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A study of radiation propagation in a medium with quadratic inhomogeneity

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***Abstract*. The propagation of Hermitian beams in a medium with a distributed quadratic inhomogeneity is studied and is shown that any solution can be represented as a function of some particular solution. This is accomplished by establishing a one-to-one correspondence between optical fields in a homogeneous medium and in a medium with an arbitrary quadratic inhomogeneity. The stability of optical resonators is studied and the condition for their stability is found. Several solutions are found using the method developed.**

Keywords: *family of beams, quasi-optic equation, characteristic planes, Hermitian beam.*

1. Introduction

The study of a two-dimensional quasi-optic equation using the theory of Lie groups showed that for some dependences of the refractive index on coordinates there exist parti-cular solutions that are analogous to a plane wave, a point source, or Hermitian beams in a homogeneous medium [1]. The characteristic feature of these particular solutions is the invariability of the radius of curvature of the wave front upon proportional variation in the perturbation of the refractive index. An exception from this rule is the case of a quadratic inhomogeneity of the refractive index, for which there exists a continuum of solutions that do not have the above properties.

To use the results of paper [1] in full measure, one should be able to obtain a general quasi-optic solution from the known particular solution. The aim of this paper is to solve this problem in the case of quadratic inhomogeneity, i.e., in the case of a linear optical system. The method of matrix optics [2 – 4] cannot be applied for this purpose because it derives a general solution without using a particular solution. In this paper, an alternative approach is applied, which is based on the fact that a general solution of any linear equation can be represented as a Fourier series for the complete system of functions that are particular solutions of this equation [5]. In the case of quadratic inhomogeneity, this series can be summed to obtain a general solution of the quasi-optic equation in the form convenient for calculations.

The method proposed in this paper allows one to find new theoretical solutions in the explicit form, which cannot be obtained using the Fresnel integrals and matrix optics. Unlike the matrix method, which directly yields a general solution of the quasi-optic equation, the method of expansion in the Fourier series is based on the construction of the solution for specific boundary conditions. The solution is obtained using a particular solution of the quasi-optic equation. In this paper, the concepts of a family of beam and of its characteristic planes are introduced. The position of characteristic planes determines the type of a given optical system and clearly solves the problem of the resonator stability.

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2. Formulation of the problem

Because in the case of quadratic inhomogeneity, one can seek the solution for the electromagnetic-field distribution by the method of separation of variables [6], it is sufficient to consider the case of two variables. In the paraxial approximation, the radiation propagation is described by the quasi-optic equation [7]

$$U''_{xx} + 2\mathrm{i}kn_0 U'_z + 2k^2 n_0 \Delta n U = 0, \quad \boldsymbol{E} = \boldsymbol{e}U\mathrm{e}^{-\mathrm{i}(\omega t - kn_0 z)}, \quad (1)$$

where $\boldsymbol{E}$ is the electric field strength; $\boldsymbol{e}$ is the unit polarisation vector; k is the wave number; ω is the circular frequency; n_0 is the unperturbed refractive index; $\Delta n = 0.5\, n''_{xx}(z) x^2$ is the refractive-index perturbation, $|\Delta n| \ll n_0$; $n(x,z) = n_0 + \Delta n$. The z-axis coincides with the propagation direction of radiation.

In this case, the solutions of equation (1) are Hermitian beams [6]

$$U^m = \left(\frac{w_0}{w}\right)^{1/2} H_m\left(\frac{x}{w}\right) \exp\left[\mathrm{i}kn_0 \frac{x^2}{2\tilde{\rho}} - \frac{\mathrm{i}(m+1)}{2}\varphi\right], \quad (2)$$

where w is the beam radius; w_0 is the beam waist radius; φ is the phase incursion; $\tilde{\rho}$ is the complex radius of the wave-front curvature [6]:

$$\frac{1}{\tilde{\rho}} = \frac{1}{r} + \frac{\mathrm{i}}{kn_0 w^2}; \quad (3)$$

r is the radius of the wave-front curvature;

$$H_m(\tau) = (-1)^m e^{\tau^2} \frac{d^m e^{-\tau^2}}{d\tau^m}$$

are Hermitian polynomials. By substituting solution (2) into quasi-optic equation (1), we obtain the equations

$$\tilde{\rho}_z' + \frac{n_{xx}''}{n_0}\tilde{\rho}^2 - 1 = 0, \quad \varphi_z' = \frac{1}{kn_0 w^2}. \tag{4}$$

