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Frequency stabilisation of a diode laser by a whispering-gallery mode

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***Abstract*. A model for stabilisation of a laser frequency is considered which takes into account the spectral splitting of the mode and the dependence of its Q factor on coupling with the laser. It is shown that the stabilisation regime is optimal when the coupling coefficient with a whispering-gallery mode is greater than the critical coefficient at which the mode is split. The presence of the amplitude–phase correlation in the active region of the laser determines the maximum value of the stabilisation parameter caused by the dynamic restriction of the feedback level. The single-mode lasing disappears in the case of a strong feedback. It is shown that the appropriate choice of the feedback-phase level suppresses the dynamic instability, which provides in principle a strong feedback with a high-Q external resonator.**

Keywords: *frequency stabilisation, high-Q microresonator, diode laser, whispering-gallery mode, stability of single-mode lasing*.

1. Introduction

An external optical feedback enriches the dynamics of diode lasers [1 – 6] and provides the control of the wavelength, polarisation, and the mode composition of radiation. In particular, the feedback allows narrowing the laser linewidth by many orders of magnitude. For this purpose, two variants of the optical feedback are typically used [7, 8]: a strong feedback in the case of a small number of transits and a weak feedback in the case of a high-Q system.

The most attractive variant of a strong feedback between a laser and a high-Q interferometer is commonly considered impossible because of the dynamic instability. In the second variant of optical feedback, three types of the reference lines formed by an external device can be used: a resonance line of the interferometer, whose width is independent of the lasing regime [9 – 13], an atomic-resonance line with the width depending on the laser-field intensity in an atomic cell [14 – 18], and the resonance line of a whispering-gallery mode of a quartz microsphere, whose width depends on the feedback strength (the cavity load) [19, 20].

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In this paper, we consider the third case, when the laser frequency is 'locked' to a whispering-gallery mode of a microsphere. The main attention is paid to the effect of the strength of coupling of the whispering-gallery mode with a matching element on its Q factor, spectral profile, and the stabilisation factor of the laser frequency.

Note that the high Q of an external resonator can be achieved either by increasing its length or by using high-quality mirrors. Both these methods have drawbacks. In the first case, a miniature laser diode becomes a large device, whereas the fabrication of a miniature resonator with high-quality mirrors is a complicated technological problem. In this connection, it is interesting to use the whispering-gallery modes of dielectric microspheres of diameter of no more than 1 mm with the Q factor of the order of 10^9 for narrowing the diode laser line and stabilising its frequency [19, 20].

2. Stabilisation scheme and mathematical model

Fig. 1 shows the principal scheme of a laser stabilised by a whispering-gallery mode of a dielectric (quartz) sphere. Laser (*1*) coupled by an input device (*2*) with a dielectric microsphere (*3*) excites a whispering-gallery wave F_+ and is itself locked by a whispering-gallery mode F_- propagating in the opposite direction. The waves F_+ and F_- are coupled via Rayleigh scattering from the density inhomogeneities in quartz [20, 21].

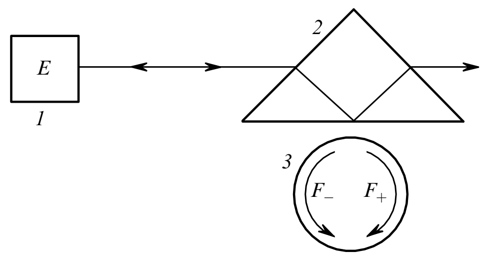

Figure 1. Stabilisation scheme: (*1*) laser; (*2*) matching prism; (*3*) external resonator (dielectric sphere).

The basic equations of the laser being stabilised described for slowly varying field amplitudes have the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{1}{2\tau}(1 + \mathrm{i}\Delta)E - \frac{1}{2\tau}(1 + \mathrm{i}\alpha)g(n)E =$$

$$= \frac{1}{2} K_1 F_-(t-\tau_1)\exp(\mathrm{i}\omega\tau_1), \quad (1a)$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} + \frac{1}{\tau_s} n = J - g(n)\frac{|E|^2}{8\pi\eta\omega}, \quad (1b)$$

$$\frac{\mathrm{d}F_-}{\mathrm{d}t} + \frac{1}{2\tau_0}(1+\mathrm{i}\delta)F_- = \frac{\mathrm{i}}{2}kF_+, \quad (1c)$$

