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Optimal control of optical soliton parameters: Part 1. The Lax representation in the problem of soliton management

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***Abstract*.** **The existence of the Lax representation for a model of soliton management under certain conditions is shown, which proves a complete integrability of the model. The exact analytic solutions are obtained for the problem of the optimal control of parameters of Schrödinger solitons in nonconservative systems with the group velocity dispersion, nonlinear refractive index, and gain (absorption coefficient) varying over the length. The examples demonstrating the nontrivial amplification dynamics of optical solitons, which are important from practical point of view, are considered. The exact analytic solutions are obtained for problems of the optimal amplification of solitons in optical fibres with monotonically decreasing dispersion and of Raman pumping of solitons in fibreoptic communication systems.**

Keywords: *optical solitons, nonlinear waves, soliton management.*

1. Introduction

The problem of control of the parameters of optical solitons has a long history. Mollenauer, Stolen, and Gordon have shown in their pioneering experiments [1] that one of the simplest methods of control of the parameters of optical solitons is a multisoliton compression followed by the spectral filtration of ultrashort emission fragments [1 – 4]. By using cascade schemes for compression of N-soliton pulses in two fragments of silica fibres with substantially different anomalous dispersions, the authors of paper [5] obtained 18 – 19-fs wave packets of extremely short duration, whose envelope contains only three optical cycles, and which are the shortest wave packets produced up to now.

Optical fibres with the dispersion characteristic continuously varying over the length (the dependence of the total dispersion of the fibre on the wavelength [6]) fabricated at the General Physics Institute, RAS, simulated the development of efficient methods for the adiabatic compression of solitons and opened the possibility for the building of generators of high-frequency trains of optical solitons for fibreoptic communication [6 – 8]. The use of various combinations of optical fibres with alternating dispersion signs (fibre dispersion management) resulted in the development of soliton wavelength-division-multiplexed communication systems with a bit rate of 40 Gbit s^{-1}.

The problem of optimal control of the parameters of optical solitons as ideal carriers of a data bit, which is also called the problem of soliton management, is at present one of the key problems. A detailed analysis of the state of the art of experiments and theory in this rapidly developing field of science and technology can be found in book [9]. In this book, the studies of all research groups and companies playing a leading role in the development of methods for data transmission using optical solitons are considered.

A mathematical problem of soliton management is a search for soliton-like solutions of a nonlinear Schrödinger equation (NSE) in a closed or open line with parameters variable over the length, such as the group velocity dispersion, nonlinearity of the refractive index, coefficients of nonlinear losses of radiation and of periodical amplification of solitons in communication systems.

In this paper, we prove that the NSE model with variable coefficients is completely integrable under certain conditions. The solutions found in the form of soliton pulses with a nontrivial law of the phase variation and having a canonical form represented by hyperbolic secant or tangent exist only in the case of a certain relation between the parameters of a nonlinear system and a soliton being channelled. These properties stimulate the technological development and fabrication of new optical fibres with a specified law of variation of basic parameters over the fibre length.

Our main purpose is to initiate new experimental studies. This determines the character of representation of basic results, which allows a reader to repeat easily the corresponding calculations and to select the so-called map of parameters for each specific experiment. The specificity of the results obtained is that an open infinite set ('ocean') of new soliton solutions of the NSE model allows one to calculate easily the required parameters of the problem if, for example, the dispersion variation over the fibre length is known from the experiment. In this case, the main problem of the experiment is to be as close as possible to the optimal map of parameters at which the problem of soliton management, as shown below, proves to be exactly integrable.

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This paper consists of two parts. In the first part, the Lax representation is obtained for the soliton management model and the conditions are found under which the problem is completely integrable. In the second part, the nonlinear Bloch theorem is obtained for optical solitons and the stability of N-soliton Bloch waves (coupled states) is analysed in a periodically nonlinear and dispersion system.

2. The Lax representation in the soliton management model

A modern progress in the theory of nonlinear waves is caused first of all by the development and application of the inverse scattering transform, which is called after its creators the spectral problem method of Zakharov – Shabat [10] or of Ablowitz – Kaup – Newell – Segur [11]. It was proved that, if the so-called Lax representation [12] is found for a nonlinear wave equation, then the solution for this equation can be found by the inverse scattering transform (see, for example, [12 – 20] and references cited in [13 – 20]). The existence of the Lax representation proves the complete integrability of the problem under study.

It should be emphasised that the problem of classification of completely integrable equations has a rather long history. The problems considered below have been discussed in detail in papers [13 – 19].

A general algebraic formulation of the problem is as follows. We will attempt to write the NSE with variable coefficients as a condition for the integrability of a pair of linear equations. Let us represent two linear differential operators, which depend on the spectral parameter λ, in the form

$$\hat{\mathscr{L}}(\lambda) = \frac{\partial}{\partial x} - \hat{L}\left(\lambda,\, q,\, \frac{\partial q}{\partial x},\, \frac{\partial^2 q}{\partial x^2}, \ldots,\, \frac{\partial^n q}{\partial x^n}\right), \tag{1}$$

$$\hat{\mathscr{T}}(\lambda) = \frac{\partial}{\partial t} - \hat{A}\left(\lambda,\, q,\, \frac{\partial q}{\partial x},\, \frac{\partial^2 q}{\partial x^2}, \ldots,\, \frac{\partial^n q}{\partial x^n}\right), \tag{2}$$

where $\hat{L}$ and $\hat{A}$ are the $n \times n$ matrices and $q(x, t)$ is the required potential. For the complex n-component vector function $\boldsymbol{\psi}(x, t) = \{\psi_1, \ldots, \psi_n\}$, differential operators (1) and (2) satisfy the system of equations

$$\hat{\mathscr{L}}(\lambda)\boldsymbol{\psi}(x, t) = 0\,, \tag{3}$$

$$\hat{\mathscr{T}}(\lambda)\boldsymbol{\psi}(x, t) = 0\,. \tag{4}$$

Equation (3) is known as the spectral problem for the operator $\hat{\mathscr{L}}(\lambda)$, while equation (4) determines the time evolution of scattering for $t > 0$. Note that hereafter in this section we use the space – time variables x and t, as accepted in quantum-mechanical problems of scattering by the potential $q(x, t)$, while the formulation of the problem itself has deep quantum-mechanical analogues [13]. On passing to the soliton management problem, we use the coordinate representation, which is commonly accepted in problems of optical solitons.