Note that by the change of variable $\tilde{\rho} = p_z' p^{-1}$, the first equation of system (4) is reduced to a second-order linear equation $n_0 p_{zz}'' = n_{xx}'' p$, which formally completely coincides with known ray equations [8]. Expressions (4) are differential analogues of the algebraic equations of matrix optics.

Let us introduce definitions that we will use below. The solution that satisfies (4) under certain boundary conditions will be called a particular solution of equation (4). A general solution satisfies system (4) under any boundary conditions and contains only two integration constants. Each particular solution of (4) corresponds to a class of Hermitian beams (2) satisfying quasi-optic equation (1) under the same boundary conditions as the initial particular solution. We will call this class of Hermitian beams a particular solution of the quasi-optic equation. Below, we will show that the knowledge of the particular solution of equation (4) allows us to find the general solution of this equation using a finite number of transformations, which is equivalent to the construction of a class of Hermitian beams (2) satisfying quasi-optic equation (1) under any boundary conditions. This method is especially convenient in the case when some separated beams exist for the given dependence $n_{xx}''(z)$, which propagate in an inhomogeneous medium in the simplest way.

3. Expansion of a Hermitian beam in the Fourier series

Because the complete orthogonal system (2) of solutions of the quasi-optic equation corresponds to any particular solution of equations (4), the general solution of equation (1) can be represented as the Fourier series in this system of functions [5]:

$$u^m = \sum_n a_n^m U^n, \tag{5}$$

where U^n, u^m are field distributions for particular and general solutions, respectively; a_n^m are Fourier coefficients, respectively. Below, we will use the following notation: U^n, $\tilde{P}$, W, W_0, R, Φ are the field distribution; the complex radius of wave-front curvature; the beam radius; the beam-waist radius; the radius of wave-front curvature; and the phase incursion for a particular solution, which is assumed known; u^m, $\tilde{\rho}$, w, w_0, r, φ are the same quantities for the general solution.

In principle, the series (5) gives a formal solution of the problem for any boundary conditions, only Fourier coefficients should be found. However, the solution of the problem by this method is inconvenient because many integrals should be calculated for determining Fourier coefficients and summing complex quantities in (5). Moreover, if we seek the eigenfunctions of the optical resonator, then expression (1) will be reduced to a set of linear equations with an infinite number of unknowns whose eigenvectors will be modes of the resonator. However, this problem can be solved without integration and subsequent summing of the Fourier series.

To do this, consider the transformation of expression (5) for a homogeneous medium (this transformation is physically equivalent to the transfer of the optical field from an inhomogeneous medium to a homogeneous medium):

$$u_h^m = \sum_n a_n^m U_h^n, \tag{6}$$

where the subscript h corresponds to the homogeneous medium. Below, as above, we use the following notation: U_h^n, $\tilde{P}_h$, W_h, W_{h0}, R_h, Φ_h are the field distribution, the complex radius of wave-front curvature, the beam radius; the beam-waist radius; the radius of wave-front curvature; and the phase incursion for the auxiliary beam; u_h^m, $\tilde{\rho}_h$, w_h, w_{h0}, r_h, φ_h are the same quantities for the reference beam. Thus, the auxiliary beam is the transform of the particular solution U^n, while the auxiliary beam is the transform of the general solution u^m of the quasi-optic equation. Therefore, to find the general solution of the quasi-optic equation, it is sufficient to find the transformation that transforms the particular solution to the auxiliary beam.