$$\frac{\mathrm{d}F_+}{\mathrm{d}t} + \frac{1}{2\tau_0}(1+\mathrm{i}\delta)F_+ = \frac{\mathrm{i}}{2}kF_- + \frac{1}{2}K_2E(t-\tau_1)\exp(\mathrm{i}\omega\tau_1), \quad (1d)$$

where E is the complex amplitude of the field inside the laser; ω is the frequency generated by the laser–external resonator system; F_+ is the complex amplitude of the whispering-gallery wave; F_- is the complex amplitude of the backward whispering-gallery wave; τ is the decay time of the field in the laser; τ_1 is the time of signal propagation from the laser to the microsphere: τ_0 is the decay time of the whispering-gallery mode; τ_s is the relaxation time of the inverse population in the laser; $(1+\mathrm{i}\alpha)g(n)$ is the complex gain of the laser active medium: $\varDelta \equiv (\omega_c - \omega)\tau$ and $\delta \equiv (\omega_0 - \omega)\tau_0$ are normalised detunings; ω_c is the natural frequency of the laser cavity at $g = 0$; ω_0 is the whispering-gallery mode frequency; K_1 and K_2 are the coupling coefficients of the fields in the laser and microresonator; and k is the coefficient of coupling between the whispering-gallery modes due to scattering.

The system of equations (1) for $F_\pm = 0$ is a standard system for studying the dynamics of a semiconductor laser [22]. For $F_\pm \neq 0$, the model describes the dynamic coupling of the laser to an external resonator. Models of this type have been widely investigated (see [1–6] and references therein). The model presented here differs from those studied earlier in that a stabilising element is a two-mode system and the coupling level with a microsphere affects the Q factor of the latter.

Let us introduce the real amplitudes and phases

$$E = A\exp[\mathrm{i}(\varphi - \omega\tau_1)],\ F_- = B\exp(\mathrm{i}\psi),\ F_+ = C\exp(\mathrm{i}\xi) \quad (2)$$

and rewrite equations (1) in the real form

$$2\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{1}{\tau}[1-g(n)]A = K_1B(t-\tau_1) \times\cos(\psi - \varphi + 2\omega\tau_1), \quad (3a)$$

$$2A\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{1}{\tau}[\varDelta - \alpha g(n)]A = K_1B(t-\tau_1) \times\sin(\psi - \varphi + 2\omega\tau_1), \quad (3b)$$

$$2\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{1}{\tau_0}B = -kC\sin(\xi - \psi), \quad (3c)$$

$$2B\frac{\mathrm{d}\psi}{\mathrm{d}t} + \frac{\delta}{\tau_0}B = kC\cos(\xi - \psi), \quad (3d)$$

$$2\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{1}{\tau_0}C = -kB\sin(\psi - \xi) + K_2A(t-\tau_1)\cos(\varphi - \xi), \quad (3e)$$

$$2C\frac{\mathrm{d}\xi}{\mathrm{d}t} + \frac{\delta}{\tau_0}C = kB\cos(\psi - \xi) + K_2A(t-\tau_1)\sin(\varphi - \xi), \quad (3f)$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} + \frac{1}{\tau_s}n = J - g(n)\frac{A^2}{8\pi\eta\omega}. \quad (3g)$$

Then, the stationary regimes of the system will be determined by the equations

$$\frac{1}{\tau}[1-g(n)]A = K_1B\cos(\psi - \varphi + 2\omega\tau_1), \quad (4a)$$

$$\frac{1}{\tau}[\varDelta - \alpha g(n)]A = K_1B\sin(\psi - \varphi + 2\omega\tau_1), \quad (4b)$$

$$\frac{1}{\tau_0}B = -kC\sin(\xi - \psi), \quad (4c)$$

$$\frac{\delta}{\tau_0}B = kC\cos(\xi - \psi), \quad (4d)$$

$$\frac{1}{\tau_0}C = -kB\sin(\psi - \xi) + K_2A\cos(\varphi - \xi), \quad (4e)$$

$$\frac{\delta}{\tau_0}C = kB\cos(\psi - \xi) + K_2A\sin(\varphi - \xi), \quad (4f)$$

$$\frac{1}{\tau_s}n = J - g(n)\frac{A^2}{8\pi\eta\omega}. \quad (4g)$$

By introducing the notation $\kappa \equiv k\tau_0$, $P \equiv K_1\tau$, $Q \equiv K_2\tau$, $\theta \equiv \tau_0/\tau$, we find from equations (4c–f)