The integrability (compatibility) condition for a pair of linear equations (3) and (4) has the form

$$\left[\hat{\mathscr{L}}(\lambda), \hat{\mathscr{T}}(\lambda)\right] = 0 \tag{5}$$

and means that the operators $\hat{\mathscr{L}}(\lambda)$ and $\hat{\mathscr{T}}(\lambda)$ are commutative operators. In the matrix form, equation (5) is written as

$$\frac{\partial \hat{L}}{\partial t} - \frac{\partial \hat{A}}{\partial x} + \left[\hat{L}, \hat{A}\right] = 0. \tag{6}$$

Equation (6) should be satisfied for any values of the spectral parameter λ and it is known as the Lax representation or the $L - A$ pair determining a system of equations for the scattering potential $q(x, t)$.

Below, we will consider the construction of the Lax pair in a more general case, taking into account the fact that variables in the potential $q(x, t)$ can be complicated, for example, mutually dependent functions. We will show below that the Lax representation allows one to obtain an heuristic 'prompt' as to what form a general solution of the problem for solitons should be sought in media with variable parameters. By making the change $x \to s(x, t)$, $t \to t$, we pass in (1), (2) to the complicated function $q[s(x, t), t]$ of configurational variables and take the matrices $\hat{L}$ and $\hat{A}$ in a standard form

$$\hat{L} = \begin{bmatrix} -\mathrm{i}\lambda & q(s, t) \\ r(s, t) & \mathrm{i}\lambda \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}. \tag{7}$$

The substitution of (7) into matrix equation (6) gives equations for potentials $q(s, t)$ and $r(s, t)$ as functions of the complicated argument $s(x, t)$:

$$\begin{aligned} &\frac{\partial q}{\partial t} + \frac{\partial q}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial B}{\partial s}\frac{\partial s}{\partial x} + 2qA + 2\mathrm{i}B\lambda\,, \\ &\frac{\partial r}{\partial t} + \frac{\partial r}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial C}{\partial s}\frac{\partial s}{\partial x} - 2rA - 2\mathrm{i}C\lambda\,, \\ &\frac{\partial A}{\partial s}\frac{\partial s}{\partial x} = qC - rB\,. \end{aligned} \tag{8}$$

By using a standard procedure for a search for the Lax pair for the usual NSE (with constant coefficients), we represent the functions $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ as polynomials of degree λ:

$$\begin{aligned} &A = A_0 + A_1\lambda + A_2\lambda^2 + \ldots, \\ &B = B_0 + B_1\lambda + B_2\lambda^2 + \ldots, \\ &C = C_0 + C_1\lambda + C_2\lambda^2 \ldots\,. \end{aligned} \tag{9}$$

To obtain the NSE, it is sufficient to consider expansions in (9) only up to quadratic terms in λ. The Korteweg – de Vries equation (together with the NSE) is obtained when cubic terms are taken into account in the expansion of matrix elements in the spectral parameter λ. To obtain a system of evolution equations for the components $q(s, t)$ and $r(s, t)$ of the potential, we substitute (9) into (8) and collect similar terms:

$$\frac{\partial A_0}{\partial s}\frac{\partial s}{\partial x} = qC_0 - rB_0,$$

$$\frac{\partial A_1}{\partial s}\frac{\partial s}{\partial x} = qC_1 - rB_1,$$

$$\frac{\partial A_2}{\partial s}\frac{\partial s}{\partial x} = 0, \tag{10}$$

$$\frac{\partial B_1}{\partial s}\frac{\partial s}{\partial x} = -2qA_1 - 2\mathrm{i}B_0\,,$$

$$\frac{\partial C_1}{\partial s}\frac{\partial s}{\partial x} = 2rA_1 + 2\mathrm{i}C_0\,,$$

$$qA_2 + \mathrm{i}B_1 = 0\,,$$

$$rA_2 + \mathrm{i}C_1 = 0\,.$$

Equations for the potential components $q(s,t)$ and $r(s,t)$ directly follow from (8) and (9):

$$\frac{\partial q}{\partial t} + \frac{\partial q}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial B_0}{\partial s}\frac{\partial s}{\partial x} + 2qA_0\,,$$

$$\frac{\partial r}{\partial t} + \frac{\partial r}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial C_0}{\partial s}\frac{\partial s}{\partial x} - 2rA_0\,. \tag{11}$$