The propagation of a Hermitian beam in a homogeneous medium is described by the expressions

$$\tilde{\rho}_h = \tilde{\rho}_{h0} + z - z_0, \quad w_h^2 = w_{h0}^2\left[1 + \frac{(z - z_0)^2}{(kn_0 w_{h0}^2)^2}\right],$$
$$r_h = \frac{(z - z_0)^2 + (kn_0 w_{h0}^2)^2}{z - z_0}, \quad \varphi_h = \arctan\frac{z - z_0}{kn_0 w_{h0}^2}. \tag{7}$$

It is assumed in (7) that the reference beam waist is located in the plane $z = z_0$. For the auxiliary beam, the relations are valid, which are similar to (7), the parameters W_{h0} and z_{h0} being arbitrary. We will assume for convenience that $z_{h0} = 0$ and $W_{h0} = W_0$. Let us represent series (5) and (6) in the form

$$u^m = \left(\frac{W_0}{W}\right)^{1/2} \exp\left(ikn_0\frac{x^2}{2\tilde{P}}\right) f^m(x, z),$$
$$f^m(x, z) = \sum_n a_n^m H_n\left(\frac{x}{W}\right) \exp\left(-i\frac{2n+1}{2}\Phi\right), \tag{8}$$

$$u_h^m = \left(\frac{W_0}{W_h}\right)^{1/2} \exp\left(ikn_0\frac{x^2}{2\tilde{P}_h}\right) f_h^m(x, z),$$
$$f_h^m(x, z) = \sum_n a_n^m H_n\left(\frac{x}{W_h}\right) \exp\left(-i\frac{2n+1}{2}\Phi_h\right). \tag{9}$$

The function $f_h^m(x, z)$ is known. Let us introduce new coordinates $\xi(x, z)$ and $\zeta(z)$ so that the equality $f_h^m(\xi(x, z), \zeta(z)) = f^m(x, z)$ is fulfilled. Comparison of expressions (8) and (9) shows that the relation

$$\frac{\xi}{W_h(\zeta)} = \frac{x}{W(z)}, \quad \Phi_h(\zeta) = \Phi(z),$$

$$\zeta(z) = kn_0 W_0^2 \tan\Phi(z).$$

should be fulfilled.

For the general solution of system (4), we obtain the expression

$$\frac{1}{r(z)} = \frac{1}{R(z)} + \frac{W_h^2(\zeta)}{W^2(z)}\left[\frac{1}{r_h(\zeta-\zeta_0)} - \frac{1}{R_h(\zeta)}\right],$$

$$w(z) = \frac{w_h(\zeta-\zeta_0)}{W_h(\zeta)}\ W(z), \quad \varphi(z) = \varphi_h(\zeta-\zeta_0). \tag{10}$$

where $\zeta_0 = \zeta(z_0)$. The propagation of the complex radius of curvature is described by the first equation of system (10), so that we do not present the corresponding expression here. To close solution (10), one should know the reference-beam radius w_{h0} and the position z_0 of the beam waist, which play the role of the integration constants in system (4). They can be found from the boundary conditions for the required beam.

The equation for the beam radius w in the second equation of system (10) can be represented in a more clear form. For this purpose, we choose the phase of the particular solution U^n so that $\zeta(0) = 0$, which can be achieved by shifting the origin of reference of the initial phase of the particular solution. Then, the relation $W_h^2(\zeta) = W_0^2[1 + \tan^2\Phi(z)]$ is valid for the auxiliary beam. After some transformations, the final expressions for w takes the form

$$w^2(z) = \frac{w_{h0}^2 W^2(z)}{W_0^2}\left\{1 + \frac{W_0^4}{w_{h0}^4}[\tan\Phi(z) - \tan\Phi(z_0)]^2\right\}\cos^2\Phi(z). \tag{11}$$

Expression (11) shows that the radius of the general solution oscillates about the radius of the particular solution. The oscillations are in general not periodic but depend on the phase of the particular solution $\Phi(z)$.