$$\sin(\xi - \psi) = -\frac{1}{(1+\delta^2)^{1/2}},\ \cos(\xi - \psi) = \frac{\delta}{(1+\delta^2)^{1/2}}, \quad (5)$$

$$\sin(\varphi - \xi) = \delta\frac{1+\delta^2-\kappa^2}{\{(1+\delta^2)[(\delta^2-1-\kappa^2)^2+4\delta^2]\}^{1/2}},$$
$$\cos(\varphi - \xi) = \frac{1+\delta^2+\kappa^2}{\{(1+\delta^2)[(\delta^2-1-\kappa^2)^2+4\delta^2]\}^{1/2}}, \quad (6)$$

$$\frac{B(\delta)^2}{A^2} = \frac{(\kappa Q)^2\theta^2}{(\delta^2-1-\kappa^2)^2+4\delta^2}, \quad (7)$$

$$\frac{C(\delta)^2}{A^2} = Q^2\theta^2\frac{1+\delta^2}{(\delta^2-1-\kappa^2)^2+4\delta^2}, \quad (8)$$

$$\varDelta - \alpha = -P\frac{B(\delta)}{A}[\alpha\cos(\varphi - \psi - 2\omega\tau_1) + \sin(\varphi - \psi - 2\omega\tau_1)]. \quad (9)$$

Expression (7) describes the resonance properties of wave B [hereafter, the waves are denoted as in (2)]. It follows from (7) that for the coupling parameter $\kappa > 1$, the resonance curve of the wave has two maxima and one minimum. The position of the maxima are determined by the relation

$$\delta = \pm(\kappa^2 - 1)^{1/2}, \tag{10}$$

and the minimum is located at the point $\delta = 0$. The parameter $\kappa = 1$ is critical: all the three extrema combine to give one maximum. When the coupling parameter κ further decreases, the resonance curve remains a single-humped curve (Fig. 2). These circumstances are well known from the theory of two-contour coupled resonance systems, which are widely used in classical radio engineering.

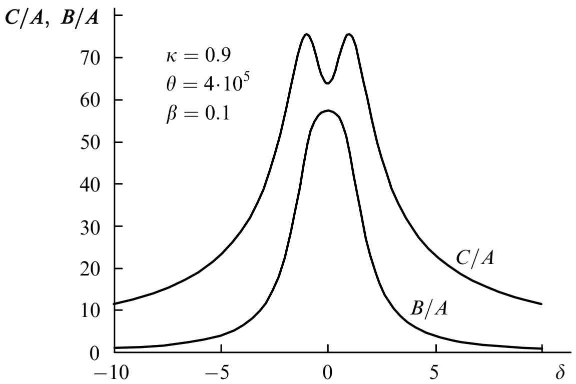

Figure 2. Resonance curves for forward (C) and backward (B) whispering-gallery waves normalised to the exciting-wave amplitude A.

The resonance curve of the wave C is described by expression (8). It also has two maxima and the minimum, whose positions are determined by the relations

$$\delta = \pm[\kappa(\kappa^2 + 4)^{1/2} - 1]^{1/2}, \quad \delta = 0. \tag{11}$$

For $\kappa = (\sqrt{5} - 2)^{1/2} \approx 0.486$, all the three extrema combine to give one maximum. As the parameter κ further decreases, the resonance curve remains a single-humped one. Therefore, the critical values of the parameter κ are different for the direct (F_+) and backward (F_-) waves.

Fig. 2 shows the resonance curves for waves B and C, which are normalised to the amplitude of the exciting wave A for $\kappa = 0.9$. For this value of κ, the resonance curve of the wave B is approximately by a factor of 1.8 narrower than the resonance curve of the wave C, the curve B being single-humped, whereas the curve C is two-humped. This circumstance is usually ignored in the literature on two-contour systems. Below, we will restrict ourselves by the case when the backward wave B is used for capturing (stabilisation) of the laser.

The Q factor of the loaded microresonator cannot exceed some maximum value $Q_{\max}$, which is determined by intrinsic losses [23]. The Q factor of the loaded cavity is determined by the relation

$$\frac{1}{Q} = \frac{1}{Q_{\max}} + \frac{1}{Q_{\rm c}}, \tag{12}$$

where $Q_{\rm c}$ is 'the load Q factor', which increases infinitely with decreasing coupling with the input element. The Q-factor components introduced above are characterised by the photon lifetimes τ_{00} and $\tau_{\rm c}$ in an unloaded microresonator and in a loaded resonator without intrinsic losses, respectively, according to

$$Q = \omega\tau_0, \quad Q_{\max} = \omega\tau_{00}, \quad Q_{\rm c} = \omega\tau_{\rm c}. \tag{13}$$

