REVIEW

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Bose condensates from the point of view of laser physics

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Contents

***Abstract*. The dynamic analogy between the Bose condensate of photons (laser), Bose condensate of atoms, and Bose condensate of Cooper pairs in a superconductor is discussed. It is pointed out that the coherent state of Bose condensates of different types can appear only due to stimulated transitions.The equations are discussed that combine into a unified dynamic system the order parameter of the Bose condensate and the concentration of atoms beyond the condensate for an ensemble of atoms captured by a trap, as well as the order parameter of Cooper pairs, the concentration of free quasiparticles, and the densities of phonons and photons in a semiconductor.**

Keywords: *Bose condensate, laser, superconductor, stimulated transitions, coherence.*

1. Coherence of Bose condensates and the inversion condition

The Bose condensate has attracted particular interest of researchers first of all in connection with the problems of superconductivity and superfluidity [1, 2]. The advent of masers and lasers [3 – 6] enlarged the family of Bose condensates because a coherent state of an electromagnetic field with a certain frequency and spatial configuration generated by a laser also can be treated as the Bose condensate of photons. The comparatively recent successful experiments [7 – 9] on cooling atoms down to extremely low temperatures of the order of 10^{-7} K have stimulated new interest in Bose condensates. At such low temperatures, the Bose condensate of atoms at a low concentration was obtained and the interference of two ensembles of Bose condensates of atoms captured by a trap was observed [10]. Note that the Bose condensate of atoms attracts first of all general physical interest. In the Bose condensate, the wave nature of matter is distinctly manifested, and an ensemble of a rather large number of particles behaves as a classical field having amplitude and phase.

As a rule, the Bose condensate of particles is *a priori* treated as a coherent state of matter. It is implicitly assumed that this coherent state is automatically formed upon Bose condensation. However, a researcher in the field of laser physics can hardly agree with this statement without any proof. Indeed, for example, photons can be accumulated in a single cavity mode in an 'underexcited' laser due to *spontaneous* transitions; however, this state of an electromagnetic field is not coherent. The coherent state of an electromagnetic field (of photons) is created in a laser due to *stimulated* transitions when the *self-excitation condition* is fulfilled. According to this condition, the electromagnetic energy emitted by an active medium should exceed losses caused by its possible absorption and scattering in the laser and by the radiation emitted by the laser.

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1.1 Laser

To specify the self-excitation condition in a laser, consider a simplest scheme of a laser consisting of an active (working) medium and a cavity (Fig. 1). In the scheme in Fig. 1, the cavity is formed by two coaxial mirrors, one of which being partially transparent to extract the generated radiation outside. We will consider two-level atoms as a model of the active laser medium. This approximation is good enough because two energy levels of the active medium, with the frequency of transition between them being close to the radiation frequency (Fig. 2), make a dominant contribution to the interaction of laser radiation with the active medium.

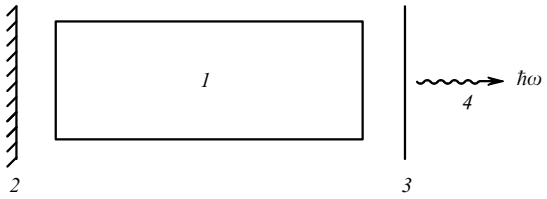

Figure 1. Simplest scheme of a laser: (*1*) active medium; (*2*) highly reflecting mirror; (*3*) partially transparent mirror; (*4*) output coherent radiation.

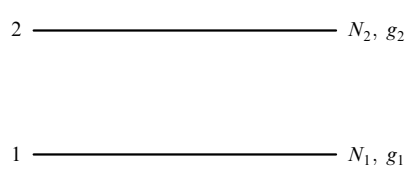

Figure 2. Scheme of a two-level atom ($N_{1,2}$, $g_{1,2}$ are populations and statistical weights of the corresponding energy levels.

The rate of photon emission upon transitions from the upper level to the lower one is described by the obvious expression

$$S_{\mathrm{em}} = W_{21} N_2 g_1 (n_{\boldsymbol{k}} + 1_{\boldsymbol{k}}). \tag{1}$$

Here, $W_{21} n_{\boldsymbol{k}}$ is the probability of the stimulated transition from the level 2 to level 1 induced by photons with the wave vector $\boldsymbol{k}$ and the density $n_{\boldsymbol{k}}$; $W_{21} \times 1_{\boldsymbol{k}}$ is the probability of the spontaneous transition from the level 2 to level 1 accompanied by emission of a photon with the wave vector $\boldsymbol{k}$; N_2 is the concentration of particles at the upper energy levels; and g_1 is the statistical weight of the lower energy level.

The rate of transition from the lower level to the upper level accompanied by absorption of a photon with the wave vector $\boldsymbol{k}$ is

$$S_{\mathrm{abs}} = W_{12} N_1 g_2 n_{\boldsymbol{k}}, \tag{2}$$

where N_1 is the concentration of particles at the lower level and g_2 is the statistical weight of the upper level.

To produce a coherent state of an electromagnetic field upon the interaction of radiation with matter, the stimulated radiation should prevail over spontaneous radiation and the emission rate should exceed the absorption rate. These conditions give the relation

$$W_{21} N_2 g_1 n_{\boldsymbol{k}} > W_{12} N_1 g_2 n_{\boldsymbol{k}}. \tag{3}$$

Because $W_{21} = W_{12}$ [11], the inequality

$$\frac{N_2}{g_2} > \frac{N_1}{g_1}. \tag{4}$$

follows from (3), which is known as the 'inverse population condition'.

For interband transitions in a semiconductor laser (Fig. 3), the condition equivalent to inequality (4) has the form

$$f_{\mathrm{e}}(\varepsilon_{\mathrm{e}})[1 - f_{\mathrm{h}}(\varepsilon_{\mathrm{h}})] > f_{\mathrm{h}}(\varepsilon_{\mathrm{h}})[1 - f_{\mathrm{e}}(\varepsilon_{\mathrm{e}})], \tag{5}$$

where $f_{\mathrm{e,h}}(\varepsilon_{\mathrm{h}})$ is the distribution function of electrons in the conduction band or holes in the valence band and $\varepsilon_{\mathrm{e,h}}$ is the energy of electrons and holes. Due to the hierarchy of the relaxation times inherent in a semiconductor, the thermodynamic quasi-equilibrium within subsystems of electrons in the conduction band and of holes in the valence band is established faster than the interband equilibrium [12]. This allows one to use the quasi-equilibrium distribution functions of electrons in the conduction band and of holes in the valence band:

$$f_{\mathrm{e,h}} = \left[\exp\left(\frac{\varepsilon_{\mathrm{e,h}} - \mu_{\mathrm{e,h}}}{kT} \right) + 1 \right]^{-1}, \tag{6}$$

where $\mu_{\mathrm{e,h}}$ is the chemical potential of electrons and holes, respectively. By substituting expression (6) into (5), we obtain the inversion condition [13]

$$\mu_{\mathrm{e}} - \mu_{\mathrm{h}} > \hbar\omega, \quad \hbar\omega = \varepsilon_{\mathrm{e}} - \varepsilon_{\mathrm{h}}, \tag{7}$$

or $\mu_{\mathrm{e}} > \hbar\omega$ because the effective mass of carriers in the conduction band in materials used in semiconductor lasers (for example, GaAs) is noticeably lower than the hole mass [14], and therefore, $\mu_{\mathrm{e}} \gg \mu_{\mathrm{h}}$.

The condition (4) [or equivalent condition (7)] is necessary but not sufficient for producing a coherent state of the filed in a laser (the coherent photon condensate). The sufficient condition is

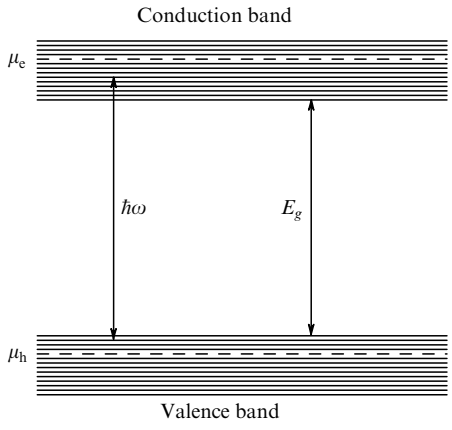

Figure 3. Energy band diagram of a semiconductor. E_g: energy gap; $\mu_{\mathrm{e,h}}$: chemical potential of charge carriers in the conduction and valence bands; $\hbar\omega$: energy of laser photons.

$$\frac{N_2}{g_2} - \frac{N_1}{g_1} > \Delta N_{\text{th}}, \tag{8}$$

where ΔN_{th} is the threshold difference of populations of energy levels, which depends on the total losses of the electromagnetic radiation in a laser and the transmission of the output mirror. Clearly, similar conditions should be satisfied for obtaining *coherent* states of any Bose particles.

1.2 Bose condensate of atoms

Recall the mechanism of Bose condensation. The chemical potential μ in the energy distribution of Bose particles in gas (the Bose–Einstein distribution)

$$n(\varepsilon) = \left[\exp\left(\frac{\varepsilon - \mu}{kT}\right) - 1\right]^{-1} \tag{9}$$

cannot be positive. For this reason, the maximum density n_{c} of particles that can be 'accumulated' by the Bose–Einstein distribution is determined by the relation [1]

$$n_{\text{c}} = 4\pi(2m^3)^{1/2} g \int_0^{\infty} \left[\exp\left(\frac{\varepsilon}{kT}\right) - 1\right]^{-1} \frac{\sqrt{\varepsilon}\,\mathrm{d}\varepsilon}{(2\pi\hbar)^3}$$

$$= 0.166 g \left(\frac{mkT}{\hbar^2}\right)^{3/2}, \tag{10}$$

where k is the Boltzmann constant; m is the particle mass; and g is the statistical weight. Expression (10) determines the minimum temperature at which the specified concentration n of particles is described by the distribution (9) with the zero (maximum) chemical potential. This is the critical temperature, below which Bose condensation begins, i.e., particles are accumulated at the lowest energy level of the system (Fig. 4). Let us present some numerical estimates for clarity. By combining laser and magnetic cooling of atoms in a trap, the authors of papers [7–9] managed to cool the atoms down to 10^{-7} K. The critical concentration of sodium atoms at which the Bose condensation begins at this temperature is 4.5×10^7 cm^{-3}. The critical concentration of excitons with a mass close to the free electron mass is $\sim 10^{16}$ cm^{-3} at temperature 1 K.

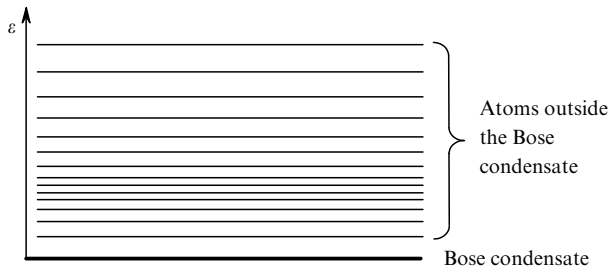

Figure 4. Energy level diagram for Bose particles.

Because in a real trap the motion of atoms is quantised, the integral relation (10) should be replaced by a sum over the quantum states of a particle in the trap:

$$N_{\text{c}} = \sum_j \left[\exp\left(\frac{\varepsilon_j}{kT}\right) - 1\right]^{-1}. \tag{11}$$

Here, N_{c} is the total number of particles in the trap. In an isotropic trap with the parabolic potential, $\varepsilon_j = \hbar\omega_{\text{t}}(j_x + j_y + j_z)$, where j_a are the quantum numbers corresponding to the quantisation of the translational motion along axes x, y, and z; ω_{t} is the oscillation frequency of atoms in the trap. For $\omega_{\text{t}} < kT$, the summation can be approximately replaced by integration, which gives [22]

$$N_{\text{c}} = 1.202 g \left(\frac{kT}{\hbar\omega_{\text{t}}}\right)^3. \tag{12}$$

The difference between expressions (10) and (12) is explained by the fact that the average volume occupied by atoms in the trap with a parabolic potential is not fixed because the amplitude of their oscillations depends on their energy. Therefore, the average radius of the atomic packet depends on the oscillation frequency of atoms in the trap and on temperature. The oscillation frequency in relation (12) can be expressed in terms of the average radius of the atomic packet and temperature, which will result in expression (10). If the volume occupied by atoms in the trap were fixed, as for example, in a three-dimensional 'potential box', we would obtain expression (10) immediately after the calculation of a sum over the states.

Consider Bose condensation from the point of view of formation of a coherent state. Note that while the inducing agent in a laser is an electromagnetic field, the inducing agents in an atomic system are atoms themselves in the Bose condensate. The relation that is equivalent to the inversion condition in the laser should follows from the requirement that the rate of the induced formation of the Bose condensate would exceed its decay rate. As a result, it is easy to obtain the relation

$$W^{\rightarrow} n_0 n(\varepsilon) > W^{\leftarrow} n_0 [n(\varepsilon) + 1], \tag{13}$$

where $W^{\rightarrow}$ and $W^{\leftarrow}$ are the probabilities of the direct and inverse processes, respectively; n_0 is the concentration of particles in the Bose condensate. The left-hand side of expression (13) is the rate of 'creation' of an additional particle in the Bose condensate induced by the condensate itself due to the transition of a particle with the energy ε to the condensate. The right-hand side of expression (13) is the rate of the inverse process, i.e., of the condensate decay and the creation of a particle with the energy ε in an ensemble of particles outside the Bose condensate. The inverse process requires the consumption of the energy ε, so that $W^{\rightarrow}/W^{\leftarrow} = \exp(-\varepsilon/kT)$ [15]. Taking this relation into account, the inequality (13) takes the form

$$n(\varepsilon) > \left[\exp\left(\frac{\varepsilon}{kT}\right) - 1\right]^{-1}. \tag{14}$$

The right-hand side of (14) is nothing but the equilibrium distribution function of particles outside the Bose condensate. This means that even a weak deviation from the equilibrium distribution resulting in an increase in the number of particles with some energy ε will lead to the appearance of a state, which can be called the inversion state using the laser terminology. Therefore, when some particles leave the Bose condensate, the condensate is immediately restored due to stimulated, i.e., coherent transitions. Therefore, below the critical temperature the Bose gas consists of the *coherent Bose condensate*, whose particles have the minimum energy, and of an ensemble of *incoherent* particles.