We should find the conditions under which the system of equations (11) is reduced to one NSE with variable coefficients (we will call it the generalised NSE). From (10), we obtain the expressions for matrix elements

$$A_0 = \frac{1}{2}A_2 qr + a_0(t)\,,$$

$$B_0 = \mathrm{i}A_1 q - \frac{1}{2}A_2\frac{\partial q}{\partial s}\frac{\partial s}{\partial x},$$

$$C_0 = \mathrm{i}A_1 r + \frac{1}{2}A_2\frac{\partial r}{\partial s}\frac{\partial s}{\partial x}, \tag{12}$$

$$B_1 = \mathrm{i}A_2 q\,,$$

$$C_1 = \mathrm{i}A_2 r\,.$$

As a result, system (11), after the reduction [16]

$$r = -\beta q^* \tag{13}$$

and under the 'strict' conditions

$$A_1^*(t) = -A_1(t),\ A_2^*(t) = -A_2(t),\ a_0^*(t) = -a_0(t) \tag{14}$$

imposed on the matrix elements, leads to the completely integrable generalised NSE model

$$\mathrm{i}\frac{\partial q}{\partial t} = \frac{1}{2}\left[D_2(t)\left(\frac{\partial s}{\partial x}\right)^2\right]\frac{\partial^2 q}{\partial s^2} + N_2(t)|q|^2 q$$

$$-\mathrm{i}\left[V(t)\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t}\right]\frac{\partial q}{\partial s} - 2\Gamma(t)q \tag{15}$$

with time-dependent coefficients

$$A_1(t) = \mathrm{i}V(t),\ A_2(t) = \mathrm{i}D_2(t),\ a_0(t) = \mathrm{i}\Gamma(t),$$

$$N_2(t) = \beta D_2(t)\,, \tag{16}$$

where the functions $V(t)$, $D_2(t)$, $N_2(t)$, and $\Gamma(t)$ are the real functions of the variable t.

Therefore, the Lax representation for the soliton management model has the form

$$\hat{L} = \begin{bmatrix} -\mathrm{i}\lambda & q \\ -\beta q^* & \mathrm{i}\lambda \end{bmatrix}, \tag{17}$$

$$\hat{A} = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}, \tag{18}$$

$$A = \left[-\frac{\mathrm{i}}{2}\beta D_2(t)|q|^2 + \mathrm{i}\Gamma(t) + \mathrm{i}V(t)\lambda + \mathrm{i}D_2(t)\lambda^2\right], \tag{19}$$

$$B = \left[-\frac{\mathrm{i}}{2}D_2(t)\frac{\partial q}{\partial s}\frac{\partial s}{\partial x} - V(t)q - D_2(t)q\lambda\right], \tag{20}$$

$$C = \left[-\frac{\mathrm{i}}{2}\beta D_2(t)\frac{\partial q^*}{\partial s}\frac{\partial s}{\partial x} + \beta V(t)q^* + \beta D_2(t)q^*\lambda\right]. \tag{21}$$

Finally, the system of linear equations solved by the method of inverse scattering problem can be written in the form

$$\frac{\partial\psi_1(s,t)}{\partial s} = -\mathrm{i}\lambda\psi_1 + q(s,t)\psi_2\,,$$

$$\frac{\partial\psi_2(s,t)}{\partial s} = -\beta q^*(s,t)\psi_1 + \mathrm{i}\lambda\psi_2\,, \tag{22}$$

$$\frac{\partial\psi_1(s,t)}{\partial t} = \left[\mathrm{i}D_2(t)\lambda^2 + \mathrm{i}V(t)\lambda - \frac{\mathrm{i}}{2}\beta D_2(t)|q(s,t)|^2 + \mathrm{i}\Gamma(t)\right]\psi_1$$

$$-\left[D_2(t)q(s,t)\lambda + V(t)q(s,t) + \frac{\mathrm{i}}{2}D_2(t)\frac{\partial q(s,t)}{\partial s}\frac{\partial s}{\partial x}\right]\psi_2\,,$$

$$\frac{\partial\psi_2(s,t)}{\partial t} = \left[\beta D_2(t)q^*(s,t)\lambda + \beta V(t)q^*(s,t) - \frac{\mathrm{i}}{2}\beta D_2(t)\right.$$

$$\left.\times\frac{\partial q^*(s,t)}{\partial s}\frac{\partial s}{\partial x}\right]\psi_1 + \left[-\mathrm{i}D_2(t)\lambda^2 - \mathrm{i}V(t)\lambda\right.$$

$$\left.+\frac{\mathrm{i}}{2}\beta D_2(t)|q(s,t)|^2 - \mathrm{i}\Gamma(t)\right]\psi_2\,. \tag{23}$$

3. Exact solutions for the soliton management model

The Lax representation (17)–(21) for equation (15) gives an heuristic method for the construction of the general solution of the Schrödinger equation with variable coefficients. Indeed, as follows from (15) and (16), the problem proves to be completely integrable if

$$D_2(t)\left(\frac{\partial s}{\partial x}\right)^2 = \beta D_2(t)\,,$$

from which it follows that $s(x,t) = \mathrm{const}\,x + f(t)$, where $f(t)$ is an arbitrary function of time, i.e., the spectral parameter λ is independent of time in the general case.

The generalised NSE in the problem of soliton management has the form

$$\mathrm{i}\,\frac{\partial\Psi}{\partial Z}=\frac{1}{2}D(Z)\,\frac{\partial^2\Psi}{\partial X^2}+N(Z)|\Psi|^2\Psi-\mathrm{i}\gamma_0\Psi+\mathrm{i}\Gamma(Z)\Psi,\quad(24)$$

where γ_0 are linear (spatially homogeneous) losses. Equation (24) is written here in the so-called canonical variables, when the dimensionless length Z of the pulse propagation are normalised to the dispersion length, and the field Ψ is expressed in units of the soliton-pulse amplitude [21–25]. The parameters introduced into (24) describe variations in the dispersion $D(Z)$, the nonlinearity $N(Z)$, and the gain $\Gamma(Z)$ over the length of interaction of emission with a spatially inhomogeneous system. According to the well-known space-time analogy, equation (24) describes spatial solitons ('slot' beams) or usual temporal solitons (stationary pulses) if the coordinate X represents time or the transverse coordinate, respectively.