The dimensionless parameter $\kappa = k\tau_0$ introduced above (we will call it the reduced Q factor) characterises the ratio of the rates of Rayleigh scattering between the modes to the total losses in the microresonator. Similarly, we introduce the parameter $\kappa_0 = k\tau_{00}$ (the intrinsic reduced Q factor) ($\kappa = Q/Q_{\rm R} \equiv \omega\tau_0/\omega k^{-1}$, where $Q_{\rm R} = \omega k^{-1}$). Microresonators made of a pure quartz studied in papers [20, 21] have the parameter κ_0 equal to 1.07 and 1.1, respectively. The coupling parameter k for the waves was 2–3 MHz.

The shadowed region IV in Fig. 3 ($k > \kappa_0$) corresponds to the nonexisting values of the reduced Q factor , while the region I corresponds to the supercritical values of the Q factor. In the region III, the condition $k < 1$ is always fulfilled and the critical coupling cannot be achieved. In other words, no splitting of the spectrum can be observed in a low-Q resonator or in the case of weak Rayleigh scattering. The region II corresponds to the subcritical values of the microresonator Q factor.

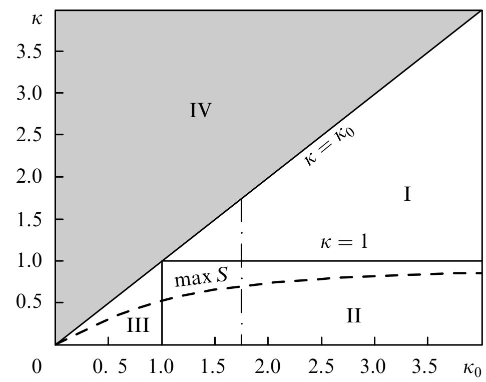

Figure 3. Diagram of reduced Q factors: (I) supercritical region; (II) subcritical region; (III) region where the critical regime is absent; (IV) region of nonexisting values of the reduced Q factor. The vertical dashed-dotted straight line is the load characteristic of the microresonator with $\kappa_0 = 1.75$; the dashed curve is the dependence of optimal values of κ on κ_0.

The dependence of the Q factor of a microsphere with the intrinsic Q factor κ_0 on the distance from the microsphere to the input prism corresponds to the load characteristic (12). The dashed-dotted vertical straight line in Fig. 3 shows the load characteristic of the microresonator with $\kappa_0 = 1.75$.

The degree of the microresonator load in real experiments is controlled by the distance d between the microresonator and a matching total internal reflection prism. It is known that the energy density of the mode field outside a sphere decreases with the distance from its surface as $\exp[-4\pi d(n_{\rm q}^2 - 1)^{1/2}/\lambda]$, where $n_{\rm q}$ is the refractive index of quartz. Therefore, it is reasonable to assume that

$$Q_{\rm c} = Q_{\rm c\,min} \exp\left[\frac{4\pi d(n_{\rm q}^2 - 1)^{1/2}}{\lambda}\right], \tag{14}$$

where a constant $Q_{\mathrm{c\,min}}$ is the minimum load Q factor of touching sphere and prism. This expression for a loaded Q factor was obtained in paper [23].

3. Stabilisation parameter

Expression (9) determines the frequency of stationary lasing of a laser with a stabilising cavity, while the relation $\varDelta - \alpha = 0$ gives the lasing frequency in the absence of coupling to an external resonator: $\omega_{\mathrm{L}} = \omega_{\mathrm{c}} - \alpha/\tau$. The optical feedback results in the deviation of the lasing frequency ω of the system from that in the free-running mode. When the optical-feedback phase is properly chosen, the lasing frequency is locked by the external resonator frequency. The degree of frequency locking is determined by the stabilisation parameter

$$S(\delta, \kappa) = \frac{\mathrm{d}\omega_{\mathrm{c}}}{\mathrm{d}\omega} = 1 - \frac{\tau_0}{\tau}\frac{\mathrm{d}\varDelta}{\mathrm{d}\delta}. \tag{15}$$

If the free-running lasing frequency ω_{c} changes by the value $\varepsilon < \Delta\Omega$, i.e., it remains within the stabilisation region $\Delta\Omega$ (Fig. 4), then the frequency ω of the stabilised laser will deviate only by ε/S.