At the absolute zero, the concentration of incoherent particles is zero. This statement is strictly true for the ideal gas of bosons. In gas of particles with a weak repulsive interaction, there are particles with the nonzero momentum even at the absolute zero whose relative concentration in gas of free particles is determined by the expression [1]

$$\frac{n-n_0}{n}=\frac{8}{3\sqrt{\pi}}\left(nl^3\right)^{1/2}, \tag{15}$$

where l is the scattering length of the interacting particles and n is their total concentration. For gas in a trap, a similar expression has the form [16]

$$\frac{N_{\rm g}-N_0}{N_{\rm g}}=0.065\left(\frac{l}{R}N_{\rm g}\right)^2, \tag{16}$$

where $N_{\rm g}$ is the total number of particles in the trap; N_0 is the number of particles in the Bose condensate; and R is the trap radius.

1.3 Superconducting Bose condensate

The interaction of electrons resulting in their pairing occurs most efficiently for the electrons whose energy is close to the Fermi energy [1, 17–19] (corresponding to the Fermi surface in the momentum space). The interaction strength decreases deep from the Fermi surface, so that the effective number of the interacting electrons becomes noticeably lower than their total number. Therefore, the energy band diagram of a superconductor can be represented as shown in Fig. 5. The superconducting condensate of Cooper pairs lies above the cushion of unpaired electrons. Due to the interaction with some agent, Cooper pairs can decompose, producing unpaired quasi-particles at the concentration n, which are separated from the condensate of Cooper pairs by the superconducting energy gap of the width Δ.

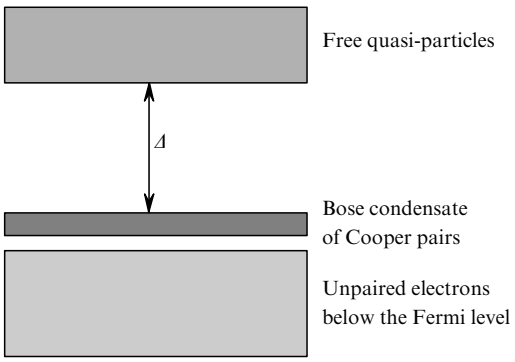

Figure 5. Energy band diagram of a superconductor (Δ is the width of the superconducting gap).

Using this scheme, we can write the expression [20–22]

$$W_+N_0n_pn_{-p}>W_-N_0(1-n_p)(1-n_{-p}) \tag{17}$$

for the superconducting condensate of Coopers pairs, which is analogous to the inequality (13). Here, n_p is the concentration of free quasi-particles with the momentum p. Assuming that a subsystem of free particles in a superconductor is quasi-equilibrium, we have

$$n_p=\left[\exp\left(\frac{\varepsilon_p-\mu_n}{kT}\right)+1\right]^{-1}, \tag{18}$$

where μ_n is the chemical potential of free quasi-particles. Taking into account that the energy $2\varepsilon_p$ is required for the decomposition of the condensate of Cooper pairs, it follows from (17) that

$$\mu_n>0\,. \tag{19}$$

Because the equilibrium between the condensate of Cooper pairs and an ensemble of quasi-particles exists when the chemical potential of quasi-particles is zero, an arbitrarily weak deviation from the equilibrium in favour of quasi-particles will result in the 'inversion'. Let us compare (19) with analogous condition (7) for a laser. The condition (7) is much more stringent: a weak deviation from the thermodynamically equilibrium distribution does not produce inversion in lasers. The matter is that laser photons carry away a large part of the energy accumulated in the active medium, whereas the condensate of Cooper pairs, like the Bose condensate of atoms, is created with the zero energy.

Consider now the dynamics of Bose condensates. We begin with the dynamics of lasers because at present it is most thoroughly studied both theoretically and experimentally due to a number of reasons. In this connection the dynamics of lasers can be used as the basis for a deeper understanding and prediction of dynamic processes proceeding in Bose condensates of other types.

2. Mathematical model of a laser

Let us assume that the active medium of a laser interacts with an electromagnetic field. Then, the propagation of the electromagnetic field in the matter can be described by the equation for the electric component $\boldsymbol{E}(\boldsymbol{r},t)$ of the electromagnetic field

$$\frac{\partial^2\boldsymbol{E}(\boldsymbol{r},t)}{\partial t^2}-c_0^2\nabla^2\boldsymbol{E}(\boldsymbol{r},t)+2\alpha c_0\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}=-4\pi\frac{\partial^2\boldsymbol{P}_{\rm tot}(\boldsymbol{r},t)}{\partial t^2}, \tag{20}$$

where c_0 is the velocity of light in vacuum; α is the coefficient of nonresonance losses; and $\boldsymbol{P}_{\rm tot}(\boldsymbol{r},t)$ is the total polarisation of the amplifier medium. It is reasonable to separate the latter into two parts: $\boldsymbol{P}_0(\boldsymbol{r},t)$ and $\boldsymbol{P}(\boldsymbol{r},t)$. The quantity $\boldsymbol{P}(\boldsymbol{r},t)$ describes the polarisation of atoms in the active medium, which are directly 'responsible' for amplification (working atoms). They are in an excited state and resonantly interact with the radiation field. To describe polarisation, a dynamic model is required, which will be discussed below. The quantity $\boldsymbol{P}_0(\boldsymbol{r},t)$ describes the polarisation of the rest of atoms in the medium, whose concentration, as a rule, is much higher than that of working atoms. This part of polarisation can be considered quasi-equilibrium and can be described by the expression

$$\boldsymbol{P}_0(\boldsymbol{r},t)=\hat{\chi}_0\boldsymbol{E}(\boldsymbol{r},t). \tag{21}$$

Below, we will assume that the operator $\hat{\chi}_0$ is a constant. Taking into account the Lorentz correction for the acting field, the total polarisation of the medium is [23]

$$\boldsymbol{P}_{\rm tot}(\boldsymbol{r},t)=\chi_0\boldsymbol{E}(\boldsymbol{r},t)+\frac{\varepsilon_0+2}{3}\boldsymbol{P}(\boldsymbol{r},t),\ \varepsilon_0=1+4\pi\chi_0. \quad (22)$$

Finally, equation (20) can be represented in the form

$$\frac{\partial^2\boldsymbol{E}(\boldsymbol{r},t)}{\partial t^2}-\frac{c_0^2}{\varepsilon_0}\nabla^2\boldsymbol{E}(\boldsymbol{r},t)+2\alpha c_0\,\frac{1}{\varepsilon_0}\,\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$
$$=-4\pi\,\frac{\varepsilon_0+2}{3\varepsilon_0}\,\frac{\partial^2\boldsymbol{P}(\boldsymbol{r},t)}{\partial t^2}. \quad (23)$$

When laser amplifiers are studied, it is reasonable to use the of Eqn (23) in the form of a travelling wave with the corresponding initial and boundary conditions. To obtain lasing, the active medium is placed inside a resonator. Because the resonator has the eigenmodes with characteristic resonance frequencies and field configurations, it is convenient to represent the field and polarisation as the expansion in the eigenmodes $\boldsymbol{U}_j(\boldsymbol{r})$ of the resonator

$$\boldsymbol{E}(\boldsymbol{r},t)=\sum_j E_j(t)\boldsymbol{U}_j(\boldsymbol{r}),\quad \boldsymbol{P}(\boldsymbol{r},t)=\sum_j P_j(t)\boldsymbol{U}_j(\boldsymbol{r}). \quad (24)$$

If necessary, the conditions can be provided when only one mode dominates. In this case, Eqn (23) can be reduced to the oscillator equation for the amplitude of the corresponding mode of the field and polarisation

$$\frac{{\rm d}^2E(t)}{{\rm d}t^2}+\frac{\omega_{\rm c}}{Q}\,\frac{{\rm d}E(t)}{{\rm d}t}+\omega_{\rm c}^2E(t)=-4\pi\,\frac{\varepsilon_0+2}{3\varepsilon_0}\,\frac{{\rm d}^2P(t)}{{\rm d}t^2}, \quad (25)$$

where $\omega_{\rm c}$ is the resonance frequency of this mode and Q is the Q factor that takes into account all linear energy losses in the resonator. To simplify notation, we retained the same letter notation for the amplitudes of the expansion of the field $E_j(t)$ and polarisation $P_j(t)$ with the index of a particular mode, but omitted the index.

The most popular model for the description of the polarisation dynamics of working atoms is the so-called two-level approximation, which we already mentioned above. In an atom interacting with the field, only two energy levels coupled by the transition that is resonant with the field frequency are considered. It can be shown [24–26] that the polarisation $\boldsymbol{P}(\boldsymbol{r},t)$ of a medium of two-level atoms satisfies the equations

$$\frac{{\rm d}^2\boldsymbol{P}(\boldsymbol{r},t)}{{\rm d}t^2}+\frac{2}{\tau_2}\,\frac{{\rm d}\boldsymbol{P}(\boldsymbol{r},t)}{{\rm d}t}+\omega_0^2\boldsymbol{P}(\boldsymbol{r},t)=-2\omega_0\,\frac{|\mu_{\rm d}|^2}{\hbar}N\boldsymbol{E}_{\rm loc}(\boldsymbol{r},t),$$
$$\frac{{\rm d}N(\boldsymbol{r},t)}{{\rm d}t}+\frac{1}{\tau_1}N(\boldsymbol{r},t)=J(\boldsymbol{r},t)+\frac{2}{\hbar\omega_0}\boldsymbol{E}_{\rm loc}(\boldsymbol{r},t)\,\frac{{\rm d}\boldsymbol{P}(\boldsymbol{r},t)}{{\rm d}t}, \quad (26)$$

where N is the difference of populations of the working levels of the active medium; $J(\boldsymbol{r},t)$ is the parameter characterising the pump rate; τ_2 is the polarisation relaxation time, which determines the width of a spectral line; τ_1 is the relaxation time of the level population; ω_0 is the frequency of transition between the working levels of atoms; $\mu_{\rm d}$ is the matrix element of the dipole moment corresponding to this transition. Equation (26) contains a local field acting on a particle, which is related to the Maxwell field entering Eqn (25) by the expression [27]

$$\boldsymbol{E}_{\rm loc}(\boldsymbol{r},t)=\frac{\varepsilon_0+2}{3}\left[\boldsymbol{E}(\boldsymbol{r},t)+\frac{4\pi}{3}\,\boldsymbol{P}(\boldsymbol{r},t)\right]. \quad (27)$$

Let us denote the difference of populations of the working levels in the absence of pumping by $N^{(0)}$. In the thermodynamic equilibrium, $N^{(0)}$ is determined by the Boltzmann distribution. As mentioned above, to obtain the gain, it is necessary (but not sufficient!) that the population of the upper level be higher than that of the lower level. This requires the supply of energy to the system resulting in transfer of atoms from the lower energy level to the upper level.

The system of Eqns (23) and (26) describes all fundamentally important processes related to the laser operation and the pulse propagation through the amplifier. Note that, unlike Eqn (23), Eqns (26) are nonlinear, which results in the appearance of an infinite system of linked equations in the case of expansion of the field and polarisation in the cavity modes. However, when one mode is excited, only one term will dominate in the expansion in modes (24). In this case, the dynamics of a single-mode laser should be described by the system of Eqns (25) and (26). The additional information of the laser dynamics is contained in Refs [28–33].

Below, we consider the features of the dynamics of single-mode lasers using the system of Eqns (25) and (26). Some features of the dynamics of amplifiers were discussed in papers [34, 35].

The system of Eqns (25) and (26) contains the parameters

$$\omega_{\rm c},\ \omega_0,\ 2\gamma_{\rm c}=\frac{\omega_{\rm c}}{Q},\ \gamma_1=\frac{1}{\tau_1},\ \gamma_2=\frac{1}{\tau_2},$$
$$\gamma_{\rm rad}=\left(2\pi\omega_0\,\frac{\mu_{\rm d}^2}{\hbar}\,N_0\right)^{1/2}. \quad (28)$$

The relation between these parameters is such that

$$\omega_{\rm c},\omega_0\gg\gamma_{\rm c},\gamma_1,\gamma_2,\gamma_{\rm rad},|\omega_{\rm c}-\omega_0|. \quad (29)$$

This makes it possible to use the method of slowly varying amplitudes in the analysis of Eqns (10), (11), in which the field and polarisation are represented in the form

$$E(t)=A(t)\exp(-{\rm i}\omega_0t),$$
$$P(t)=\frac{3}{\varepsilon_0+2}B(t)\exp(-{\rm i}\omega_0t), \quad (30)$$

where $A(t)$ and $B(t)$ are the functions slowly varying in time compared to a rapidly oscillating exponential. The difference of populations $N(t)$ can be also considered a function slowly varying in time. The procedure of substituting (30) into Eqns (25) and (26) was described, for example, in Refs [30–33]. This substitution gives the system of equations

$$\frac{{\rm d}A}{{\rm d}t}+(\gamma_{\rm c}+{\rm i}\Delta_{\rm c})A={\rm i}\beta B,\ \Delta_{\rm c}=\omega_{\rm c}-\omega,\ \beta=\frac{2\pi\omega_0}{\varepsilon_0},$$
$$\frac{{\rm d}B}{{\rm d}t}+(\gamma_2+{\rm i}\Delta_0)B=-\frac{{\rm i}\tilde{\mu}^2}{\hbar}NA,\ \Delta_0=\omega_0-\omega,$$
$$\tilde{\mu}_{\rm d}^2=|\mu_{\rm d}|^2\left(\frac{\varepsilon+2}{3}\right)^2, \quad (31)$$

$$\frac{dN}{dt} + \gamma_1 N = \tilde{J}(t) + \frac{i}{2\hbar}(AB^* - A^*B),\ \tilde{J} = J + \gamma_1 N^{(0)}.$$

Here, we neglected the term $4\pi(\varepsilon_0 + 2)\mu_d^2 NB/(9\hbar)$ because it is small compared to the term B/τ_2 for most laser media.