4. Determination of parameters of the reference beam

The boundary conditions for parameters of the beams can be imposed by two methods, either by specifying the field distribution on some surface or imposing two boundary conditions on the radius of wave-front curvature. The boundary conditions of the first type are characteristic of the propagation of radiation in space, while those of the second type are typical for radiation in optical resonators. Consider the methods for providing these boundary conditions in turn.

(1) Let us assume that the characteristic radius of the amplitude distribution (beam radius) $w^* = w(z^*)$ and the radius of wave-front curvature $r^* = r(z^*)$ are specified on some plane $z = z^*$. Then, we find from (10) the radius of wave-front curvature and the radius of the reference beam at the point $\zeta = \zeta^* - \zeta_0$, where $\zeta^* = \zeta(z^*)$:

$$\frac{1}{r_h^*} = \frac{1}{r_h(\zeta^*-\zeta_0)} = \frac{1}{R_h(\zeta^*)} + \frac{W^2(z^*)}{W_h^2(\zeta^*)}\left[\frac{1}{r^*} - \frac{1}{R(z^*)}\right],$$

$$w_h^* = w_h(\zeta^*-\zeta_0) = \frac{w^*}{W(z^*)}W_h(\zeta^*). \tag{12}$$

Because the right-hand parts of expressions (12) are known, we can find the parameters of the reference beam from expressions (7):

$$w_{h0} = \left[\frac{(r_h^* w_h^*)^2}{r_h^{*2} + (kn_0 w_h^{*2})^2}\right]^{1/2},$$

$$\zeta_0 = \zeta^* - \frac{r_h^*\left(kn_0 w_h^{*2}\right)^2}{r_h^{*2} + \left(kn_0 w_h^{*2}\right)^2}. \tag{13}$$

It follows from (13) that the reference beam with real parameters can be found for any boundary conditions.

(2) Consider the boundary conditions of the second type. Let us assume that spherical mirrors with radii of curvature r_1 and r_2 are located at points $z_1 < z_2$, respectively. The radii of focusing and scattering mirrors are considered positive and negative, respectively. Then, two conditions $r(z_1) = -r_1$ and $r(z_2) = r_2$ are imposed on the radius of wave-front curvature. From the first equation of system (10), we find two radii of reference-beam curvature $r_{h1} = r_h(\zeta_1 - \zeta_0)$ and $r_{h2} = r_h(\zeta_2 - \zeta_0)$, which are described by the expressions

$$\frac{1}{r_{h1,2}} = \frac{1}{R_h(\zeta_{1,2})} + \frac{W^2(z_{1,2})}{W_h^2(\zeta_{1,2})}\left[\frac{1}{r_{1,2}} - \frac{1}{R(z_{1,2})}\right]. \tag{14}$$

The solution of equations (7) in this case has the form

$$\zeta_0 = \frac{\zeta_1(\zeta_1 - r_{h1}) - \zeta_2(\zeta_2 - r_{h2})}{2\zeta_1 - 2\zeta_2 + r_{h2} - r_{h1}},$$

$$w_{h0}^2 = \frac{\left[r_{h1}(\zeta_1-\zeta_0) - (\zeta_1-\zeta_0)^2\right]^{1/2}}{kn_0}. \tag{15}$$

It is obvious that the second equation of system (15) does not always have real roots, i.e., the reference Hermitian beam with a real radius of the amplitude distribution exists not for all boundary conditions. Therefore, not any optical resonator is stable, the stability condition having the form

$$r_{h1}(\zeta_1-\zeta_0) \geqslant (\zeta_1-\zeta_0)^2. \tag{16}$$

A similar inequality can be written for r_{h2} and ζ_2. The condition (16) means that the modulus of the radius of wave-front curvature of the reference beam at some point is not smaller than the distance from the reference-beam waist to this point (the equality is achieved for a point source). This condition is always fulfilled for Hermitian beams in a homogeneous medium, whose propagation is described by expressions (7). Therefore, expression (16) in the parametric form completely solves the problem of stability of a resonator with an arbitrary quadratic inhomogeneity.