When the lasing frequency varies within the microresonator resonance $\omega_0 \pm \pi/2\tau_0$, the phase of the backward wave changes by the value of the order of unity. The phase shift $2\omega\tau_1$ entering expression (9) gives the addition of the order of τ_1/τ_0. When the resonator Q factor exceeds 10^8 and the distance between the sphere and laser is less than 10 cm, this addition does not exceed 1 %, so that we will consider this phase a constant.

Expression (9) becomes cumbersome after the substitution of the values of trigonometric functions. Consider the case $2\omega\tau_1 = -\arctan(1/\alpha) + \pi n$, $n = 0, 1, 2, \ldots$. The choice of this phase delay provides the optimum influence of the in-phase and quadrature components of the feedback signal (see Section 4). In this case, the stabilisation curve $\delta(\varDelta_{\mathrm{L}})$ is antisymmetric, while the stabilisation parameter is a symmetric function of the detuning, with an extremum at $\delta = 0$. Using the results obtained in Appendix and also (12) and (13), we obtain

$$\varDelta_{\mathrm{L}} = -2\beta(\alpha^2+1)^{1/2}\frac{\kappa\delta}{(\delta^2-1-\kappa^2)^2+4\delta^2}\left(1-\frac{\kappa}{\kappa_0}\right), \tag{16}$$

where $\varDelta_{\mathrm{L}} \equiv (\omega_{\mathrm{c}} - \alpha/\tau - \omega)\tau_0$; β is a parameter of geometrical matching [see (A7)]. In this case, the stabilisation parameter is

$$S(\delta,\kappa) = 1 + 2\beta(\alpha^2+1)^{1/2}\theta\kappa\left(1-\frac{\kappa}{\kappa_0}\right) \times \frac{(\delta^2-1-\kappa^2)^2 - [4(\delta^2-1-\kappa^2)+1]\delta^2}{\left[(\delta^2-1-\kappa^2)^2+4\delta^2\right]^2}. \tag{17}$$

Fig. 4 shows the dependences $\omega(\omega_{\mathrm{c}})$ and $S(\omega_{\mathrm{c}})$ described by expressions (16) and (17) for $\kappa_0 = 1.5$. The parameters β and α in this case and below are 0.1 and 4, respectively.

The stabilisation parameter for the zero detuning

$$S(\delta=0,\kappa) = 1 + 2\beta(\alpha^2+1)^{1/2}\left(1-\frac{\kappa}{\kappa_0}\right)\frac{\kappa\theta}{(1+\kappa^2)^2} \tag{18a}$$

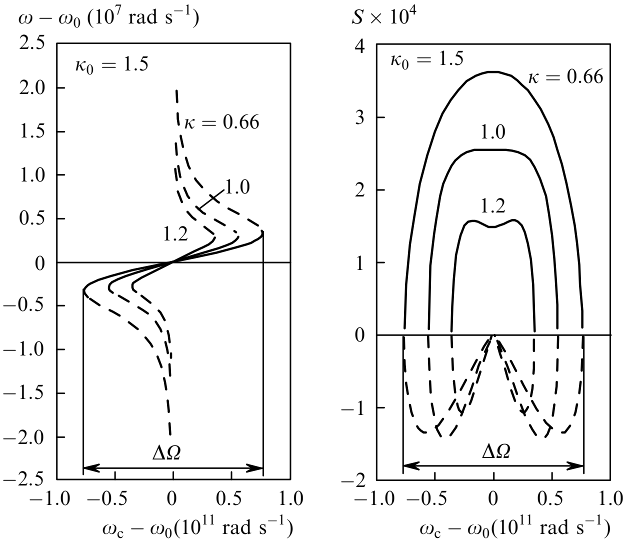

Figure 4. Stabilisation curve $\omega(\omega_{\mathrm{c}})$ and the stabilisation parameter $S(\omega_{\mathrm{c}})$ for the microresonator with $\kappa_0 = 1.5$ for different loading conditions: optimal over the stabilisation parameter ($\kappa = 0.66$), critical ($\kappa = 1.0$), and supercritical ($\kappa = 1.2$); $\theta \approx 4 \times 10^5$, $\beta = 0.1$, $\alpha = 4$, $\tan(2\omega\tau_1) = -1/\alpha$; $\Delta\Omega$ is the stabilisation region.

reaches the maximum at the point that is the only real solution of the cubic equation $\kappa^3 - 2\kappa_0\kappa^2 - 3\kappa + 2\kappa_0 = 0$ in the interval $\kappa \subset [0, \kappa_0]$. The curve $\max S$ in Fig. 3 presents the dependence of the optimal values of κ on κ_0.