Depending on the relation between parameters γ_c, γ_1, γ_2, γ_{rad}, $|\omega_c - \omega_0|$, lasers can be divided into four dynamic classes.

Lasers for which the above parameters are of the same order of magnitude belong to the class A. Their dynamics is described by a complete system of Eqns (31).

Most existing lasers belong to the class B. For these lasers, $\gamma_2 \gg \gamma_1$, γ_c, γ_{rad}, and $|\omega_c - \omega_0|$. Such a relation between parameters allows one to simplify the system of Eqns (31) by neglecting the derivative of polarisation and excluding from equations the polarisation as a dynamic variable. As a result, we obtain

$$\frac{dA}{dt} + (\gamma_c + i\Delta_c)A = \frac{1}{2}\sigma c\left(1 - i\frac{\Delta_0}{\gamma_2}\right)NA,$$
$$\frac{dN}{dt} + \gamma_1 N = \tilde{J}(t) - 2\sigma c N\frac{|A|^2}{8\pi\hbar\omega}, \tag{32}$$

where

$$\sigma = 4\pi\,\frac{\omega}{c}\,\frac{\tilde{\mu}_d^2}{\hbar}\,\frac{1}{\gamma_2^2 + \Delta_0^2}$$

is the cross section for stimulated transitions.

Lasers for which $\gamma_2, \gamma_1 \gg \gamma_c, \gamma_{rad}, |\omega_c - \omega_0|$, belong to the class C. In this case, the derivatives in two last equations of the system (31) can be neglected, and both polarisation and the difference of populations N of energy levels can be excluded as dynamic variables. The system of equations is reduced to one equation for the complex amplitude A

$$\frac{dA}{dt} + (\gamma_c + i\Delta_c)A = \frac{1}{2}\sigma c\left(1 - i\frac{\Delta_0}{\gamma_2}\right) \times \frac{N_0}{1 + 2\sigma c|A|^2/8\pi\hbar\omega}A. \tag{33}$$

For lasers of the class D, we have $\gamma_c \gg \gamma_1, \gamma_2, \gamma_{rad}$, $|\omega_c - \omega_0|$, which allows one to exclude the field as a dynamic variable:

$$\frac{dB}{dt} + (\gamma_2 + i\Delta_0)B = \beta\,\frac{\tilde{\mu}_d^2}{\hbar}\,\frac{\gamma_c - i\Delta_c}{\gamma_c^2 + \Delta_c^2}\,BN,$$
$$\frac{dN}{dt} + \gamma_1 N = \mathcal{J} - \frac{\beta}{\hbar}\,\frac{\gamma_c}{\gamma_c^2 + \Delta_c^2}|B|^2. \tag{34}$$

3. Properties of the dynamics of class A lasers

To simplify an analysis performed below, we consider the case when $\omega_0 = \omega_c$, which gives $\Delta_0 = \Delta_c = 0$. Equation (31) has two nonzero stationary solutions

$$|A|^2 = A_{sat}^2(r - 1),\ iB = \pm\frac{\gamma_c}{\beta}A,\ A_{sat}^2 = \frac{\hbar^2\gamma_1\gamma_2}{\tilde{\mu}^2}, \tag{35}$$

where

$$r = 2\pi\,\frac{\mu_d^2}{\hbar}\,\frac{\omega_c\tilde{J}}{\gamma_c\gamma_1\gamma_2}.$$

These solutions correspond to the monochromatic lasing. Two stationary states C_+ and C_- of the laser correspond to them in the phase space A, B, N (Fig. 6). They cannot be distinguished from each other upon detection of the radiation field because they differ only in the phase difference for the field and polarisation. The lasing regimes (35) become unstable [36, 37] if

$$r > r^* \equiv \frac{\gamma_c}{\gamma_2}\,\frac{\gamma_c + \gamma_1 + 3\gamma_2}{\gamma_c - \gamma_1 - \gamma_2}. \tag{36}$$

To find out what regime develops instead of stable lasing with the constant field amplitude, it is necessary to analyse Eqns (31) numerically. The result of such a study, which was first performed in papers [36, 37], is shown in Fig. 7. For $r > r^*$, a complicated self-modulation process was developed instead of steady-state lasing. Gradually increasing oscillations were observed around one of the stationary points shown in Fig. 6 ($C_\pm$), and then the system jumped to another equilibrium region, etc. Fig. 8 shows the time dependence of the self-modulation process [38]. The residence time of the system in the vicinity of one of the stationary points is a random quantity, so that the process is a random one as a whole.

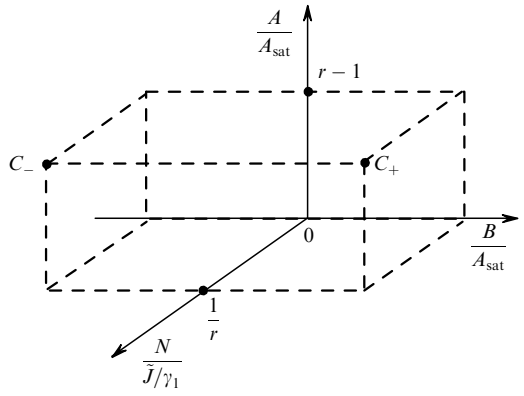

Figure 6. Phase plane of a laser (C_+ and C_- are stationary states of the laser).

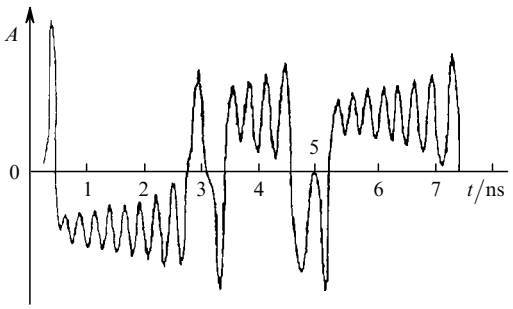

Figure 7. Time dependence of the laser-field amplitude in the regime of random oscillations (calculation [36]).

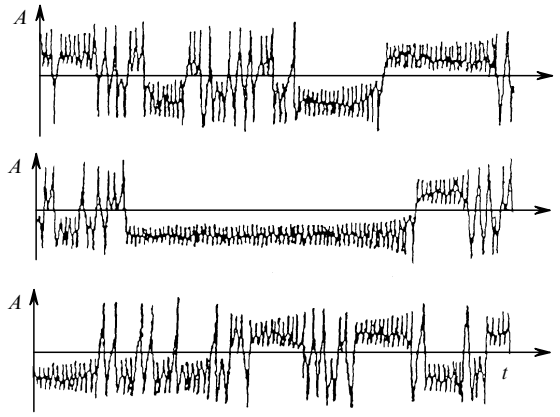

Figure 8. Time dependence of the laser-field amplitude in the regime of random oscillations (calculation [38]).

Let us analyse the behaviour of the system in the amplitude–phase representation. Fig. 9 shows the time dependences of the radiation intensity and of the sine of the phase difference [39]. One can see sharp jumps of the phase difference, which occur when the radiation intensity vanishes. Fig. 10 presents spatial phase trajectories in the dynamic chaos regime.

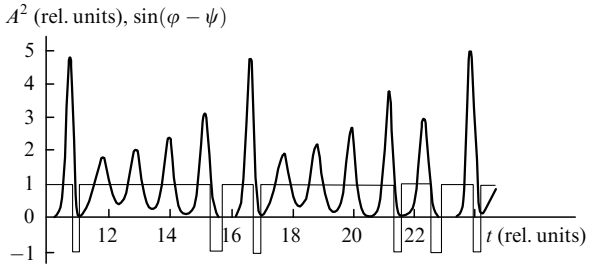

Figure 9. Variation in the phase difference between the field and polarisation of laser radiation in the regime of random oscillations without noise effects (calculation [39]). The thick curve is the laser-field amplitude, the thin curve is the sine of the phase difference.

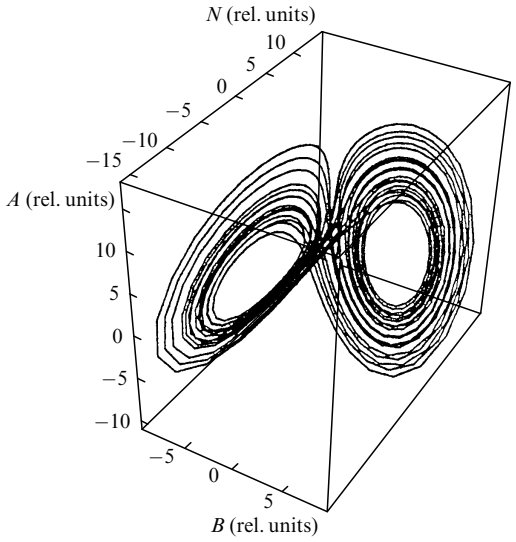

Figure 10. Phase portrait of the laser trajectory in the regime of random oscillations.

It follows from expression (36) that to observe the dynamic chaos experimentally, the condition $\gamma_c - \gamma_1 - \gamma_2 > 0$ should be satisfied. The dependence of r^* on the parameter γ_c/γ_2 is shown in Fig. 11. One can see that r^* has a minimum, which depends on the ratio γ_1/γ_2 and is approximately 15 for $\gamma_1/\gamma_2 = 1$. Therefore, to observe the regime of random oscillations, a laser is required in which the excitation threshold can be exceeded more than an order of magnitude. Such conditions can be obtained in a NH_3 laser [40–42].

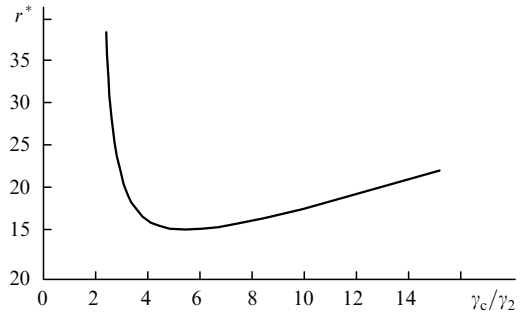

Figure 11. Critical value of the excitation parameter r^* as a function of γ_c/γ_2.

Let us discuss in more detail the results of paper [41]. In this paper, $^{15}NH_3$ molecules were excited by a $^{13}CO_2$ laser. The purely rotational 153-μm $aR(4, 4)$ transition was used for lasing. It can be expected that the gain line in the active medium will be homogeneously broadened at comparatively

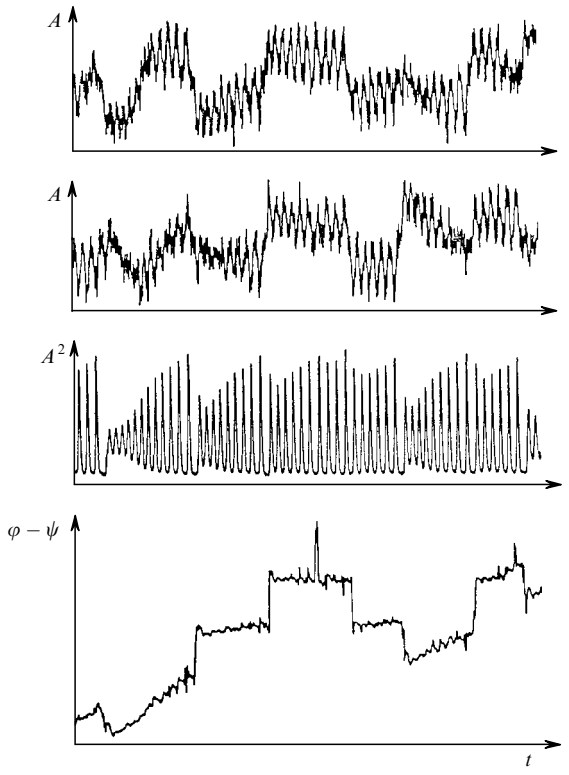

Figure 12. Time dependence of the field amplitude and the phase difference between the field and polarisation of laser radiation in the regime of random oscillations (experiment [42]).

high gas pressures, and the lasing regime will be described by the theoretical model (31). These expectations were completely confirmed. Fig. 12 shows the oscillations that were observed in the $^{15}NH_3$ laser for $\gamma_c/\gamma_2 = 4.5$, $\gamma_1/\gamma_2 = 0.25$, and $r > 10.8$ in the absence of mismatch. One can see that the pattern of random oscillations is completely consistent with theoretical predictions (see Figs 7–9).

At comparatively low pressures, noticeable deviations from the predictions of the two-level model were observed. The reader can find the details in original papers [40–42] and references therein.