When the system described by equation (24) is conservative (losses and amplification are absent), the condition of its complete integrability is defined, according to (15) and (16), as

$$N(Z)=\beta D(Z)\quad(25)$$

and the simplest solution for a soliton without the phase modulation (i.e., a soliton with a trivial phase, which linearly depends on X) has the form

$$\Psi(X,Z)=\frac{1}{\sqrt{\beta}}\begin{cases}\eta\,\mathrm{sech}(\eta X)\exp\left[-0.5\mathrm{i}\eta^2\int_0^Z D(\zeta)\mathrm{d}\zeta\right]\\ \eta\tanh(\eta X)\exp\left[-\mathrm{i}\eta^2\int_0^Z D(\zeta)\mathrm{d}\zeta\right],\end{cases}\quad(26)$$

in the intrinsic coordinate system, where an arbitrary parameter η is introduced for the definition of the soliton form factor. The upper expression in (26) corresponds to a bright soliton obtained under the initial conditions $N(Z=0)=\beta D(Z=0)$, while the lower expression corresponds to a dark soliton obtained under the initial conditions with the opposite sign $N(Z=0)=-\beta D(Z=0)$.

Consider in more detail the case of a nonconservative system described by the complete equation (24). By using the substitution

$$\tilde{\Psi}(X,Z)=\frac{\Psi(X,Z)}{G(Z)},\quad(27)$$

$$R(Z)=N(Z)G^2(Z),\quad(28)$$

where the function $G(Z)$ satisfies the equation

$$\frac{\partial G(Z)}{\partial Z}=-\gamma_0 G(Z)+\Gamma(Z)G(Z),\quad(29)$$

we write equation (24) in the form

$$\mathrm{i}\,\frac{\partial\tilde{\Psi}}{\partial Z}=\frac{1}{2}\,D(Z)\,\frac{\partial^2\tilde{\Psi}}{\partial X^2}+R(Z)|\tilde{\Psi}|^2\tilde{\Psi}.\quad(30)$$

One can easily see that equation (30) proves to be completely integrable then and only then when the condition

$$R(Z)=\beta D(Z)\quad(31)$$

is satisfied.

Therefore, there exist the following completely integrable models for solitons in nonconservative systems:

$$\mathrm{i}\,\frac{\partial\Psi}{\partial Z}=\frac{1}{2}\,CN(Z)\exp\left[-2\gamma_0 Z+2\int_0^Z\Gamma(\zeta)\mathrm{d}\zeta\right]\frac{\partial^2\Psi}{\partial X^2}$$
$$+N(Z)|\Psi|^2\Psi-\mathrm{i}\gamma_0\Psi+\mathrm{i}\Gamma(Z)\Psi,\quad(32)$$

$$\mathrm{i}\,\frac{\partial\Psi}{\partial Z}=\frac{1}{2}\,D(Z)\frac{\partial^2\Psi}{\partial X^2}+\frac{1}{C}\,D(Z)\exp\left[2\gamma_0 Z-2\int_0^Z\Gamma(\zeta)\mathrm{d}\zeta\right]$$
$$\times|\Psi|^2\Psi-\mathrm{i}\gamma_0\Psi+\mathrm{i}\Gamma(Z)\Psi,\quad(33)$$

$$\mathrm{i}\,\frac{\partial\Psi}{\partial Z}=\frac{1}{2}\Phi(Z)F(Z)\frac{\partial^2\Psi}{\partial X^2}+\Phi(Z)|\Psi|^2\Psi$$
$$+\frac{\mathrm{i}}{2}\Psi\frac{\partial}{\partial Z}\ln F(Z),\quad(34)$$

where the functions $N(Z)$, $\Gamma(Z)$, $D(Z)$, $\Phi(Z)$, and $F(Z)$ are arbitrary integrable and differentiable functions.

For the complete analysis, we present the example of the simplest solution of equation (34) for a soliton without the phase modulation in the intrinsic coordinate system:

$$\Psi(X,Z)=$$
$$F^{1/2}(Z)\begin{cases}\eta\,\mathrm{sech}(\eta X)\exp\left[-0.5\mathrm{i}\eta^2\int_0^Z\Phi(\zeta)F(\zeta)\mathrm{d}\zeta\right]\\ \eta\tanh(\eta X)\exp\left[-\mathrm{i}\eta^2\int_0^Z\Phi(\zeta)F(\zeta)\mathrm{d}\zeta\right].\end{cases}\quad(35)$$

Note that the soliton solutions of equations (32)–(34) without the phase modulation have a unique feature of not changing their duration in the process of amplification (absorption).

Another nontrivial solution for the model (24) is phase-modulated solitons, which were predicted in papers [26–28]. Consider these solutions in more detail.

We will write the solution of equation (30) in the form

$$\tilde{\Psi}(X,Z)=P^{1/2}(Z)Q[S(X,Z),Z]\exp\left[\mathrm{i}\,\frac{C(Z)}{2}X^2\right],\quad(36)$$

where the variable S is a complicated function $S=S(X,Z)$. Note that we purposely use similar designations $s(x,t)$ and $S(X,Z)$ for two complicated functions in (7), (8) and (36) in order to emphasise a deep relation between the two theoretical methods being developed here.