Fig. 5 presents the parametric family of the dependences of the stabilisation parameter $S(\delta = 0, \kappa)$ on the distance d between the sphere and the resonator for different κ_0. The load Q factor was described by expression (14). The asterisks mark the points at which the stabilisation parameter reaches the maximum for the zero delay $S_{\max}(\delta = 0, \kappa_0)$ in the given resonator. The circles mark the points at which the critical coupling is achieved, and the stabilisation parameter is

$$S_{\mathrm{cr}}(\delta=0,\kappa=1) = 1 + \frac{\beta(\alpha^2+1)^{1/2}}{2k\tau}\left(1-\frac{1}{\kappa_0}\right) < S_{\max}(\delta=0). \tag{18b}$$

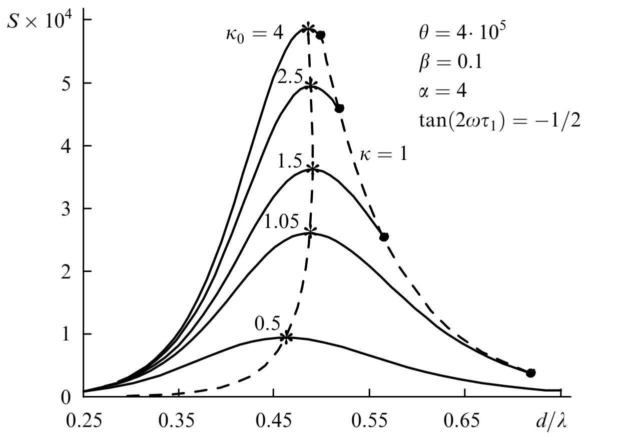

Figure 5. Dependences of the stabilisation parameter S at the zero detuning on the distance d between the matching prism and microresonator for different Q factors of the microresonator.

It can be shown that the stabilisation parameter reaches the maximum in the subcritical region for microresonators with any intrinsic reduced Q factor λ_0 (Figs. 3 and 5). For distances exceeding the critical distance ($\kappa > 1$), the dependence of the stabilisation parameter $S(\delta, \kappa)$ on the detuning becomes two-humped, in accordance with the spectral splitting of the backward wave B (Fig. 4). In this case, expression (18a) no longer describes the maximum value of S. Nevertheless, the value of S in the maxima of spectral components (10) does not exceed $S_{\max}(\delta = 0, \kappa_0)$ and rapidly decreases with increasing distance.

Such a dependence of the stabilisation parameter on the distance can be simply explained. When the distance between the prism and sphere is small, the Q factor of the sphere is also small, whereas at large distances, it is close to maximum; however, the coupling with the laser becomes weaker. Therefore, there exists an optimal distance at which the value of S is maximum. Note (see Fig. 5) that this optimal distance does not virtually change over a comparatively broad range of intrinsic Q factors and has a weak maximum.

Stabilisation not only results in the laser-frequency locking to the resonator frequency ω_0 but also reduces the frequency fluctuations in a stabilised laser compared to those in a free-running mode, resulting in the narrowing of the laser line. The square of the stabilisation parameter characterises the narrowing of the feedback-laser line compared to the line in the absence of stabilisation (for the case of white frequency noise). For $\beta \approx 0.1$ and the external resonator Q factor of $\sim 10^8 - 10^9$, the stabilisation parameter can be as high as $\sim 10^4 - 10^5$. However, the stability of the stabilisation regime restricts the possible value of this parameter.

4. Stability of the locking regime

Stability of the locking regime of the laser by a single-mode high-Q resonator was analysed in papers [11, 12]. The stationary lasing proved to be stable if

$$\frac{S}{\tau_0} < \Omega_{\rm r}\,, \tag{19}$$

where $\Omega_{\rm r}$ is the frequency of the relaxation pulsations of the laser [24].

This condition is fulfilled when the laser frequency is equal to the resonance frequency of a reference interferometer (in our case, to the frequency of a whispering-gallery mode of the sphere) and the phase shift $2\omega\tau_1$ provides the optimal suppression of frequency fluctuations. Upon narrowing of the laser line in the subcritical regime of interest, the resonance remains single-humped and the condition (19) determines the maximum coupling level. However, in the experiment [25] with a spherical microresonator, whose distance from the laser was reduced to ~ 1 cm, it was found that the admissible coupling level can be greater than it follows from (19). This results in a noticeable change in the lasing threshold and in the appearance of the additional selectivity of a composite resonator. We will show below that the mechanism of instability caused by the amplitude–phase coupling in an amplifying medium [11, 12] can be 'switched off'.