4. Properties of the dynamics of lasers of classes B, C, D

The class B lasers described by Eqns (32) have obviously the same stationary solutions as the class A lasers. However, we can expect beforehand that the dynamics of class B lasers is simpler. The mater is that, because the system of Eqns (32) contains only the field-energy density in the cavity, the equations for class B lasers have in fact the two-dimensional phase space, so that the field phase does not play the role of a dynamic variable. In an autonomous system with a two-dimensional phase space, no dynamic chaos can appear. One can easily verify that the stationary solutions for class B lasers are always stable: the relations between relaxation parameters for these lasers do not admit the existence of the critical value of the excitation parameter r^* [see (36)]. The dynamic possibilities of class B lasers are seemingly rather limited. However, these lasers have 'hidden' dynamic resources, as we will show below.

For the dimensionless variable $X = |A|^2/A_{sat}^2(r-1)$ at the dimensionless time scale $\tau - \Omega_0 t$, where $\Omega_0 = [\gamma_1\gamma_c(r-1)]^{1/2}$, Eqns (32) are equivalent to the second-order equation

$$X'' - \frac{(X')^2}{X} + X(X-1) = \left(\frac{\gamma}{\gamma_c}\right)^{1/2} f(X', X), \tag{37}$$

where the prime means differentiation over τ and

$$f(X', X) = -X'[1 + (r-1)X]. \tag{38}$$

For most class B lasers, the parameter $\gamma_1/\gamma_c \ll 1$, so that the solutions of Eqn (37) are close to the solutions of the conservative equation

$$X'' - \frac{(X')^2}{X} + X(X-1) = 0. \tag{39}$$

After the change of the dynamic variable $Y = \ln X$, equation (39) can be written inn the form

$$Y'' + U(Y) = 0, \quad U(Y) = e^Y - 1. \tag{40}$$

It is obvious that (40) is the equation of motion in a constant potential $U(Y)$. The study of Eqn (39) shows [43, 44] that it describes a peculiar oscillator whose oscillation period depends on the oscillation amplitude. Small-amplitude oscillations about the stationary state $X = 1$ are close to sinusoidal oscillations with the period $T_0 = 2\pi[\gamma_1\gamma_c(r-1)]^{-1/2}$ at the dimensional time scale. As the oscillation amplitude increases, they are transformed to a sequence of pulses whose duration decreases with increasing maximum oscillation amplitude X_{max} as $\sim 3.525X_{max}^{1/2}$, while the time interval between the pulses increases as $(8X_{max})^{1/2}$. Taking the right-hand side of Eqn (37) into account, we see that the decay time of the oscillations is close to $1/r\gamma_1$. However, even a weak external periodic perturbation is sufficient to excite undamped strong oscillations of the field amplitude.

Let us demonstrate this by the example of a weak modulation of the loss factor (the photon lifetime) in the cavity. Let us introduce a variable loss factor into (32) and write the equations for the photon density $W = |A|^2/(8\pi\hbar\omega)$ in the cavity:

$$\frac{dW}{dt} + \frac{1}{\tau_c}(1 + \delta\cos\Omega t)W = -\sigma NW,$$
$$\frac{dN}{dt} + \gamma_1 N = I - 2\sigma NW. \tag{41}$$

The loss modulation factor δ is assumed small. The modulation frequency Ω is of the order of $\Omega_0 = [\gamma_1\gamma_c(r-1)]^{1/2}$, but can be several times smaller than this value. Fig. 13 shows the time dynamic of the photon density W and the population difference N. Note that a weak modulation of the loss factor results in a weak modulation of the population, but in a *strong* modulation of the photon density. As δ increases or the modulation frequency becomes higher than $\Omega_0 = [\gamma_1\gamma_c(r-1)]^{1/2}$, the regular oscillations occurring through a successive doubling of the period become random. This does not contradict to the above statement that a chaos is impossible in a system in a two-dimensional phase space because the system (41) is not autonomous. All these theoretical conclusions were confirmed experimentally [45].

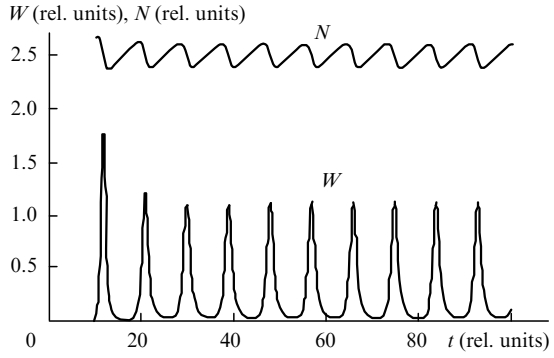

Figure 13. Time dependences of the photon density W and the population difference N in the class B laser in the case of a weak periodic modulation of the loss factor in the laser cavity.

The class B lasers (as class A lasers) can generate the so-called *giant* pulse. Due to a long lifetime of particles at the upper level, they can be accumulated at this level provided the cavity mirrors are blocked because otherwise a sufficient amount of particles cannot be accumulated due to stimulated emission. If, after the accumulation of particles at the upper level, the mirrors are rapidly unblocked, the lasing threshold will be strongly exceeded and the energy stored at

the upper level will rapidly transfer to the cavity in the form of an electromagnetic radiation and will be emitted by the laser as a short high-power pulse. In practice, the cavity mirrors are unblocked in time of the order of 10^{-8} s using the Kerr or Pockels effects [29].

The dynamics of class D lasers is similar as a whole to that of class B lasers. The only difference is that a giant-pulse class D laser ($\gamma_{rad} \gg \gamma_2$) can emit in the giant pulse virtually 100 % of the energy stored in its active medium, whereas the class B laser ($\gamma_{rad} \ll \gamma_2$) can emit in the giant pulse only up to 50 % of the stored energy.

This is explained as follows. For $\gamma_{rad} \gg \gamma_2$, the radiative processes dominate over relaxation processes, and a two-level quantum system (a two-level atom) can transfer from one quantum state to another under the action of the resonance field with the 100 % probability [46]. The relaxation shifts the phase of polarisation and the level population inversion in the field becomes impossible. Figs 14 and 15 show pulses emitted by class B and D lasers. The pulse emitted by class B laser is asymmetric. The difference of the energy level populations tends to zero, half the stored energy being not emitted. The pulse emitted by the D class laser is symmetric, and the level population difference changes its sign to the pulse end. The pulse shape is close to hyperbolic secant, which is typical for a soliton [46].

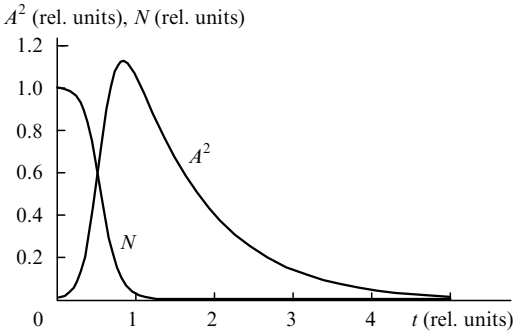

Figure 14. Giant-pulse shape for the class B laser and the time dependence of the population difference.

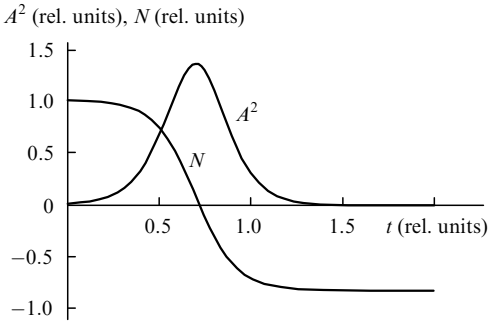

Figure 15. Giant-pulse shape for the class D laser and the time dependence of the population difference.

The class C lasers have a simple dynamics. The dynamic phase space of Eqn (33) is one-dimensional. The stationary states of class C lasers are stable, and a weak modulation of their parameters causes a weak modulation of the field amplitude in these lasers.

5. On the quantum theory of a laser

In the quantum theory of a laser, as in quantum electrodynamics, an electromagnetic field is considered as a set of quantum oscillators. The field can be naturally expanded in the eigenmodes of the laser cavity:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{i} \sum_k \omega_k \left[\hat{a}_k^+(t) \boldsymbol{U}_k^*(\boldsymbol{r}) - \hat{a}_k(t) \boldsymbol{U}_k(\boldsymbol{r})\right], \tag{42}$$

where the expansion coefficients $\hat{a}_k^+, \hat{a}_k$ represent the creation and annihilation operators for a photon in the corresponding mode of the cavity. The commutative relations between the operators corresponds to the commutative relations for an oscillator:

$$\hat{a}_k \hat{a}_k^+ - \hat{a}_k^+ \hat{a}_k = 1; \tag{43}$$

where $\hat{n}_k = \hat{a}_k^+ \hat{a}_k$ is the number operator for photons in a given cavity mode. However, a problem arises in the description of a coherent field. The coherent field corresponds to a nonzero average field value, however, averaging of the field (42) over a state with the specified number of photons gives zero. This problem can be solved by introducing coherent states $|\alpha_k\rangle$ of the field, which are the eigenstates of the photon annihilation operator [11, 47–49]

$$\hat{a}_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle. \tag{44}$$

The field

$$\langle \boldsymbol{E}(\boldsymbol{r}, t) \rangle = \mathrm{i} \sum_k \omega_k [\alpha_k^*(t) \boldsymbol{U}_k^*(\boldsymbol{r}) - \alpha_k(t) \boldsymbol{U}_k(\boldsymbol{r})] \tag{45}$$

averaged over these functions is nonzero and is a quantum analogue of a classical coherent field. The matter is that the quantum coherent state $|\alpha_k\rangle$ corresponds to a minimal indeterminacy in the phase and the number of photons.

The expansion of the coherent state in the states $|n_k\rangle$ with a certain number of photons has the form [11, 49]

$$|\alpha_k\rangle = \sum_{n_k} \exp\left(-\frac{|\alpha_k|^2}{2}\right) \frac{\alpha_k}{n_k^{1/2}!} |n_k\rangle. \tag{46}$$

One can see that the number of photons in the coherent state is not determined. The probability of detection of the number n_k of photons in the coherent state is [11, 49]

$$w(n_k) = \exp\left(-|\alpha_k|^2\right) \frac{|\alpha_k|^2}{n_k!}. \tag{47}$$

Expression (47) is a well-known Poisson distribution. The study of photon statistics in a He–Ne laser [50] has demonstrated very good agreement of expression (47) with experimental results. However, one should not believe that any laser always generates only coherent states of the type (44).

The quantum theory of a laser can be constructed based on Eqns (31) by considering the dynamic variables in them as operators. The presence of relaxation terms in the equations necessitates the introduction of external fluctuation forces (Langevin forces) into the right-hands sides of the equations. This approach to the description of the quantum theory of a laser is discussed in [51]. In addition, the density-matrix formalism can be used [52].

6. System of equations for the Bose condensate of atoms

The dynamics of the Bose condensate of atoms is described by the equation [1, 53–57]

$$i\hbar \frac{\partial}{\partial t}\psi(\boldsymbol{r},t) =$$

$$\left(-\frac{\hbar^2}{2m}\Delta + V_{\rm int}(\boldsymbol{r},t) + U|\psi(\boldsymbol{r},t)|^2\right)\psi(\boldsymbol{r},t), \qquad (48)$$

where $V_{\rm int}(\boldsymbol{r},t)$ is the energy of interaction of the condensate with an external field, for example, with the field of a trap in which atoms are captured; $U = (4\pi\hbar^2 a_{\rm coh})/m$ is the energy of the pair interaction between particles; and $a_{\rm coh}$ is the scattering length. Eqn (48) is called in the literature either the Pitaevskii–Gross equation or the nonlinear Schrödinger equation. It was obtained from the Schrödinger equation for a system of interacting particles by the method of self-consistent field. Therefore, the dynamic variable $\psi(\boldsymbol{r},t)$ represents initially the wave function of a subsystem of atoms forming the Bose condensate. In the limit of a sufficiently large numbers in the condensate, this function is treated as a classical quantity and is called the order parameter. Similarly to the laser coherent field, the wave function $\psi(\boldsymbol{r},t)$ can be considered as a classical limit of *the average value of the annihilation operator for particles in the Bose condensate*.

The dynamic meaning of equation (48) is similar to that of equation (33) for class C lasers. Within the framework of equation (48), the dynamics of particles outside the Bose condensate (incoherent particles) remains unexplored. However, equation (33) contains parameters that determine the appearance and existence of a coherent laser field, whereas equation (48) has no quantities that would determine the conditions of the appearance and existence of the order parameter. Therefore, it is clear that equation (48) describes only an ensemble of particles under stationary (quasi-stationary) conditions at the temperature below the critical one. In this sense, the existence of the term containing the time derivative in this equation can be justified only in the case when the time dependences of the parameters entering Eqn (48) do not result in the fallout of some incoherent particles from the condensate. The dynamics of incoherent particles becomes fundamentally important when atoms enter the trap and leave it and also upon the perturbation of the fields in the trap, which distort the equilibrium distribution of particles. In other words, the dynamic coupling between Eqn (48) and the equation for atoms outside the Bose condensate is required.

To find such a coupling, we will use the law of conservation of the number of particles. According to papers [58, 59], we rewrite equation (48) in the form of two real equations for the modulus $|\psi|$ of the order parameter and its phase φ ($\psi = |\psi|e^{i\varphi}$):

$$\frac{\partial|\psi|^2}{\partial t} = -\mathrm{div}\,\boldsymbol{j}, \quad \boldsymbol{j} = i\frac{\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \equiv \frac{\hbar}{m}|\psi|^2\nabla\varphi \qquad (49)$$

$$\frac{\partial\varphi}{\partial t}|\psi| = \frac{\hbar}{2m}\left[\nabla^2|\psi| - (\nabla\varphi)^2|\psi|\right] - \frac{V_{\rm int}(\boldsymbol{r},t)}{\hbar}|\psi| - \frac{U}{\hbar}|\psi|^3,$$

where $\boldsymbol{j}$ is the current of coherent particles. The first equation in (49) represents the law of conservation of the number of particles in the Bose condensate, which does not interact with other subsystems. Upon the interaction of these particles with a subsystem of incoherent particles, the particles will transfer from one subsystem to another. To take this interaction into account, this equation should be supplemented with a corresponding term describing such transitions.