5. Families of beams and characteristic planes

We will call a family of beams the set of Hermitian beams that not necessarily belong to the same class but have the same radius of wave-front curvature on some plane. We will call such planes the characteristic planes. Let us assume that the particular solution U^n and the required beam u^m on the plane $z = 0$ (the plane position is inessential) have the same radius of wave-front curvature $R_0 = r_0$. Assume also that the radius of the amplitude distribution for the required beam is w_0 at the point $z = 0$. As above, we assume that $\zeta(0) = 0$. Then, as follows from (13), the reference-beam waist will be located at the point $z_0 = 0$ and $w_0 = w_{h0}$, and the propagation of the beam family is described by the expressions

$$w^2 = \frac{w_0^2\,W^2(z)}{W_0^2}\left[\cos^2\Phi(z) + \frac{W_0^4}{w_0^4}\sin^2\Phi(z)\right],$$
$$\frac{1}{r} = \frac{1}{R} + \frac{\tan\Phi(z)}{kn_0W^2(z)}\,\frac{W_0^4 - w_0^4}{W_0^4\tan^2\Phi(z) + w_0^4}. \qquad (17)$$

It follows from (17) that, except the plane $z_0 = 0$, other characteristic planes can also exist. By using the second equation of system (17), we obtain that $r(z_j) = R(z_j) = r_j$ at points $\Phi(z_j) = \pi j/2$, where j is an integer. The position of characteristic planes coincides with the extrema of the radius w of the amplitude distribution for the required beam relative to the radius W of the beam for the particular solution: $\mathrm{d}w/\mathrm{d}W = 0$ at points $z = z_j$. Additional characteristic planes can appear only when the radius of the required beam oscillates relative to the particular solution. If the oscillations are absent, only one characteristic plane $z_0 = 0$ exists. Fig. 1 illustrates the concept of characteristic planes of the beam family.

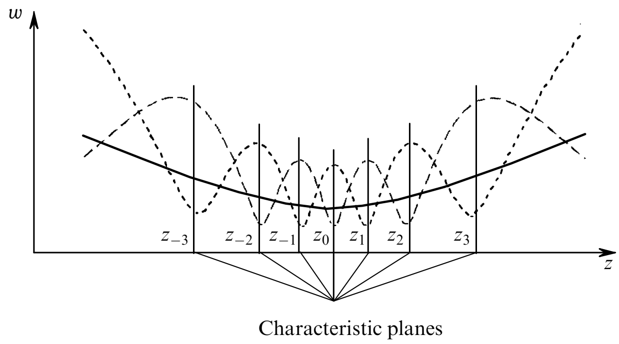

Figure 1. Characteristic planes (z_j) of an optical system and dependences $w(z)$ for three beams belonging to the same family.

Let us study qualitatively the resonator stability. Let the left mirror with the radius r_0 be located at the point $z_0 = 0$ and the right mirror with the radius r_L be located at the point $z = L$. It is obvious that the mode of such a resonator can be only a beam belonging to two beam families, the first family being determined by the left mirror of the resonator, and the second by the right mirror. For some values of r_0 and r_L, such a beam cannot exist and the resonator is unstable.

We can find from (17) the positions z_j of characteristic planes for each radius r_0. It follows from expression (4) for the beam-phase incursion that the phase is a monotonic function of the coordinate z, and the relation $\operatorname{sign}\Phi(z) = \operatorname{sign} z$ is valid because $\Phi(0) = 0$. This means that a set of characteristic planes $\{z_j\}$ is ordered as $z_j < z_{j+1}$. Therefore, the radius of the right mirror of a stable resonator satisfies the inequalities ($R_L = R(L), W_L = W(L)$)

$$\left.\begin{matrix}-\tan\Phi(z)\\ \cot\Phi(z)\end{matrix}\right\} < kn_0W_L^2\left(\frac{1}{r_L} - \frac{1}{R_L}\right) < \left\{\begin{matrix}\cot\Phi(z),\ z \in (z_{2j}, z_{2j+1}),\\ -\tan\Phi(z),\ z \in (z_{2j-1}, z_{2j}).\end{matrix}\right. \qquad (18)$$

It follows from (18) that the instability regions of the resonator are grouped near characteristic planes. For example, in the case of a homogeneous medium, the only characteristic plane $z_0 = 0$ exists and there are only two instability regions.