We will restrict ourselves to the case of zero detuning, when the laser frequency coincides with the resonance frequency of a whispering-gallery mode. First, following [11, 12], we explain qualitatively the limitation of the coupling level (19). The field in the laser is a sum of the intrinsic field E and the field Φ_-, which returns to the laser from the external resonator. Upon tuning to the resonance maximum, the phase of the field Φ_- is shifted relative to the phase of the intrinsic field by $2\omega\tau_1 - \pi/2$. Fluctuations of the laser frequency cause only fluctuations of the phase of the field Φ_- relative to the phase of the intrinsic field E, whereas the amplitude fluctuations of the field Φ_- in the first approximation are absent. Under the condition $2\omega\tau_1 = 2\pi n$, the field Φ_- is in quadrature to the intrinsic field. Phase fluctuations of the filed Φ_- in the first approximation are transformed to the amplitude fluctuations of the total field. However, because of the amplitude–phase coupling, these fluctuations are transformed to phase fluctuations with the conversion coefficient α. Thus, due to the feedback, noise fluctuations of the laser frequency cause fluctuations of its phase and, hence, frequency. When the transmission resonance was used, the feedback sign was negative, i.e., secondary fluctuations suppressed initial fluctuations. Note that such a suppression of the frequency noise is accompanied by the increase in the amplitude noise.

The mechanism of fluctuation suppression described above is analogous to the operation of the AFC system. The active region of the laser plays the role of a controlling element, which transforms fluctuations of the field amplitude to the fluctuations of the field phase. This controlling element has its own resonance at the frequency of relaxation oscillations, and when the gain (the coupling level) increases, the system is self-excited at the relaxation-oscillation frequency. The condition (19) reflects this 'relaxation α-instability'.

If the phase shift between the microresonator and the laser satisfies the condition $2\omega\tau_1 = 2\pi n + \pi/2$, then the field Φ_- is on average in-phase with the field E. Phase fluctuations of the field Φ_- caused by the frequency noise of the laser produce, in the first approximation, only phase fluctuations of the total field. In the case of in-phase configuration of the fields, a secondary change in the phase reduces its first change, and the frequency noise is suppressed. It is important that the amplitude–phase trans-formation is not involved in this process, the factor α does not give any contribution, and the restriction (19) is not valid.

The model described above gives for $2\omega\tau_1 = 2\pi n$

$$S \approx 2\beta\left(1 - \frac{\kappa}{\kappa_0}\right)\frac{\alpha\kappa\theta}{(1+\kappa^2)^2} < \Omega_{\rm r}\tau_0. \tag{20}$$

If $2\omega\tau_1 = 2\pi n + \pi/2$, then

$$S \approx 2\beta\left(1 - \frac{\kappa}{\kappa_0}\right)\frac{\kappa\theta}{(1+\kappa^2)^2} < \Omega_{\rm r}\tau_0. \tag{21}$$

Note that for the specified coupling level β, the maximum value of S is obtained in the case of the optimum combination of contributions from the in-phase and quadrature components $2\omega\tau_1 = -\arctan(1/\alpha)$ considered above, when the amplitude – phase coupling and the condition (19) are manifested most strongly.

Thus, the feedback phase determines the stability of the stabilisation regime (the feedback sign) and the mechanism of its violation. When the condition $2\omega\tau_1 = -\arctan(1/\alpha)$ is fulfilled, the maximum feedback level and the optimised stabilisation parameter S are limited. On the other hand, an appropriate choice of the feedback phase can exclude the amplitude–phase correlation, thereby removing the requirement (19). This opens up the possibility of combining a high Q factor of the resonator and a strong feedback, i.e., to obtain even greater values of S and the maximum noise suppression provided by the optical method.

The use of common confocal interferometers results in a great distance between them and the laser. The unavoidable drift of this distance changes the feedback phase and destroys the stabilisation regime. For this reason, the distance is additionally stabilised. The working point upon such stabilisation does not correspond to the in-phase condition, and the condition (19) remains valid.

A small distance between the laser and the microresonator in paper [25] allowed the use of passive stabilisation of the phase incursion $2\omega\tau_1$. The system was 'self-determined' in the choice of the feedback phase. In this case, we managed to obtain stable lasing when the feedback level was much higher than the condition (19) admits. We can show within the framework of the model described above that in the case of nonzero detuning of the system frequency within the resonance of the whispering-gallery mode, the feedback phase can be also chosen so that the mechanism of relaxation α-instability will be significantly weakened in some vicinity of the phase.