It is convenient to proceed further using the language of *quasi-particles* introduced by Bogolyubov [1, 60]. The creation ($\hat{B}_p^+$) and annihilation ($\hat{B}_p$) operators for quasi-particles with the momentum $\boldsymbol{p}$ are coupled with the corresponding operators $\hat{A}_p^+$ and $\hat{A}_p$ of noninteracting atoms by the relations

$$\hat{A}_p = \frac{\hat{B}_p + L_p\hat{B}_{-p}^+}{(1-L_p^2)^{1/2}}, \quad \hat{A}_p^+ = \frac{\hat{B}_p^+ + L_p\hat{B}_{-p}}{(1-L_p^2)^{1/2}}, \qquad (50)$$

$$L_p = \frac{1}{mu^2}\left[\varepsilon(\boldsymbol{p}) - \frac{p^2}{2m} - mu^2\right],$$

$$\varepsilon(\boldsymbol{p}) = \left[u^2p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}, \quad u = \left(\frac{UN_{\rm g}}{m}\right)^{1/2}, \qquad (51)$$

where $\varepsilon(p)$ is the kinetic energy of quasi-particles with the momentum p; $N_{\rm g}$ is the number of particles in the system. The use of quasi-particles is convenient because, in particular, their equilibrium energy distribution is described by the Bose–Einstein law with the zero chemical potential

$$n_p = \left[\exp\left(\frac{\varepsilon(p)}{kT}\right) - 1\right]^{-1}. \qquad (52)$$

By using the law of conservation of particles, we can write

$$\frac{\partial|\psi|^2}{\partial t} = \Sigma - \mathrm{div}\,\boldsymbol{j}, \quad \Sigma = \int S_p^{\rm rec}(\boldsymbol{r},t)\frac{d^3p}{(2\pi\hbar)^3}, \qquad (53)$$

where

$$S_p^{\rm rec}(\boldsymbol{r},t) = W^{\rightarrow}(\boldsymbol{p})\left(|\psi|^2+1\right)n_p(\boldsymbol{r},t)$$

$$-W^{\leftarrow}(\boldsymbol{p})|\psi|^2[n_p(\boldsymbol{r},t)+1] \qquad (54)$$

describes the particle exchange between the subsystems. The meaning of equation (54) is quite clear. The term $W^{\rightarrow}(\boldsymbol{p})(|\psi|^2+1)n_p(\boldsymbol{r},t)$ describes the transition resulting in an increase in the concentration of particles in the Bose condensate and a decrease in the concentration $n_p(\boldsymbol{r},t)$ of quasi-particles by unity in some incoherent subsystem. Therefore, this term is proportional to the concentration of particles in the Bose condensate plus unity. The term $W^{\leftarrow}(\boldsymbol{p})|\psi|^2[n_p(\boldsymbol{r},t)+1]$ describes the inverse transition. The functions $W^{\rightarrow}(\boldsymbol{p})$, $W^{\leftarrow}(\boldsymbol{p})$ depend on the specific interaction between atoms.

Because atoms involved in the condensate interact not only with each other but also with incoherent particles, we should make the replacement

$$\frac{U}{\hbar}|\psi|^3 \to \frac{U}{\hbar}\left(|\psi|^2 + n\right)|\psi| \tag{55}$$

in the right-hand side of the second equation of system (49), where

$$n = \int n_p(\boldsymbol{r}, t)\,\frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \tag{56}$$

is the total concentration of incoherent particles as a function of the coordinate and time.

The second equation from (49) and equation (53), in which the replacement (55) was performed, can be combined to form a single complex equation using the following idea. The exchange term (54) can be naturally separated into two components. One of them is proportional to $|\psi|^2$ and describes the induced (coherent) part of the process of the particle exchange between subsystems. This component is included into the equation as a regular term. Another component represents the spontaneous incoherent process and is included into the equation in the form of a random Langevin force $\eta(\boldsymbol{r}, t)$. As a result, we obtain the equation

$$\frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) = \frac{1}{\mathrm{i}\hbar}\left[-\frac{\hbar^2}{2m}\Delta + V_{\mathrm{int}}(\boldsymbol{r}, t) + U\left(|\psi(\boldsymbol{r}, t)|^2 + n(\boldsymbol{r}, t)\right)\right]\psi(\boldsymbol{r}, t) = \Sigma(n)\psi(\boldsymbol{r}, t) + \eta(\boldsymbol{r}, t), \tag{57}$$

$$\Sigma(n) = \frac{1}{2}\int\{W^{\rightarrow}(\boldsymbol{p})n_p(\boldsymbol{r}, t) - W^{\leftarrow}(\boldsymbol{p})[n_p(\boldsymbol{r}, t) + 1]\}\,\frac{\mathrm{d}^3 p}{(2\pi\hbar)^3},$$

$$\langle\eta(\boldsymbol{r}, t)\eta(\boldsymbol{r}, t')\rangle = f(t - t')\int W^{\rightarrow}(\boldsymbol{p})n_p(\boldsymbol{r}, t)\,\frac{\mathrm{d}^3 p}{(2\pi\hbar)^3}, \tag{58}$$

where $f(t - t')$ describes the correlation properties of the Langevin force. The time dependence of the correlation function is determined by physical conditions. It is often assumed in the literature that a random Langevin force is δ-correlated [see, for example, 61, 65, 66].

We will describe the dynamics of an ensemble of incoherent quasi-particles using the kinetic equation [63, 64]

$$\frac{\partial}{\partial t}\,n_p(\boldsymbol{r}, t) + \frac{\partial}{\partial \boldsymbol{r}}\,n_p(\boldsymbol{r}, t)\frac{\partial \varepsilon}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}}\,n_p(\boldsymbol{r}, t)\frac{\partial \varepsilon}{\partial \boldsymbol{r}} = -S_p^{\mathrm{rec}}(\boldsymbol{r}, t) + S_p^{\mathrm{col}}(\boldsymbol{r}, t) + Q_p^{\mathrm{in}}(\boldsymbol{r}, t) - Q_p^{\mathrm{out}}(\boldsymbol{r}, t), \tag{59}$$

where $S_p^{\mathrm{col}}(\boldsymbol{r}, t)$ is the collision integral for quasi-particles; and $Q_p^{\mathrm{in}}(\boldsymbol{r}, t)$ is the input of incoherent particles to the system and $Q_p^{\mathrm{out}}(\boldsymbol{r}, t)$ is their output from the system. While the input of particles is specified by external circumstances, their output depends on the distribution function of quasi-particles. The input and output of particles can be also determined by the boundary conditions.

Equations (57) and (59) represent the desired system. This system can describe cooling (heating) of a system of particles and Bose condensation. It is evident that Bose condensation occurs when the exchange integral (58) is positive. The positive sign of this integral gives the 'inversion' condition (5). Equations (57) and (59) yield the law of conservation of particles in the form

$$\frac{\partial}{\partial t}\left[|\psi(\boldsymbol{r}, t)|^2 + \int n_p(\boldsymbol{r}, t)\,\frac{\mathrm{d}^3 p}{(2\pi\hbar)^3}\right] = -\mathrm{div}\,\boldsymbol{j} + \int\left[Q_p^{\mathrm{in}}(\boldsymbol{r}, t) - Q_p^{\mathrm{out}}(\boldsymbol{r}, t)\right]\frac{\mathrm{d}^3 p}{(2\pi\hbar)^3}. \tag{60}$$

The system can contain particles that play the role of a buffer and do not experience Bose condensation under the conditions under study. Their role can be taken into account in the particle exchange term and the collision integral.

Kinetic Eqn (59) is valid when the motion of particles is not quantised. Because the motion in a trap is quantised, Eqn (59) is valid only when the energy quantum of a particle in the trap is less than kT, and hence, of kT_{c} (T_{c} is the critical temperature of Bose condensation). Otherwise Eqn (59) should be replaced by its quantum analogue, which is a purely technical problem.

Let us assume that an ensemble of atoms is cooled to the absolute zero and there is neither inflow nor outflow of atoms. In this case, the number of atoms in the condensate is strictly defined, and the condensate cannot be in the coherent state. However, this is valid for an ideal gas. In a gas of interacting particles, there are always a certain number of real particles (not quasi-particles!) with nonzero energy even at the absolute zero. Their distribution is described by the relation ([1], Ch. 3, (23.18))

$$N_p = \frac{1}{2}\,\frac{m^2 u^4}{[\varepsilon(p) + p^2/2m + mu^2]\varepsilon(p)}. \tag{61}$$

Therefore, the number of real particles in the Bose condensate is not strictly defined. This specifies the degree of uncertainty of its phase (the degree of coherence):

$$\Delta N_0 \Delta\varphi = \frac{1}{2},\ \Delta N_0 = \left(\langle N_p^2\rangle - \langle N_p\rangle^2\right)^{1/2},$$

$$\langle N_p\rangle = \frac{8}{3\sqrt{\pi}}\left(\frac{N_{\mathrm{g}} a^3}{V}\right)^{1/2}. \tag{62}$$

7. 'Atomic laser' and its dynamics

If the conditions are provided under which incoherent atoms captured by a trap leave it in the form of a coherent condensate, then the trap will resemble a laser generating a coherent electromagnetic wave. Traps that are used for producing the atomic Bose condensate are commonly constructed so that the atoms with the negative projection of the total momentum on the direction of the magnetic field of the trap (for example, $m_F = -1$) are located in the attractive potential (Fig. 16), while the potential energy of atoms with $m_F \geqslant 0$ in the trap has a maximum and decays to its periphery, so that the atoms with $m_F \geqslant 0$ are forced out from the trap. For this reason, the Bose condensate is formed by atoms with the negative value of m_F. In real experiments, the sodium, rubidium, and lithium atoms are used for which the total momentum is $F = 1$. By applying a radio-frequency field, which transfers atoms from the state $m_F = -1$ to the state $m_F = 0$ [10], coherent atoms can be extracted from the trap. Of course, the field should be sufficiently monochromatic lest the mutual coherence of the atoms leaving the trap be destroyed.

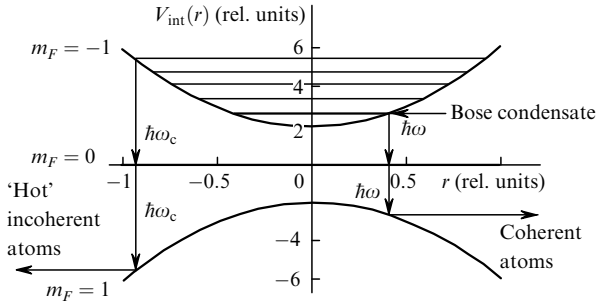

Figure 16. Scheme of a source of a coherent atomic beam – 'an atomic laser' (ω is the radio-frequency used to extract a beam of coherent atoms from a trap; ω_c is the radio-frequency used to extract a beam of incoherent atoms for cooling an ensemble of atoms in the trap; stable potential with $m_F = -1$; unstable potentials with $m_F = 0$ and 1).

Atoms in the trap are usually cooled below the critical temperature by applying a radio-frequency field, in the same way as upon extracting coherent atoms from the trap (Fig. 16). The frequency of this field is chosen so that the high-energy atoms, whose frequency substantially differs from the frequency of low-energy atoms with, would be in the resonance. For this reason, the high-energy atoms transfer to the state with an unstable potential and leave the trap, providing cooling of the atoms remaining in the trap. The cooling rate can be controlled by varying the intensity and frequency of the applied field. By choosing different frequencies for the cooling and extracting fields, the conditions can be provided when these two processes will not appreciably affect each other.

The dynamics of an atomic laser was described in paper [10] by the system of equations

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_{\pm}(\boldsymbol{r},t) = \left(-\frac{\hbar}{2m}\Delta + V_{\pm}^{\rm int}(\boldsymbol{r},t) + U|\psi(\boldsymbol{r},t)|2\right) \times \psi_{\pm}(\boldsymbol{r},t) + \hbar\Omega\,\psi_{\mp}(\boldsymbol{r},t)\exp(\pm\mathrm{i}\omega t), \qquad (63)$$

$$|\psi(\boldsymbol{r},t)|^2 = |\psi_+(\boldsymbol{r},t)|^2 + |\psi_-(\boldsymbol{r},t)|^2.$$

Here, ψ_+ and ψ_- are the order parameters for the Bose condensate of atoms in the attractive and repulsive potentials of the trap, respectively; $\hbar\Omega(\boldsymbol{r}) = \mu H(\boldsymbol{r})$ is the coupling coefficient between the condensates caused by radio-frequency radiation; μ is the magnetic dipole moment of an atom for the $m_F = -1 \to m_F = 0$ transition; H and ω are the amplitude and frequency of the radio-frequency field, respectively. The coupling coefficient can be changed by varying the radio-frequency field amplitude. The latter relation in (63) was presented in paper [10]. In our opinion, a more correct relation is the superposition of two order parameters, which we will take into account below.

The system of Eqns (63) does not reflect the fact that upon generation of a coherent atomic beam, the atoms should enter the trap in the incoherent state and leave it in the form of the coherent Bose condensate. Therefore, incoherent atoms are also involved in the generation process, and a consistent description of an 'atomic laser' should be based on a system of equations unifying the equations for the Bose condensate and a subsystem of incoherent atoms.