The substitution of (36) into (30) gives the nonlinear evolution equation for the function $Q(S,Z)$ of the form

$$\mathrm{i}\,\frac{\partial Q}{\partial Z}=\frac{1}{2}DP^2\,\frac{\partial^2 Q}{\partial S^2}+RPQ|Q|^2$$
$$-\frac{\mathrm{i}}{2P}\,Q\left(\frac{\partial P}{\partial Z}+DCP\right)-\mathrm{i}T\,\frac{\partial Q}{\partial S}\left(\frac{\partial P}{\partial Z}+DCP\right)$$
$$-T^2\,\frac{Q}{2}\left(\frac{\partial C}{\partial Z}+DC^2\right).\quad(37)$$

One can easily see that the Lax representation (17)–(20) for equation (37) exists then and only then when the two conditions

$$\frac{\partial P}{\partial Z} + DCP = 0, \tag{38}$$

$$\frac{\partial C}{\partial Z} + DC^2 = 0 \tag{39}$$

are simultaneously satisfied.

Now, we should write in the explicit form the compatibility (solubility) condition for the system of equations (38) and (39):

$$C(Z) = \pm\Theta P(Z), \tag{40}$$

where Θ is an arbitrary constant, which can assume any values, including zero. Therefore, we obtain an infinite 'ocean' of soliton solutions for the soliton management model (24), which was found for the first time in papers [26–28].

The relation with the Lax representation (17)–(20) is established by the mutual dependence of the main parameters:

$$N_2(Z) = R(Z)P(Z), \quad D_2(Z) = D(Z)P^2(Z). \tag{41}$$

As shown in the previous section, the Lax representation (17)–(20), which means the complete integrability of the NSE model (15), exists then and only then when dispersion and nonlinearity are connected by the relation

$$N_2(Z) = \beta D_2(Z). \tag{42}$$

In the soliton menegement model considered the Lax representation similar to (17)–(20) exists then and only then when more complicated mathematical conditions are satisfied, which, however, open up the wide opportunities for the experimental realisation of the effects under study. The parameters of a nonlinear system, for example, a soliton communication system, the data storage system, or a soliton laser, which are also well described by the model (36), (40), cannot be arbitrary and should be chosen in accordance with two main conditions.

First, all the basic spatially dependent parameters of the model (24) should be interrelated:

$$R(Z) = \beta D(Z)P(Z) = N(Z)G^2(0) \times \exp\left[-2\gamma_0 Z + 2\int_0^Z \Gamma(\zeta)\mathrm{d}\zeta\right]. \tag{43}$$

Second, the solutions in the form of bright and dark quasi-soliton pulses of the type (36) with the function

$$Q(S) = \begin{cases} \eta\,\mathrm{sech}[\eta P(Z)X] \\ \eta\tanh[\eta P(Z)X] \end{cases} \tag{44}$$

and the nontrivial phase exists then and only then when the conditions

$$P = P_0\left[1 + P_0\int_0^Z D(Z')\mathrm{d}Z'\right]^{-1}, \quad R = \frac{P(Z)D(Z)}{C_0} \tag{45}$$

are satisfied.

It is assumed in (45) that the dispersion characteristic $D(Z)$ of the system is an arbitrary integrable function, which an experimenter may choose at will. Then, the rest of the parameters of the system – the gain and nonlinearity – should satisfy the condition (45).

When it is more convenient to choose in a particular experiment, for example, the specified profile of the effective nonlinearity parameter (or the gain profile in a distributed periodic system), the rest of the parameters should satisfy the condition

$$P = P_0\exp\left[-C\int_0^Z R(Z')\mathrm{d}Z'\right], \quad D = \frac{C_0 R(Z)}{P(Z)}. \tag{46}$$

Therefore, the problem of the optimal control of soliton parameters with the help of spatially inhomogeneous systems proves to be integrable and has the solutions of the type (36), (44) under the conditions (45), (46).

To illustrate the nontrivial dynamics of the solutions obtained, we consider a number of specific examples. The NSE model with variable coefficients (15) is completely integrable under the conditions (16) imposed on the basic parameters of the model. Within the framework of this model, solitons (26) without the phase modulation, both bright and dark, interact elastically and do not change their form. They only acquire different accelerations, as one can see from Figs 1 and 2. Figs 1 and 2b show the space–time dynamics of the 'trappon' type for bright and dark solitons, which was calculated in the parametric region

$$D_2(t) = \cos t, \quad N_2(t) = \beta\cos t. \tag{47}$$

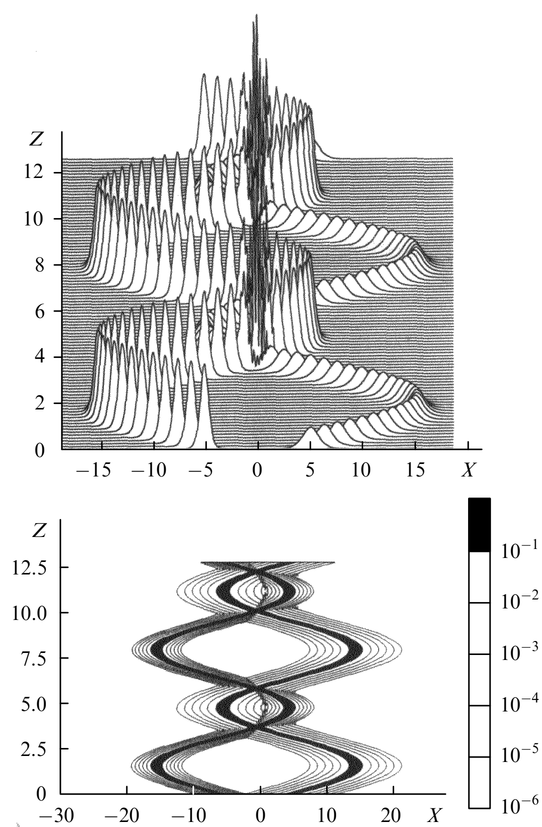

Figure 1. Interaction of NSE (15) soliton solutions (26) in the parametric region (47) for $\beta = 4.0$ and initial soliton velocities 10.0 and −10.0.