Consider now a change in the resonator stability caused by a continuous (quasi-stationary) variation in the refractive index, which can be produced by the heating of the medium. First, until the appearance of additional characteristic planes, the stability diagram resembles that for the resonator in the case of a homogeneous medium. Then, as the degree of inhomogeneity increases, several new characteristic planes can appear, which approach the point z_0 with increasing $|n''_{xx}|$. The resonator becomes unstable when the successive characteristic plane z_0 is found in the vicinity of the plane $z = L$. Fig. 2 shows the dependence of the stability diagram on the phase $\Phi(L)$. The purely imaginary phase $\Phi(L)$ is plotted to the left from zero in the region of negative values. The unstable regions are hatched. The stability (or instability) of the resonator can be determined from Fig. 2 from the dependence of $\Phi(L)$ on the refractive index.

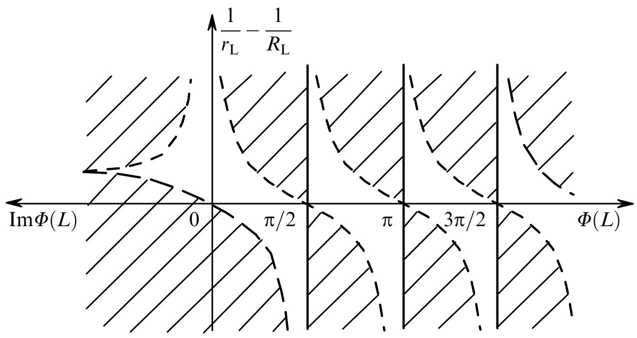

Figure 2. Stability diagram of an optical resonator (unstable regions are hatched).

6. Examples of the solution construction

(1) The simplest particular solution can be constructed for a homogeneous distributed lens with $n''_{xx} = \text{const}$. Because in this case the variable z does not enter explicitly the quasi-optic equation, equation (1) is invariant with respect to displacements along the z-axis ($z \to z + \varepsilon$). Therefore, there exists the solution of equation (1), which is invariant with respect to this one-parametric transformation [10]. Such a solution for the refractive index $n''_{xx} = -\alpha < 0$ is a plane beam with parameters

$$W_\infty^2 = \frac{1}{k(\alpha n_0)^{1/2}}, \quad R_\infty = \infty, \quad \Phi_\infty = z\left(\frac{\alpha}{n_0}\right)^{1/2}. \qquad (19)$$

We will call this inhomogeneity of the refractive index the α-lens. Consider the propagation of a beam satisfying the boundary conditions $r(0) = \infty$, $w(0) = w_0$. For the radius of the amplitude distribution (11), we obtain the known relation [6, 8, 11]

$$w^2 = w_0^2\left\{\cos^2\left[z\left(\frac{\alpha}{n_0}\right)^{1/2}\right] + \frac{W_\infty^4}{w_0^4}\sin^2\left[z\left(\frac{\alpha}{n_0}\right)^{1/2}\right]\right\}. \qquad (20)$$

For the distributed α-lens, the radius w of any beam oscillates about W_∞.