The equation for an 'atomic laser' obtained using this unified system has the form [59]

$$\frac{\partial}{\partial t}\psi_{\pm}(\boldsymbol{r},t) = -\frac{\mathrm{i}}{\hbar}\left\{-\frac{\hbar^2}{2m}\Delta + V_{\pm}^{\rm int}(\boldsymbol{r},t) + U\left[|\psi(\boldsymbol{r},t)|^2 + N(\boldsymbol{r},t)\right]\right\}\psi\pm(\boldsymbol{r},t) + \Sigma(n_{\pm})\psi_{\pm}(\boldsymbol{r},t) - \mathrm{i}\Omega(\boldsymbol{r})\psi_{\mp}(\boldsymbol{r},t)\exp(\pm\mathrm{i}\omega t) + \eta(\boldsymbol{r},t), \qquad (64)$$

$$\frac{\partial}{\partial t}n_{p\pm}(\boldsymbol{r},t) + \frac{\partial}{\partial \boldsymbol{r}}n_{p\pm}(\boldsymbol{r},t)\frac{\partial\varepsilon}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}}n_{p\pm}(\boldsymbol{r},t)\frac{\partial\varepsilon}{\partial \boldsymbol{r}} = -S_{p\pm}^{\rm rec}(\boldsymbol{r},t) + S_p^{\rm col}(\boldsymbol{r},t) + Q_{p\pm}^{\rm in}(\boldsymbol{r},t) - Q_{p\pm}^{\rm out}(\boldsymbol{r},t),$$

where

$$\psi(\boldsymbol{r},t) = \psi_+(\boldsymbol{r},t) + \psi_-(\boldsymbol{r},t).$$

Strictly speaking, there are three potentials in the trap for atoms with the total momentum $F = 1$. Following paper [10], we restricted ourselves to a model case of two potentials. The writing of a system of equations for three and more potentials presents no substantial problems.

A comprehensive analysis of the system of equations (64) is a complicated problem. This analysis can be based on the results of studies of the dynamics of lasers, which has been developed well enough in papers [24, 25, 30–33, 40 45]. By simplifying the system (64), we can elucidate some important questions.

Let us assume that the trap potential is time-independent and find the eigenfunctions $\Phi_{\pm}(\boldsymbol{r})$ of the trap potentials by solving the equations

$$\left(-\frac{\hbar^2}{2m}\Delta + V_{\pm}^{\rm int}(\boldsymbol{r},t)\right)\Phi_{\pm}(\boldsymbol{r}) = E_{\pm}\Phi_{\pm}(\boldsymbol{r}), \qquad (65)$$

where $E_{\pm}$ are the energy eigenvalues for the corresponding potentials. We assume that $E_+ = \varepsilon_+$ is a purely real quantity for a stable potential, and $E_- = \varepsilon_- + \mathrm{i}\gamma$ is a complex quantity for an unstable potential.

Let us represent the required order parameters in the form

$$\psi_+(\boldsymbol{r},t) = A(t)\Phi_+(\boldsymbol{r})\exp\frac{\mathrm{i}}{\hbar}\left[\varepsilon_+ t - \int u_+(t)\mathrm{d}t\right],$$
$$\psi_-(\boldsymbol{r},t) = B(t)\Phi_-(\boldsymbol{r})\exp\frac{\mathrm{i}}{\hbar}\left[\varepsilon_- t - \int u_-(t)\mathrm{d}t\right], \qquad (66)$$

where

$$u_{\pm}(t) = U\int\left[|\psi(\boldsymbol{r},t)|^2 + n(\boldsymbol{r},t)\right]|\Phi_{\pm}(\boldsymbol{r})|^2\mathrm{d}^3r\,.$$

We replace the equations for the distribution of particles over momenta by the equation for the average number density of particles. We represent the term describing the output of incoherent particles from the trap in the relaxation form $Q_{\pm}^{\rm out} = v_{\pm}n_{\pm}$. By substituting (66) into the first equation of the system (64) and by multiplying the obtained equation by $\Phi_{\pm}^*(\boldsymbol{r})$, we integrate all the terms of the equation over the trap volume. As a result, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} - \bar{\sigma}_+ A + \mathrm{i}\bar{\Omega} B \exp[\mathrm{i}\delta(t)] + \bar{\eta}_+(t),$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = (\bar{\sigma}_- - \gamma)B - \mathrm{i}\bar{\Omega}^* A \exp[-\mathrm{i}\delta(t)] + \bar{\eta}(t), \tag{67}$$

$$\frac{\mathrm{d}n_\pm}{\mathrm{d}t} = -2\bar{\sigma}_\pm |A|^2 - v_\pm n_\pm + Q_\pm ,$$

where

$$\bar{\sigma}_\pm = \int \Sigma(n_\pm)|\Phi_\pm|^2 \mathrm{d}^3 r, \quad \bar{\Omega} = \int \Omega(\boldsymbol{r})\Phi_+^* \Phi_- \mathrm{d}^3 r,$$
$$\delta(t) = \frac{1}{\hbar}\left\{(\varepsilon_+ - \varepsilon_-)t - \int [u_+(t) - u_-(t)]\mathrm{d}t\right\} - \omega t , \tag{68}$$

$$\left\langle \eta_\pm^*(\boldsymbol{r}, t)\eta_\pm(\boldsymbol{r}, t')\right\rangle = f(t - t')$$
$$\times \int\int W^{\rightarrow}(\boldsymbol{p}) n_p(\boldsymbol{r}, t)|\Phi_\pm(\boldsymbol{r})|^2 \frac{\mathrm{d}^3 p\, \mathrm{d}^3 r}{(2\pi\hbar)^2}. \tag{69}$$

Despite the simplifications made above, the basic processes responsible for the formation of Bose condensate in the trap are retained in the equations.

For convenience, we will use the notation

$$A(t) = |A(t)| \exp[\mathrm{i}\varphi(t)], \quad B(t) = |B(t) \exp[\mathrm{i}\xi(t)] \tag{70}$$

and rewrite equations (67) in the form

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = \bar{\sigma}_+|A| + \bar{\Omega}|B| \sin(\delta + \xi - \varphi)$$
$$+ \mathrm{Re}[\bar{\eta}_+(t) \exp(-\mathrm{i}\varphi)],$$
$$\frac{\mathrm{d}|B|}{\mathrm{d}t} = (\bar{\sigma}_- - \gamma)|B| - \bar{\Omega}|A| \sin(\delta + \xi - \varphi) \tag{71}$$
$$+ \mathrm{Re}[\bar{\eta}_-(t) \exp(-\mathrm{i}\xi)],$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}|A| = -\bar{\Omega}|B| \cos(\delta + \xi - \varphi) + \mathrm{Im}[\bar{\eta}_+(t) \exp(-\mathrm{i}\varphi)],$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t}|B| = -\bar{\Omega}|A| \cos(\delta + \xi - \varphi) + \mathrm{Im}[\bar{\eta}_-(t) \exp(-\mathrm{i}\xi)], \tag{72}$$

$$\frac{\mathrm{d}n_\pm}{\mathrm{d}t} = -2\bar{\sigma}_\pm |A|^2 - v_\pm n_\pm + Q_\pm .$$

By considering the dynamics of atoms in the trap, we can neglect fluctuations in the first approximation. Then, stationary regimes are possible at which all derivatives in equations (71) and (72) are zero. One can easily see that $\cos(\delta + \xi - \varphi) = 0$ in the stationary regime, and the moduli of amplitudes are

$$|A_{\mathrm{st}}| = |B_{\mathrm{st}}| = 0, \quad (n_\pm)_{\mathrm{st}} = \frac{Q_\pm}{v_\pm} \tag{73}$$

or

$$|A_{\mathrm{st}}| \neq 0, \quad |B_{\mathrm{st}}| \neq 0 . \tag{74}$$

It is obvious that in the case (73), the coherent Bose condensate does not exist.

Let us study the stability of the regime (73) by the usual method of linearisation of the equations with respect to the stationary values. The linearised equations

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = \bar{\sigma}_+|A| + \bar{\Omega}|B|, \quad \frac{\mathrm{d}|B|}{\mathrm{d}t} = (\bar{\sigma}_- - \gamma_-)|B| - \bar{\Omega}|A| \tag{75}$$

are simple enough in order to describe immediately the conditions of the instability of the zero state (73):

$$\bar{\sigma}_+ + \bar{\sigma}_- - \gamma > 0, \quad \bar{\sigma}_+(\sigma_- - \gamma_-) + \bar{\Omega}^2 < 0 . \tag{76}$$

Because atoms in the state (−) rapidly leave the trap, $\bar{\sigma}_-$ is small compared to γ. Therefore, to generate the Bose condensate in the trap, it is necessary and sufficient that

$$\bar{\sigma}_+ > \min\left(\gamma, \frac{\bar{\Omega}^2}{\gamma}\right). \tag{77}$$

The condition (77) for formation of the coherent Bose condensate is more stringent than condition (14), which means only that $\bar{\sigma}_+$ should be positive. Therefore, for

$$0 < \bar{\sigma}_+ < \min\left(\gamma, \frac{\bar{\Omega}^2}{\gamma}\right) \tag{78}$$

the situation appears in the 'atomic laser' when Bose condensation will occur only due to spontaneous transitions [according to Eqns (67), due to fluctuations] even below the critical temperature. In this case, the Bose condensate will not be coherent. The inequality (77) is physically equivalent to the self-excitation condition for a common laser, which, as mentioned above, is more stringent than the condition for the population inversion in the laser medium, and depends on absorption of photons in the laser and emission of laser photons. If v_+ is small enough to be neglected, then a stationary state with the zero amplitudes of the order parameter can exist only in the absence of the particle inflow to the trap ($Q_\pm = 0$). The inflow of particles to the trap at $v_+ = 0$ results in the so-called *non-threshold* Bose condensation if the atoms captured by the trap are cool enough for the total temperature in the system to be below the critical temperature. The maximum critical temperature is obviously determined by the maximum number of particles that can be confined in the trap. The non-threshold lasing is also studied in the field of laser physics [67, 68].

If we are interested in the dynamics of the order parameter rather than in the details of the distribution function of incoherent particles, we can use a simple approximation for the functional $\bar{\sigma}_+$, because $\bar{\sigma}_+$ depends first of all on the total concentration of incoherent particles in the trap and does not change substantially upon continuous variations in the energy distribution of particles. For example, the approximation

$$\bar{\sigma}_+ = \sigma_0(n - n_0) \tag{79}$$

is possible, where σ_0 is a constant and n_c is the equilibrium density of incoherent quasi-particles for the zero chemical potential and the specified temperature of particles in the trap. It is determined by expression (10). By using relation

(79), we can calculate the nonzero values of the stationary amplitudes $|A_{st}|$ and $|B_{st}|$ of the order parameter and find the flux of coherent atoms leaving the trap. By neglecting $\bar{\sigma}_-$ and γ_+, we obtain

$$|A_{st}| = \frac{1}{2}\left[\frac{\gamma}{\bar{\Omega}^2}(Q_+ - v_+ n_0) - \frac{v_+}{\sigma_0}\right], \qquad (80)$$

$$|B_{st}| = \frac{\bar{\Omega}}{\gamma}|A_{st}|, \quad I = 2\gamma|B_{st}|V,$$

where V is the trap volume. The condition of stationary phases $\cos(\delta + \xi - \varphi) = 0$ determines the frequency of a radio-frequency field at which the stationary regime is possible. This frequency should satisfy the relation

$$\delta(t) = \frac{1}{\hbar}\left\{(\varepsilon_+ - \varepsilon_-)t - \int[u_+(t) - u_-(t)]\mathrm{d}t\right\} - \omega t = 0, \quad (81)$$

which can be fulfilled at constant amplitudes of the order parameter. If the condition (81) is not satisfied, the regime with constant amplitude is impossible. Upon continuous entering of incoherent particles to the trap, the order parameter of the Bose condensate will be an oscillating function of time.

If γ exceeds the other parameters having the dimensionality of the inverse time, the dynamic variable B can be excluded from the system of Eqns (67) and this system will take the form

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = \left(\bar{\sigma}_+ - \frac{\bar{\Omega}^2}{\gamma}\right)|A|, \qquad (82)$$

$$\frac{\mathrm{d}n_+}{\mathrm{d}t} = -2\bar{\sigma}_+|A|^2 - v_+ n_+ + Q_+ .$$

After the substitution of expression (79) into (82) instead of $\bar{\sigma}_+$, the system of equations for the 'atomic laser' becomes isomorphous to the system of equations (32) for the class B laser. Therefore, the theory of relaxation oscillations [43, 44], which was briefly described in section 4, can be applied to the 'atomic laser'. In particular, by modulating the amplitude of a radio-frequency field, strong oscillations of the order parameter of atoms in the trap can be excited, resulting in the appearance of a beam of coherent atoms outside the trap in the form of periodically repeated bunches. The 'atomic laser' can be also brought to the dynamic chaos regime, as has been done with a CO_2 laser [45]. In this case, the atomic bunches are no longer periodic in time. The experimental study of the dynamic chaos in an atomic trap is a very interesting problem.

The 'atomic laser' can be used as a source of a single pulse of coherent atoms. For this purpose, in the absence of a radio-frequency field, which transfers atoms in the Bose condensate from a stable potential to an unstable potential, the Bose condensate should be created in the trap and then a radio-frequency field should be rapidly switched on.

As mentioned above, using the system of Eqns (67), we can describe both the dynamics of the Bose condensate and the process of its formation upon cooling atoms in the trap. In the simplest form, this can be fulfilled upon slow cooling, when the formation of the quasi-equilibrium distribution function occurs faster than a change in the mean energy of atoms in the trap. In this case, the mean energy of the gas can be expressed in terms of its temperature, which can be treated as an additional dynamic variable. This approach is used in the theory of a semiconductor laser, when the temperature of carriers is considered separately from the lattice temperature as an additional dynamic variable [69, 70].