The interaction dynamics of solitons calculated for the parameters

$$D_2(t) = N_2(t) = 1 - \alpha t, \tag{48}$$

is presented in Fig. 2 (dynamics of the 'boomeron' type).

It should be emphasised that the qualitative features of the propagation of solutions presented in Figs 1, 2 correspond to trappons and boomerons (see Ref. [17], pp. 336, 337), which were, however, obtained for much more complicated models.

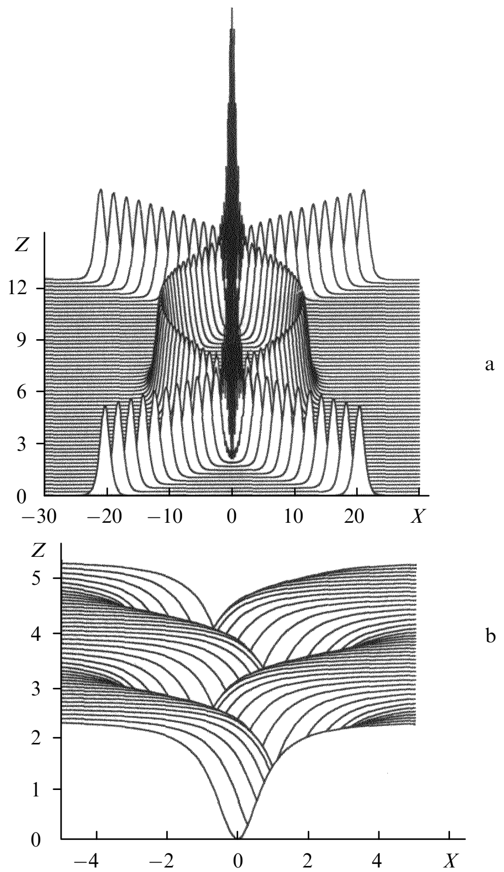

Figure 2. Dynamics if the bright (a) and dark (b) solitons (26) in the parametric region (48).

The dynamics of solitons (26) calculated using the model (47), (48) should be compared with the dynamics of phase-modulated soliton solutions (36) obtained for the parameters

$$R(Z) = \cos Z, \quad D(Z) = \cos Z \exp(\sin Z),$$

$$P(Z) = \exp(-\sin Z) \tag{49}$$

or

$$R(Z) = 1 - \alpha Z, \; D(Z) = (1 - \alpha Z) \exp\left(Z - \frac{\alpha Z^2}{2}\right). \tag{50}$$

Solutions (36) with parameters (49), (50) presented in Fig. 3 were obtained in the regions of complete integrability of the model (24). They interact elastically, but are substantially different from solutions (26) with parameters (47), (48) in that they not only acquire the acceleration but also change their duration and amplitude. This important difference is caused by the fact that solutions (36), (49), and (50) have the nontrivial phase modulation, which is determined by the parameter $P(Z)$. In this connection, in order to distinguish them from solutions (26), (47), and (48) shown in Figs 1 and 2, it is expedient to call new solutions the quasi-solitons, emphasising that they can change their amplitude, duration, and phase in inhomogeneous media.

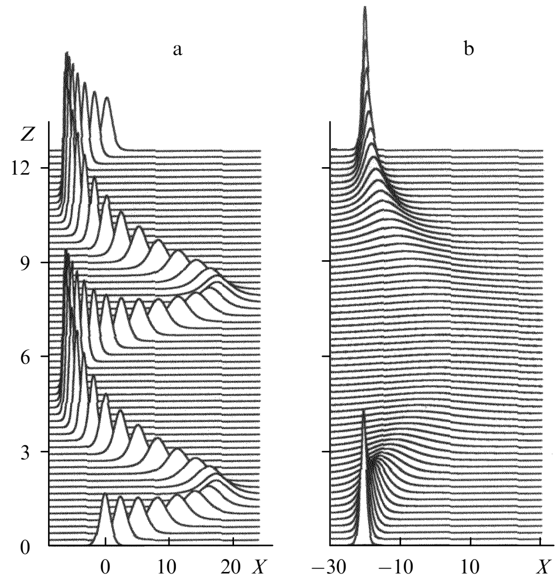

Figure 3. Nonlinear dynamics of a quasi-soliton (36) of the 'trappon' type with parameters (49) (a) and of a quasi-soliton (36) of the 'boomeron' type with parameters (50) (b).

The large-scale numerical calculations showed that the gain and interaction dynamics of solitons without the phase modulation (32)–(35) has all the features of the interaction of 'true' solitons. The results obtained for the functions $D(Z)$, $N(Z)$, $\Gamma(Z)$, $F(Z)$, and $\Phi(Z)$ of different forms will be described in a separate paper. Here, we present the calculation of the gain and interaction dynamics for in-phase and out-of-phase solitons of the model (32) in the simplest case of $\Gamma(Z) = \Gamma(0) = \text{const}$ (Fig. 4). The characteristic feature of the soliton solutions of equations (32)–(34) is that the soliton duration is constant during the increase of its energy. Note also that the self-consistence of the dispersion, nonlinearity, and gain in the model (32)–(34) is accompanied by the averaging of the spatial stochastic variations in the gain (absorption coefficient), and vice versa, the stochastic variations in the dispersion should be taken into account in the gain (absorption coefficient) according the expression

$$\Gamma(Z) = \frac{1}{2F(Z)} \frac{\partial F}{\partial Z}.$$

Consider two practically important cases. Let us present an exact analytic solution for the problem of Raman pumping of solitons. Consider the most interesting case, when a wave of the molecular vibrations of the medium is excited by two laser beams from the opposite ends of an

