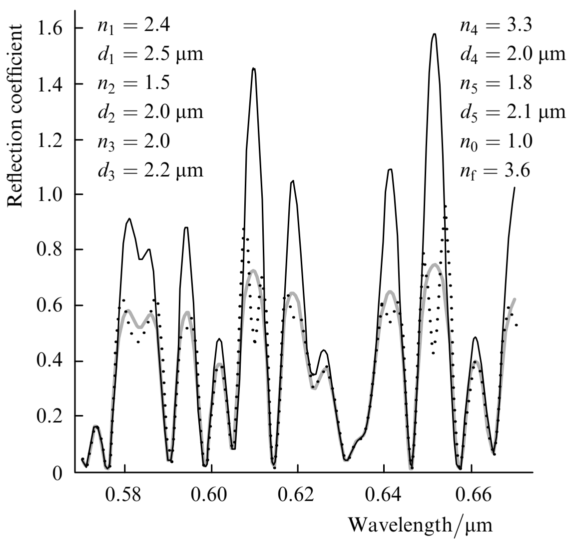

Figure 6. Dependence of the reflection coefficient on the wavelength of an incident wave for a stack consisting of five layers; solid line - first approximation, dashed line – second approximation, grey line – exact result.

The above examples show good agreement between the exact results and the second approximation of the multiple reflection method for the case of normal incidence of an optic wave on a layered system. Very often there is satisfactory agreement even with the first approximation. However, if an incoming wave impinges on a layered system with a big difference in the refractive indexes between neighbouring layers, the first and second approximations fail in the region of wavelengths where the reflection coefficient reaches a maximum, and higher order approximations are needed. The exact criteria requiring the use of higher order approximations will be given in our future publication, including the case of oblique incidence as well.

Note that homogeneous absorption in a stack only improves agreement between the approximate analytical and exact numerical results because the appearance of complex parts in wave vectors k_j decreases higher order terms additionally. If absorption in a stack sharply varies from layer to layer, however, the modification of the suggested multiple reflection method is needed. This will be also considered in our future publication.

References

1. Brekhovskikh L M *Waves in Layered Media, 2nd edition* (New York: Academic Press, 1980; Moscow: Nauka, 1973)
2. FinkY et al. *Science* **282** 1679 (1998)
3. Levin L M *Zh. Tekh. Fiz.* **18** 1399 (1948)
4. Morozov G V, Maev R G, Drake G W F *Kvantovaya Electron.* **25** 1003 (1998) [*Quantum Electron.* **28** 977 (1998)]
5. Morozov G V, Maev R G, Drake G W F *Phys. Rev. E* **60** 4860 (1999)
6. Abeles F *Ann. Phys. (Paris)* **3** 504 (1948)
7. Thomson W T *J. Appl. Phys.* **21** 89 (1950)
8. Haskell N A *Bull. Seism. Soc. Am.* **43** 17 (1953)
9. Lowe M J S *IEEE Trans. Ultrason. Ferr. Freq. Con.* **42** 526 (1995)
10. Stolyarov S N *Kvantovaya Electron.* **15** 1637 (1988) [*Sov. J. Quantum Electron.* **18** 1021 (1988)]
11. Morales J J, Nuevo M J *Am. J. Phys.* **59** 1140 (1991)
12. Polak W J *Appl. Phys.* **74** 777 (1993)
13. Landau L D, Lifshitz L M *Electrodynamics of Continuous Media, 2nd edition* (Oxford: Permagon, 1984, sec.83; Moscow: Nauka, 1982)
14. Morales J J *Eur. J. Phys.* **13** 222 (1992)
15. Parmigiani F *Am. J. Phys.* **51** 245 (1983)
16. Abdulhalim I et al. *J. Non-Cryst. Solids* **163** 1251 (1993)