Note in conclusion that the dynamics of Bose condensate of atoms in a trap has been described by different methods [71–81]. The approach presented above is physically clear and reflects many important features of the dynamics of Bose condensate.

8. Superconducting Bose condensate

Similarly to an electromagnetic field and atomic Bose condensate, we can introduce the creation ($\hat{A}^+$) and annihilation ($\hat{A}$) operators for a Cooper pair and define a coherent state as the eigenfunction of the equation

$$\hat{A}|\Psi\rangle = \Psi|\Psi\rangle. \qquad (83)$$

Then, in the limit of a great number of particles, the function Ψ can be identified with the order parameter of a superconductor, which was first introduced in paper [82] where the stationary equation

$$\left(a - b|\Psi(\boldsymbol{r})|^2\right)\Psi(\boldsymbol{r}) + \left(\nabla - \mathrm{i}\frac{2e}{\hbar c}\boldsymbol{A}(\boldsymbol{r})\right)^2\Psi(\boldsymbol{r}) = 0 \qquad (84)$$

was proposed for the order parameter. Here, $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential of a magnetic field; and a and b are parameters describing the properties of the material, the parameter a being temperature-dependent:

$$a(T) = \alpha\frac{T_c - T}{T_c}. \qquad (85)$$

To construct the theory of a superconducting state based on Eqn (84), this equation should be supplemented with the expression for the superconducting current

$$\boldsymbol{j}(\boldsymbol{r}) = \mathrm{i}\frac{e\hbar}{2m}[\Psi(\boldsymbol{r})\nabla\Psi^*(\boldsymbol{r}) - \Psi^*(\boldsymbol{r})\nabla\Psi(\boldsymbol{r})]$$

$$-\frac{2e^2}{mc}|\Psi(\boldsymbol{r})|^2\boldsymbol{A}(\boldsymbol{r}) \qquad (86)$$

and with boundary conditions.

Equation (84), which was initially obtained by minimising the free-energy functional [82], was then substantiated using the microscopic approach [83]. This equation is assumed to be valid near the critical temperature. At present, equation (84) forms the basis for the macroscopic theory of superconductivity (the Ψ-theory, according to the terminology used in [2]), which is discussed in books [1, 84].

The nonstationary equation for the order parameter was obtained only for the so-called zero-gap superconductors [85] containing paramagnetic impurities at sufficiently high concentrations. It has the form [84]

$$\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = D\left(a - b|\Psi(\boldsymbol{r},t)|^2\right)\Psi(\boldsymbol{r},t)$$

$$+D\left(\nabla-\mathrm{i}\,\frac{2e}{\hbar c}\,\boldsymbol{A}(\boldsymbol{r})\right)^2\Psi(\boldsymbol{r},t)-\mathrm{i}\,\frac{2e}{\hbar}\,\chi(\boldsymbol{r},t)\Psi(\boldsymbol{r},t). \qquad (87)$$

Here, D is the diffusion coefficient of electrons in the normal state; χ is the electrochemical potential per electron charge. It follows from this equation that

$$\frac{\partial|\Psi(\boldsymbol{r},t)|^2}{\partial t}=D\bigl(a-b|\Psi(\boldsymbol{r},t)|^2\bigr)|\Psi(\boldsymbol{r},t)|^2$$

$$+2\mathrm{Re}\bigl[\Psi^*(\boldsymbol{r},t)L\Psi(\boldsymbol{r},t)\bigr], \qquad (88)$$

$$L\equiv D\left(\nabla-\mathrm{i}\,\frac{2e}{\hbar c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2.$$

The term $2\mathrm{Re}[\Psi^*(\boldsymbol{r},t)\hat{L}\Psi(\boldsymbol{r},t)]$ in the right-hand side of equation (88) is not the divergence of the current. Equation (48) does not seemingly satisfy the law of conservation of the number of particles. However, such a difficulty does not appear in equation (57) for the atomic Bose condensate. Formally, this is explained by the fact that the operator ∇^2 in equation (57) has an imaginary unit factor. A similar factor is absent in equation (87). How can it be understood? This question will be elucidated in the next section devoted to the derivation of equations that combine the order parameter and free quasi-particles into a unified dynamic system.

9. Dynamic system of equations for a superconductor

Let us compare equation (87) with dynamic equations for a laser. Based on this equation, a superconductor would be assigned to class C lasers. Recall that relaxation processes in the active medium of class C lasers proceed much faster than the relaxation of photon in the laser cavity. In superconductor, free particles play the role of an active medium, so that we can assume that equation (87) is obtained from a complete dynamic system of equations for a superconductor by adiabatically excluding the concentration of particles as a dynamic variable. This is possible only when the quasi-equilibrium in a subsystem of quasi-particles is established much faster than in a subsystem of Cooper pairs. Otherwise a complete dynamic system of equations is needed to describe the dynamics of a superconductor.

The derivation of a complete nonstationary coupled system of equations for the order parameter and free quasi-particles, based on a consistent microscopic theory, is a difficult problem, which has not been solved so far [62]. For this reason an attempt was made [58] to solve this problem using simulation based on the energy band diagram of a superconductor (Fig. 5). According to [58], we will derive this system of equations using the law of conservation of particles and consider, as a first step, the case when the order parameter is independent of coordinates. Then, the time dependence of the modulus of the order parameter is completely determined by the particle exchange with the m and n electron subsystems, so that

$$\frac{\partial|\Psi(\boldsymbol{r},t)|^2}{\partial t}=\Sigma_m=+\Sigma_n\,, \qquad (89)$$

where Σ_m is the rate of particle exchange between a subsystem of Cooper pairs and the m electron subsystem and Σ_n is the rate of particle exchange between subsystems of Cooper pairs and free quasi-particles. In essence, these are the same recombination integrals that were discussed in section 6. Dynamically, they are similar to recombination integrals discussed in the literature [63, 64].

The rate Σ_m can be divided into two rates describing the induced and spontaneous transitions, respectively: $\Sigma_m=\Sigma_m^{\mathrm{ind}}+\Sigma_m^{\mathrm{sp}}$. The quantum-mechanical rate of induced transitions between subsystems is

$$\Sigma_m=|\Psi(\boldsymbol{r},t)|^2\int W_m(p)[m_pm_{-p}-(1-m_p)$$

$$\times(1-m_{-p})]\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}. \qquad (90)$$

The function $W_m(p)$ is a matrix element of the Hamiltonian of interaction between particles for a transition from a coherent state of Cooper pairs (the Ψ subsystem) to a state of the m subsystem. As mentioned above, the electrons with momenta close to the Fermi momentum p_{F} efficiently interact with each other in a superconductor. For this reason, $W_m(p)$ rather rapidly decreases with increasing difference between p and p_{F}, so that the effective number of electrons involved in the formation of the Cooper pair condensate becomes noticeably lower than the total number of electrons in the conduction band. Accordingly, we introduce the total effective number of electrons involved in superconductivity,

$$N_{\mathrm{s}}=\frac{1}{W_m(p_{\mathrm{F}})}\int W_m(p)\,\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}, \qquad (91)$$

and the effective number of electrons in the m subsystem

$$N_m=\frac{1}{W_m(p_{\mathrm{F}})}\int m_pW_m(p)\,\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}. \qquad (92)$$

According to the law of conservation of particles,

$$n+N_m+2|\Psi(\boldsymbol{r},t)|^2=N_{\mathrm{s}}\,, \qquad (93)$$

where n is the total number of free quasi-particles in the n subsystem. By using (93) and excluding N_m from expression (90), we obtain

$$\Sigma_m^{\mathrm{ind}}=\frac{2}{\tau_m}\left(1-2\frac{n}{N_{\mathrm{s}}}-4\frac{|\Psi(\boldsymbol{r},t)|^2}{N_{\mathrm{s}}}\right)|\Psi(\boldsymbol{r},t)|^2,$$

$$\frac{2}{\tau_m}=W_m(p_{\mathrm{F}})N_{\mathrm{s}}\,. \qquad (94)$$

Spontaneous transitions from the Ψ subsystem to the m subsystem are described by the quantity

$$\Sigma_m^{\mathrm{sp}}=\int W_m(p)m_pm_{-p}\frac{\mathrm{d}^3p}{(2\pi\hbar)}. \qquad (95)$$

Assuming that m_p is described by the Fermi distribution and using relation (93), we will express (95) in terms of the total concentration of quasi-particles and Cooper pairs. We will not perform these calculations explicitly because there is no need for this below.

The parameter

$$\Sigma_n = \int\int W_n(p,\boldsymbol{p}')\left[n_p n_{p'}(M_q+1)(|\psi|^2+1)\right.$$

$$\left.-(1-n_p)(1-n_{p'})M_q|\psi|^2\right]\mathrm{d}^3p'\mathrm{d}^3p \qquad (96)$$

is usually called the recombination integral. It is constructed similarly to the exchange integral in an atomic system between the Bose condensate and incoherent particles. The only difference is that Bose condensation of Cooper pairs occurs due to the recombination of two Fermi particles involving phonons whose spectral concentration is denoted by M_q, where $q = p - p'$ due to the law of conservation of momentum.

If we now assume that the order parameter and the magnetic field depend on coordinates, the term $2\mathrm{Re}[\Psi^*(\boldsymbol{r},t)\hat{L}\Psi(\boldsymbol{r},t)]$ should by included into equation (89), so that

$$\frac{\partial|\Psi(\boldsymbol{r},t)|^2}{\partial t} = \frac{2}{\tau_m}\left(1-2\,\frac{n}{N_\mathrm{s}}-4\,\frac{|\Psi(\boldsymbol{r},t)|^2}{N_\mathrm{s}}\right)|\Psi(\boldsymbol{r},t)|^2$$

$$+2\mathrm{Re}\Psi(\boldsymbol{r},t)\hat{L}\Psi^*(\boldsymbol{r},t).$$

This elucidates the problem of conservation of the number of particles. One can see that not only free quasi-particles but also 'under-condensate' unpaired electrons are involved in the dynamics of the superconductor. This circumstance can be interpreted as follows.

Let us assume that a superconducting sample at the temperature close to the absolute zero is placed in a magnetic field, which adiabatically increases with time. As the magnetic field increases, it will penetrate deep in the sample, decreasing the order parameter. The decrease in the modulus of the order parameter is related to the destruction of Cooper pairs. The electrons produced in this process cannot be free quasi-particles because this requires the energy equal to the width Δ of the superconducting gap, which can be borrowed from nowhere at the temperature close to the absolute zero. These electrons can come only to the system of under-condensate electrons. Therefore, if we wrote the explicit equation for the concentration m of particles in the subsystem, then the expression $2\mathrm{Re}[\Psi^*(\boldsymbol{r},t)\hat{L}\Psi(\boldsymbol{r},t)]$ should appear in the left-hand side of this equation but with the minus sign. This solves the problem of conservation of the number of particles in equation (87).

As a result, we can pass to the equation for the complex order parameter Ψ, by including into it the term with the chemical potential

$$\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = -\mathrm{i}\,\frac{2e}{\hbar}\,\chi(\boldsymbol{r},t)\Psi(\boldsymbol{r},t)+\frac{1}{\tau_m}\left(1-2\,\frac{n}{N_\mathrm{s}}\right.$$

$$\left.-4\,\frac{|\Psi(\boldsymbol{r},t)|^2}{N_\mathrm{s}}\right)\Psi(\boldsymbol{r},t)+D\left(\nabla-\mathrm{i}\,\frac{2e}{\hbar c}\,\boldsymbol{A}(\boldsymbol{r},t)\right)^2\Psi(\boldsymbol{r},t)$$

$$+\frac{1}{2}\,S^{\mathrm{rec}}\Psi(\boldsymbol{r},t)+\Xi(\boldsymbol{r},t), \qquad (97)$$

$$S^{\mathrm{rec}}(\boldsymbol{r},t) = \int W_n(p,\boldsymbol{p}')[n_p n_{p'}(M_q+1)$$

$$-(1-n_p)(1-n_{p'}M_q]\mathrm{d}^3p'\mathrm{d}^3p\,. \qquad (98)$$

The quantity Σ_n contains terms proportional to $\Psi(\boldsymbol{r},t)$. These terms correspond to induced transitions and they appear in Eqn (97) as the coherent term $(1/2)S^{\mathrm{rec}}\Psi(\boldsymbol{r},t)$, which is proportional to the order parameter. The remaining term corresponds to spontaneous transitions. It is included into Eqn (97) as the fluctuation force $\Xi(t)$. This fluctuation force contains the term caused by spontaneous $\Psi \to m$ transitions. The correlation function $\Xi(t)$ is described by the expression

$$\langle\Xi^*(t)\Xi(t')\rangle = f(t-t')\left[\Sigma_m^{\mathrm{sp}}+\int\int W_n(p,p')n_p n_{p'}M_q\mathrm{d}^3p\mathrm{d}^3p'\right]$$

[see expression (59) and comments to it].