The solution (19) can be formally generalised to the region of positive $n''_{xx} = \beta > 0$, i.e., for the β-lens. After substitution of the refractive index $n''_{xx} = \beta > 0$ to system

(4), we obtain

$$W_\infty^2 = \mp \frac{\mathrm{i}}{k(\beta n_0)^{1/2}}, \quad R_\infty = \infty, \quad \Phi_\infty = \pm \mathrm{i} z \left(\frac{\beta}{n_0}\right)^{1/2}. \tag{21}$$

Although the solution (21) does not satisfy the conditions of boundedness at infinity, it can be used to construct the general solution for the β-lens. As above, consider the propagation of a beam with boundary condition $r(0) = \infty$, $w(0) = w_0$. By substituting (21) into (11) and using the relation $\tan(\mathrm{i}q) = \mathrm{i}\tanh q$, where q is a real number, we obtain

$$w^2 = w_0^2 \left\{ \cosh^2 \left[z \left(\frac{\beta}{n_0}\right)^{1/2} \right] + \frac{|W_\infty^4|}{w_0^4} \sinh^2 \left[z \left(\frac{\beta}{n_0}\right)^{1/2} \right] \right\}. \tag{22}$$

Therefore, any beam for β-lens diverges exponentially [6, 8, 11].

(2) As shown in Ref. [1], for the quadratic inhomogeneity of the form

$$\Delta n = \frac{l^4}{(z^2 + l^2)^2} \frac{n''_{xx}(0,0)}{2} x^2 \tag{23}$$

the particular solution

$$W_l^2 = W_{l0}^2 \left(1 + \frac{z^2}{l^2}\right), R_l = \frac{z^2 + l^2}{z}, \Phi_l = (1 - \delta)^{1/2} \arctan \frac{z}{l},$$
$$W_{l0}^2 = \frac{l}{k n_0 (1 - \delta)^{1/2}}, \quad \delta = \frac{l^2 n''_{xx}(0,0)}{n_0} < 1 \tag{24}$$

exists. We will call the inhomogeneous distributed lens (23) the l-lens. The l-lens power achieves the maximum at the point $z = 0$ and decreases with increasing z. The parameter l determines the width of the inhomogeneity distribution over the coordinate z. The l-lens power decreases by a factor of four at the distance $z = l$.

To extend the solution (24) to the region $\delta > 1$, we use the procedure from the previous section. Let us define the beam-waist radius and the phase as

$$W_{l0}^2 = \frac{l}{k n_0 |1 - \delta|^{1/2}}, \quad \Phi_l = |1 - \delta|^{1/2} \arctan \frac{z}{l}. \tag{25}$$

The final result of calculations has the form

$$w^2 = w_{h0}^2 \left(1 + \frac{z^2}{l^2}\right) \left\{1 + \left(W_{l0}^4 / w_{h0}^4\right) \times \begin{cases} [\tan \Phi_l(z) - \tan \Phi_l(z_0)]^2 \Big\} \cos^2 \Phi_l(z), & \delta < 1, \\ [\tanh \Phi_l(z_0) - \tanh \Phi_l(z_0)]^2 \Big\} \cosh^2 \Phi_l(z), & \delta > 1. \end{cases} \tag{26}$$

The position z_0 of the beam waist and the radius w_{h0} of the intensity distribution for the reference beam are determined from expressions (13) or (15), depending on the type of boundary conditions. It is obvious that any beam for the l-lens diverges linearly at infinity, as a Hermitian beam in a homogeneous medium. The exclusion from this rule is the case $l \to \infty$, when the beam radius for the α-lens oscillates relative to the radius W_∞ of a plane-parallel beam, while for the β-lens any beam diverges exponentially. Such a behaviour is explained by the fact that for $l < \infty$ the l-lens power decreases to zero upon removing from the point $z = 0$. For this reason, at large distances from the point $z = 0$ the beams are not virtually affected by the inhomogeneities of the refractive index and become similar to Hermitian beams in a homogeneous medium. For $l \to \infty$, the l-lens power is constant, and, therefore, the beams are affected by the inhomogeneity at any distances from the coordinate origin. When $l \to 0$, the l-lens is equivalent to a concentrated thin lens. Therefore, the l-lens is the generalisation of a distributed homogeneous lens and a thin lens. Fig. 3 illustrates the behaviour of a Gaussian beam ($m = 0$) for the l-lens at $r(0) = \infty$ in focusing ($\delta < 0$) and scattering ($\delta > 1$) media.