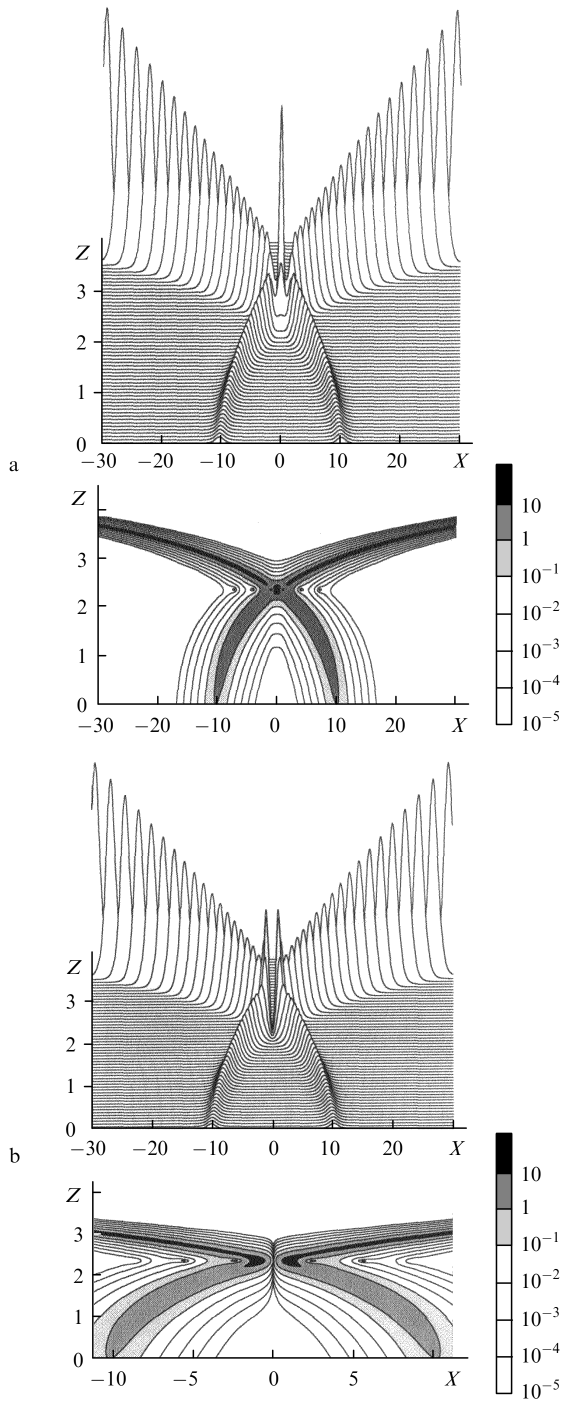

Figure 4. The gain and interaction dynamics for in-phase (a) and out-of-phase (b) solitons (32) calculated for $\Gamma(0) = 1.0$, initial soliton velosities 1.0 and -1.0.

optical fibre [24]. The gain distribution is described in this case by the function

$$\Gamma(Z) = \exp(-\gamma Z) + \exp[\gamma(Z - L)]\,. \qquad (51)$$

The problem proves to be completely integrable if the dispersion is determined by the expression

$$D(Z) = \exp\{2[A\sinh(\gamma Z) - B\cosh(\gamma \mathrm{Z}) + \mathrm{B}]\}, \qquad (52)$$

where

$$A = \frac{1}{\gamma}[1 + \exp(-\gamma L)];\;\; B = \frac{1}{\gamma}[1 - \exp(-\gamma L)]; \qquad (53)$$

and γ are linear losses at the Raman frequency.

It should be emphasised that two substantially different situations appear in the problem of the Raman pumping of solitons, which are illustrated in Figs 5a and 5b. In the first case, the pump decays rather strongly so that the gain at the fibre centre is virtually zero, and the optimal function $D(Z)$ has two inflections. In the second case, on the contrary, the pump decays weakly, and the gain is a monotonic function. The optimal dispersion characteristic is a linear increasing function.

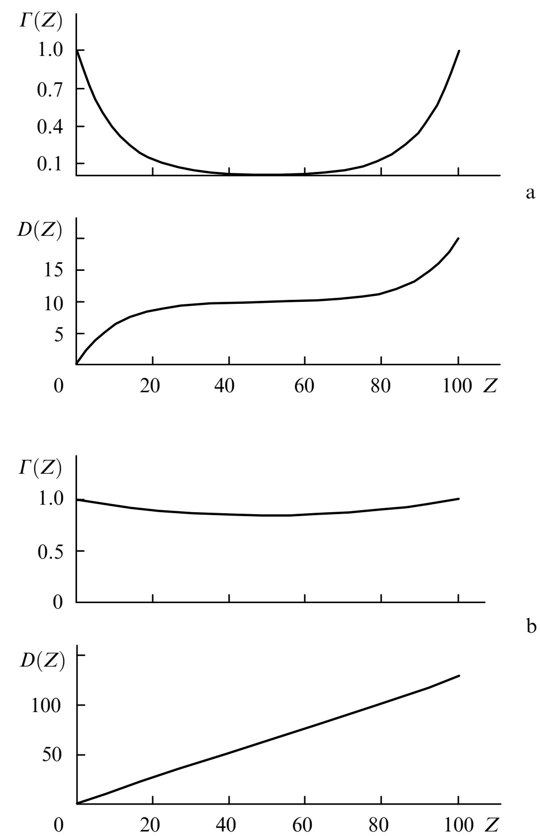

Figure 5. Optimal map for the parameters of an optical fibre upon Raman pumping of solitons (51) calculated for $\gamma = 0.1$ (a) and 0.01 (b).