5. Conclusions

Thus, the model presented in this paper describes the splitting of the whispering-gallery mode spectrum. In microspheres made of a high-quality silica, the splitting of the forward wave in the resonator occurs at the resonator Q factor exceeding $(0.4-0.6)\times10^9$; for the backward wave, the splitting occurs when the Q factor exceeds $(0.8-1.2)\times10^9$. We estimated the parameter of stabilisation of the semiconductor-laser frequency by a highly coherent radiation of the backward wave of the whispering-gallery mode. The maximum stabilisation parameter $\sim 10^4$ is achieved for the subcritical resonator load, i.e., for the Q factor lower than the splitting threshold in the backward wave. The maximum value of the stabilisation parameter in the case of the traditional choice of the phase is restricted by the relaxation α-instability. However, for the specified lasing frequency of the system near the resonance of the whispering-gallery mode, there exists the range of values of the optical feedback phase in which this mechanism of instability is strongly weakened. Therefore, it is possible for the first time to combine a high Q factor of an external interferometer and a strong coupling with it.

The model considered above ignores the shift of the microresonator resonance caused by variations in temperature, pressure, and a change in the gap between the microsphere and the prism (the proximity of a prism with another refractive index changes the effective refractive index in the matching region). We will consider these questions and the stability in the case of a strong feedback in our next studies.

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Appendix

We will determine the real coupling coefficient K_1 in the stationary case, when (a) resonance losses in the diode-laser resonator are compensated by the gain, i.e., $g = 0$, and (b) a wave with the amplitude F_- is excited in the microresonator, and the energy of this wave enters the laser resonator. Then, we obtain from the stationary equation (1a) under the condition (a) and zero detuning $\Delta = 0$

$$|E| = K_1\tau|F_-|. \tag{A1}$$

On the other hand, the condition (b), which expresses the energy conservation in the system, gives the equality

$$\gamma^2\,\frac{|F_-|^2}{8\pi}\,\frac{V_0}{\tau_c} = \frac{|E|^2}{8\pi}\,\frac{V_L}{\tau}, \tag{A2}$$

where V_0 is the volume of the microresonator mode; V_L is the volume of the laser mode; γ is a coefficient that takes into account the transmission of matching optics and the overlap of mode fields of the laser and microresonator in the matching prism; and τ_c is the photon lifetime [see expressions (12) and (13)]. It follows from (A1) and (A2) that

$$K_1 = \gamma\left(\frac{V_0}{V_L}\right)^{1/2}\frac{1}{(\tau\tau_c)^{1/2}}. \tag{A3}$$

To determine K_2, we consider excitation of a microresonator mode by the field E generated in the laser. We will find the field amplitude E_0 in the laser input spot of cross section S_c on the prism hypotenuse from the equality of the output laser power and the incident-beam power in the input spot:

$$\vartheta\gamma^2\,\frac{|E|^2}{8\pi}\,\frac{V_L}{\tau} = \frac{|E_0|^2}{8\pi}\,S_c c, \tag{A4}$$

where ϑ is the efficiency of laser radiation coupling, which takes into account reflectivities of laser mirrors and distributed losses; and c is the speed of light. The expression for the coefficient of coupling K_2' between the incident-field amplitude E_0 and the wave-amplitude F_+ was obtained for the model of a ring resonator [26]. It follows from expressions (3) and (4) of paper [26] that (in our notation)

$$K_2' \approx \frac{1}{(\tau_c\tau_0')^{1/2}}, \tag{A5}$$

where τ_0' is the round-trip transit time in the microresonator.

From expressions (A4) and (A5) for K_2 and the product of coupling coefficients, we have

$$K_2 = K_2'\,\frac{|E|}{|E_0|} = \gamma\left(\frac{\vartheta\tau_c'}{\tau_0'}\right)^{1/2}\left(\frac{V_L}{V_0}\frac{1}{(\tau\tau_c)^{1/2}}\right),\quad \tau_c' = \frac{V_0}{S_c c}, \tag{A6}$$

$$K_1K_2 = \frac{\beta}{\tau\tau_c},\quad \beta = \gamma\,2\left(\frac{\vartheta\tau_c'}{\tau_0'}\right)^{1/2}. \tag{A7}$$

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