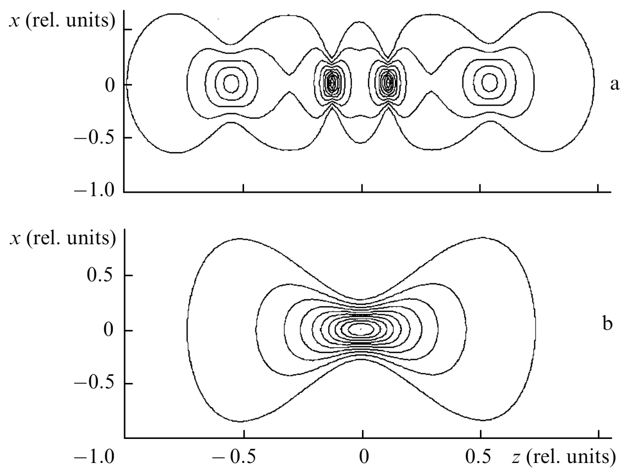

Figure 3. Equal-intensity lines of a Gaussian beam ($m = 0$) for the l-lens for $\delta < 0$ (a) and $\delta > 1$ (b).

Consider the case $\delta = 1$ ($\Phi_l = 0$). After the passage to the limit $\delta \to 1$ in (26), we obtain

$$w^2 = w_{h0}^2 \left(1 + \frac{z^2}{l^2}\right) \times \left[1 + \frac{l^2}{(k n_0 w_{h0}^2)^2} \left(\arctan \frac{z}{l} - \arctan \frac{z_0}{l}\right)^2\right]. \tag{27}$$

One can see from (27) that in this case, neither periodic oscillations of the beam radius about the particular solution for the l-lens occur nor exponential approach of the radius to this solution is observed. For $l \to \infty$, expression (27) describes the propagation of the reference beam. The same result is obtained by substituting $\delta = 0$ into equation (26). Therefore, the reference beam can be really considered the transform of the required beam in the homogeneous medium, while the required beam can be considered the result of deformation of the reference beam in the inhomogeneous medium. The position $z = z_0$ of the reference-beam waist is a mathematical centre of the required beam because upon a continuous decrease of the inhomogeneity of the medium to zero, the beam waist is found at this point.

7. Conclusions

Thus, any solution of the quasi-optic equation can be represented as a function of some particular solution. The search for the solution for given boundary conditions is reduced to the construction of the reference beam in a homogeneous medium, which is locally equivalent to the required beam. Because the parameters of the reference beam are written in terms of the boundary conditions as fraction – irrational expressions, we can conclude that if some particular solution of the quasi-optic equation is expressed in terms of elementary functions, this is also valid for the general solution.

The representation of the solution as a function of some particular solution leads to the concept of the family of beams and its characteristic planes. The characteristic planes correspond to the extrema of radii of the amplitude distribution of the beams for the given family. The position and the number of characteristic planes qualitatively describe the behaviour of Hermitian beams in a linear optical system. The resonator modes can be obtained by finding a beam that belongs to two families, the first of them being determined by the left resonator mirror, and the second one by the right mirror. If such a beam is absent, the resonator is unstable. The stability condition is also equivalent to the condition of the existence of the reference beam for the given mode. The instability regions of the optical resonator are grouped near characteristic planes, whose position is determined by one of the resonator mirrors.

As an application, the solutions were obtained for a homogeneous distributed lens and for the l-lens, which generalises a distributed homogeneous lens and a concentrated thin lens. The study of the solution for the l-lens showed that, depending on the parameter δ, three types of the beam behaviour are possible. For $\delta < 0$, the beam radius oscillates relative to the particular solution (24); for $0 < \delta < 1$, the beam behaves similarly to Hermitian beams in a homogeneous medium; and for $\delta > 1$, the beam expands exponentially relative to the particular solution (24). In two latter cases, the only characteristic plane exists, whereas for $\delta < 0$, the number of characteristic planes is equal to the integer of the expression $2(1 - \delta)^{1/2}$.

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