Consider the exact analytic solution for the problem of the soliton propagation in a fibre with the decreasing dispersion [6]

$$D(Z) = \frac{1}{1 + \alpha Z}. \qquad (54)$$

In this case, the expression for the optimal gain has the form

$$\Gamma(Z) = \frac{1}{2(1 + \alpha Z)}\left[\frac{1 - \alpha C_0 + \ln(1 + \alpha Z)}{C_0 - (1/\alpha)\ln(1 + \alpha Z)}\right]. \qquad (55)$$

The optimal coefficient $P(Z)$ is determined by the soliton duration

$$T(Z) = \frac{1}{P(Z)} = C_0 - \frac{1}{\alpha}\ln(1 + \alpha Z). \qquad (56)$$

The typical dependences $D(Z)$, $T(Z)$, and $\Gamma(Z)$ are shown in Fig. 6. The dispersion decreasing as a hyperbolic function requires the production of a very interesting gain profile in the fibre, when absorption first dominates and then the gain dominates. This can be interpreted (Fig. 5) as the counter Raman pumping of a soliton. Fig. 7 shows the interaction dynamics of solitons with optimal and non-optimal phase profiles.

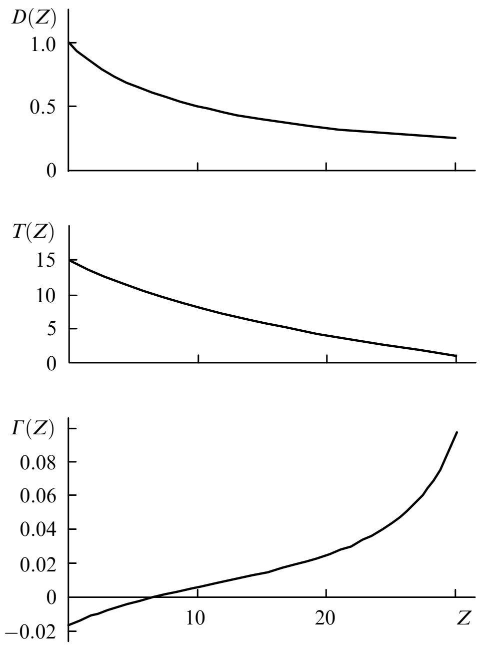

Figure 6. Optimal map for the parameters of an optical fibre with the dispersion (54) for $\alpha = 0.1$ and $C_0 = 15.0$.

Note in conclusion that the new class of solutions obtained in this paper is a more general and contains canonical NSE solitons with constant coefficients. The passage to the limit occurs for the parameter $\Theta \to 0$, when the soliton amplitude and duration take stationary values according to (31)–(33).

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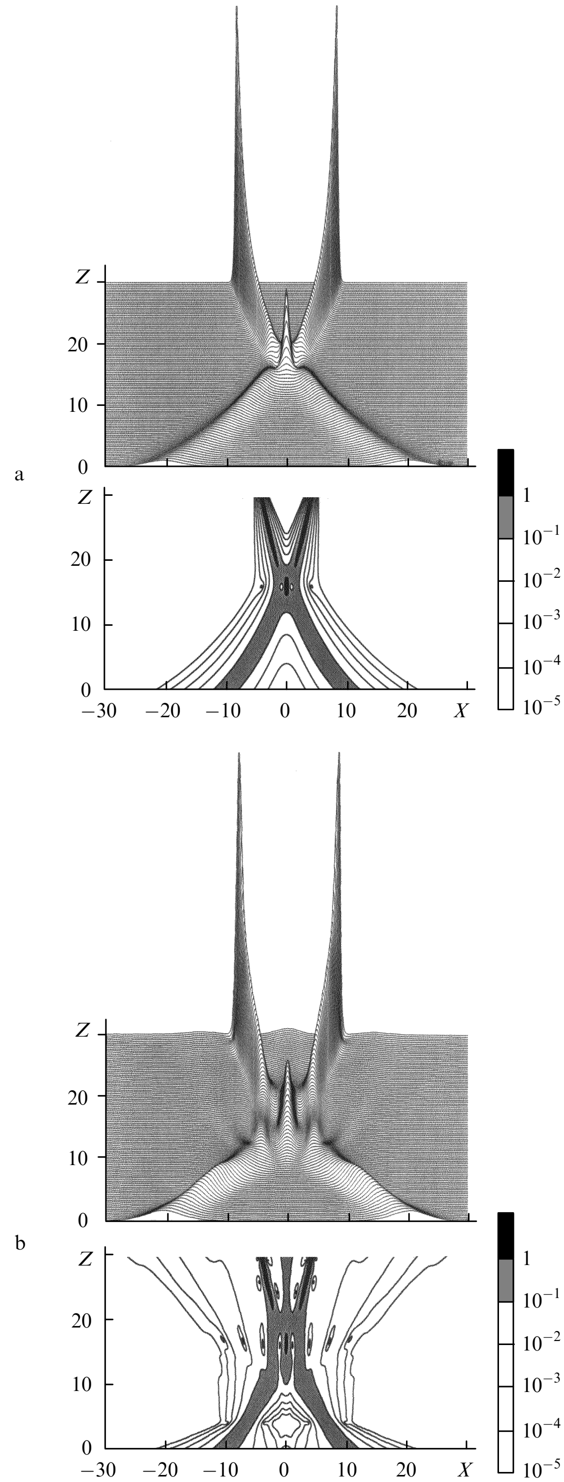

Figure 7. Interaction of quasi-solitons (36) in an optical fibre in the parametric region (54)–(56) with the optimal phase profile for $\alpha = 0.1$ and $C_0 = 15.0$ (a), and the structural instability of solitons upon violation of optimal relations (54)-(56) of the type $P(Z) = 3P_{\mathrm{opt}}(Z)$ (b).

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