The dynamics of free quasi-particles was considered in papers [58–60]. It was described by the kinetic equation

$$\frac{\partial}{\partial t}\,n_p(\boldsymbol{r},t)+\frac{\partial}{\partial\boldsymbol{r}}\,n_p(\boldsymbol{r},t)\,\frac{\partial\varepsilon}{\partial\boldsymbol{p}}-\frac{\partial}{\partial\boldsymbol{p}}\,n_p(\boldsymbol{r},t)\,\frac{\partial\varepsilon}{\partial\boldsymbol{r}}$$

$$= S_p^{\mathrm{col}}(\boldsymbol{r},t)-S_p^{\mathrm{rec}}(\boldsymbol{r},t)\,, \qquad (99)$$

in which the energy of quasi-particles is determined by the relation

$$\varepsilon = \left\{\Delta^2+\frac{1}{4m^2}\left[p_\mathrm{F}^2-\left(\boldsymbol{p}-\frac{e}{c}\,\boldsymbol{A}\right)^2\right]^2\right\}^{1/2}. \qquad (100)$$

The quantity $S_p^{\mathrm{col}}(\boldsymbol{r},t)$ in equation (99) is a common collision integral, which describes the momentum and energy redistributions of quasi-particles in a subsystem. The recombination integral $S_p^{\mathrm{rec}}(\boldsymbol{r},t)$ describes the exchange of particles in the subsystem n with particles in the condensate of Cooper pairs and in the system m. This recombination integral is presented in review [64] in the form

$$S_p^{\mathrm{rec}}(\boldsymbol{r},t) = \int U_\mathrm{e}(p,\boldsymbol{p}',\Delta)[n_p n_{p'}(M_q+1)$$

$$-(1-n_p)(1-n_{p'})M_q]\mathrm{d}^3p'. \qquad (101)$$

It contains, along with the concentration of free particles, the phonon concentration M_q. The function U_e, as W_n in expression (98), is nothing but a matrix element of the interaction Hamiltonian of the particles. The difference between these functions is that W_n is the matrix element of the interaction Hamiltonian for the transition between the subsystem n and the Cooper pair condensate, whereas U_e corresponds to the transition between the subsystem n and the combined $m-\Psi$ system, which was considered in papers [63, 64] as a specific vacuum in which free quasi-particles are created.

If the phonon concentration is nonequilibrium, is should be also described by a dynamic equation. This equation is often written in a simple form [64, 86]

$$\frac{\partial M_q}{\partial t} = -\frac{1}{\tau_q}(M_q-M_{q0})+S_q^{\mathrm{rec}}+S_q^{\mathrm{col}}+Q_q\,, \qquad (102)$$

where M_{q0} is the equilibrium phonon concentration at a given temperature. The first term in the right-hand side of Eqn (102) describes the escape of phonons from a superconducting sample to a substrate, the second term describes the creation of phonons upon recombination of free quasi-

particles, the third term is caused by scattering of quasi-particles, and the fourth term describes excitation of phonons by an external source. The recombination term is

$$S_q^{\mathrm{rec}}(t) = \frac{1}{V_\mathrm{s}} \int U(\boldsymbol{p}, \boldsymbol{p}-\boldsymbol{q}, \Delta)[n_p n_{p'}(M_q+1) - (1-n_p)(1-n_{p'})M_q]\mathrm{d}^3p'\mathrm{d}^3r, \tag{103}$$

where V_s is the sample volume.

Therefore, the order parameter (97), free quasi-particles (99) and phonons (102) are unified into a dynamic system due to recombination processes. If necessary, the recombination integral for photons instead of phonons (or along with the latter) can be written. Obviously, the system of equations (97), (99), (102) should be supplemented with the Maxwell equations, which will be coupled with this system by the carrier current and the density of photons if the latter interact with the superconductor. In a nonstationary state, the current includes the non-superconducting component along with the superconducting component (86), so that the total current is

$$\boldsymbol{J}(\boldsymbol{r},t) = \mathrm{i}\frac{e\hbar}{2m}[\Psi(\boldsymbol{r},t)\nabla\Psi^*(\boldsymbol{r},t) - \Psi^*(\boldsymbol{r},t)\nabla\Psi(\boldsymbol{r},t)] - \frac{2e^2}{mc}|\Psi(\boldsymbol{r},t)|^2\boldsymbol{A}(\boldsymbol{r},t) - \sigma\left(\nabla\chi + \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right). \tag{104}$$

10. Stationary state of a superconductor

Let us return to the stationary state when the order parameter is time-independent. Strictly speaking, this state takes place when fluctuations are neglected. It is described by the equation

$$\frac{1}{D\tau_m}\left(1 - 2\frac{n}{N_\mathrm{s}} - 4\frac{|\Psi(\boldsymbol{r},t)|^2}{N_\mathrm{s}}\right)\Psi(\boldsymbol{r},t) + \left(\nabla - \mathrm{i}\frac{2e}{\hbar c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2\Psi(\boldsymbol{r},t) = 0. \tag{105}$$

Equation (105) differs from equation (84) in that the coefficient $a = (1/D\tau_m)(1-2n/N_\mathrm{s})$ in it acquired the dynamic sense. In the stationary state,

$$n = \int\left(\exp\frac{\varepsilon}{kT}+1\right)^{-1}\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}, \tag{106}$$

ε being determined by expression (100). Therefore, the coefficient a depends not only on temperature but also on a magnetic field and, finally, on the spatial coordinates; although, the dependence on a magnetic field is negligible if $(e/c)A \ll p_\mathrm{F}$. In the absence of a magnetic field, the coefficient a depends only on temperature. To calculate this dependence, we should know the temperature dependence of the energy gap, which is determined by the Bardeen–Cooper–Schrieffer equation [1, 17]

$$\frac{g}{2(2\pi\hbar)^3}\int\frac{\tanh(\varepsilon/2kT)}{\varepsilon}\,\mathrm{d}^3p = 1. \tag{107}$$

The function $\Delta(T)$ can be calculated analytically [1] only in the limiting cases of low temperatures,

$$\Delta = \Delta^{(0)}\left[1 - \left(\frac{2\pi kT}{\Delta^{(0)}}\right)^{1/2}\exp\left(-\frac{\Delta^{(0)}}{kT}\right)^{1/2}\right] \tag{108}$$

and at temperatures close to the critical temperature,

$$\Delta = 3.06kT_\mathrm{c}\left(1 - \frac{T}{T_\mathrm{c}}\right)^{1/2}. \tag{109}$$

The critical temperature T_c and the energy gap $\Delta^{(0)}$ at the absolute zero are related by the expression $T_\mathrm{c} = 0.57\Delta^{(0)}$.

The function $a(T)$ calculated numerically over the entire temperature range is shown in Fig. 17. As expected, it differs from the linear dependence (85), which takes place only at temperatures close to the critical temperature. If for $\Delta(T_\mathrm{c}) = 0$, the critical temperature is defined by the condition $a(T_\mathrm{c}) = 1$, then we obtain

$$N_\mathrm{s} = \frac{0.693mp_\mathrm{F}kT_\mathrm{c}}{\pi^2\hbar^3}, \tag{110}$$

and the coefficient $a(T)$ can be represented in the form

$$a(T) = \frac{1}{D\tau_m}\left\{1 - 1.44\frac{T}{T_\mathrm{c}} \times\int\left[\exp\left(\frac{\Delta^2}{(kT)^2}+\xi^2\right)^{1/2}+1\right]^{-1}\mathrm{d}\xi\right\}. \tag{111}$$

Note finally that the coefficient $a(T)$ can be also nonzero when the energy gap is zero. Therefore, equation (97) also describes zero-gap superconductors, which were predicted in paper [85].

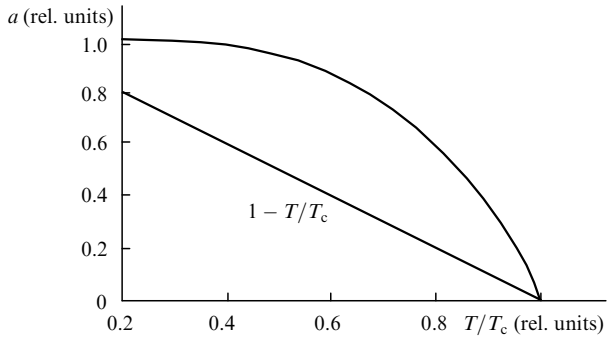

Figure 17. Temperature dependence of the parameter a.

Equation (105) in the absence of a magnetic field takes the form

$$\nabla^2\Psi(\boldsymbol{r},t) + \frac{1}{D\tau_m}\left(1 - 2\frac{n}{N_\mathrm{s}} - 4\frac{|\Psi(\boldsymbol{r},t)|^2}{N_\mathrm{s}}\right)\Psi(\boldsymbol{r},t) = 0. \tag{112}$$

Except the obvious solutions $\Psi = 0$ and $|\Psi|^2 = (N_\mathrm{s}-2n)/4 \equiv D\tau_m a(T)$, equation (112) has other solutions, which depends on coordinates and can be either periodic or 'asymptotic' functions, i.e., they can tend to infinity on a certain surface (at point, in the one-dimensional case). The

periodic functions exist at positive values of $a(T)$, while the asymptotic functions exist at negative values of $a(T)$ [87]. One of the interesting problems in the physics of the superconducting state is the problem of the superconductor–metal contact, which was considered earlier using the microscopic theory [88].

In paper [87], the method was proposed for calculating the superconductor–metal contact using the Ψ theory and the asymptotic solutions. One of the results of this paper is the conclusion that the order parameter in a superconducting layer at the temperature below the critical temperature initiates the order parameter in another contacting superconducting layer whose temperature is higher than the critical temperature. In calculations in paper [87], a linear temperature dependence of the coefficient a was used. However, the dependence $a(T)$ shown in Fig. 17 is more favourable for the appearance of this effect because the coefficient $a(T)$ achieves the value close to the maximum at higher temperatures.

11. On nonstationary processes in superconductors

Let us combine the equations describing the dynamics of a superconductor in a single system

$$\frac{\partial \Psi(\boldsymbol{r},t)}{\partial t} = -\mathrm{i}\,\frac{2e}{\hbar}\,\chi(\boldsymbol{r},t)\Psi(\boldsymbol{r},t) + \frac{1}{\tau_m}\left(1 - 2\,\frac{n}{N_\mathrm{s}} - 4\,\frac{|\Psi(\boldsymbol{r},t)|^2}{N_\mathrm{s}}\right)\Psi(\boldsymbol{r},t) + D\left(\nabla - \mathrm{i}\,\frac{2e}{\hbar c}\,\boldsymbol{A}(\boldsymbol{r},t)\right)^2\Psi(\boldsymbol{r},t) + \frac{1}{2}\,S^{\mathrm{rec}}\Psi(\boldsymbol{r},t) + \Xi(\boldsymbol{r},t),$$

$$\frac{\partial}{\partial t}\,n_p(\boldsymbol{r},t) + \frac{\partial}{\partial \boldsymbol{r}}\,n_p(\boldsymbol{r},t)\,\frac{\partial \varepsilon}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}}\,n_p(\boldsymbol{r},t)\,\frac{\partial \varepsilon}{\partial \boldsymbol{r}} = S_p^{\mathrm{col}}(\boldsymbol{r},t) - S_p^{\mathrm{rec}}(\boldsymbol{r},t)\,, \tag{113}$$

$$\frac{\partial M_q}{\partial t} = -\frac{1}{\tau_q}(M_q - M_{q0}) + S_q^{\mathrm{rec}} + S_q^{\mathrm{sc}} + Q_q\,,$$

where S_q^{sc} describes the scattering of photons. When a superconducting sample is irradiated by a high-frequency electromagnetic field (for example, with frequency that is equal to or higher than $\Delta/\hbar$), a family of dynamic equations should be supplemented with the nonstationary Maxwell equations. As a result, we obtain the developed dynamic system with the multidimensional phase space. Because of this, a variety of dynamic regimes can occur in the superconductor, which are of interest both theoretically and for practice. In particular, it would be interesting to find and study the regimes with a deep modulation of the order parameter caused by a comparatively weak modulation of an external magnetic field of sound.

12. Concluding remarks

Until recently, the phenomena in which the Bose condensate of particles with a nonzero mass and laser radiation would be the components of a single dynamic process have not been discussed in the literature. Only very recently, the assumptions were made that such a process can occur. The studies of stimulated emission from a strongly overexcited injection GaAs laser [89] revealed a number of interesting and nontrivial circumstances, which were discussed in detail at the scientific session of the Division of General Physics and Astronomy of RAS [90]. A circumstance that is most important for the topic considered here is that the stimulated emission spectrum is shifted to the red compared to the emission spectrum of a normally excited laser. Such behaviour of the emission spectrum can be explained by the fact that carriers in the overexcited semiconductor laser are 'pressed down' to the bottom of the corresponding bands. Based on this fact and some other facts, which are described in detail in papers [89, 90], the author of these papers assumed that the Bose condensate was formed in the electron–hole plasma of a strongly excited semiconductor [90].

The question of the possibility of Bose condensation of photoexcited electron–hole pairs in semiconductors has been the focus of attention for the 20th-century scientists [91–94]. It has been found in these studies that the electron–hole liquid produced upon powerful laser excitation is not superfluid and has little in common with the Bose condensate of excitons. When the density of carriers is high and the mean distance between them is smaller than the exciton radius, the concept of an exciton as a particle has no sense. However, when the density of nonequilibrium carriers is high, we deal with *a correlated state of the electron–hole plasma*. In this case, similarly to a superconductor, the correlation increases with increasing carrier density and should appear at higher temperatures. There are two energy branches in the valence band of GaAs, which correspond to light and heavy holes with larger and smaller effective masses. The appearance of the correlated state at nearly room temperature is quite possible for the electron–hole plasma in GaAs consisting of light holes.

Let us emphasise once more that stimulated transitions play a decisive role in the formation of the coherent Bose condensate of any nature. We believe that these transitions underlie *the universal mechanism of spontaneous violation of the symmetry in Nature* [20, 21]. To do this requires only the conditions under which the rate of the spontaneous process of producing some object with characteristic properties would exceed the rate of its decay. It is likely that stimulated processes have played a crucial role in the formation of our Universe, whose symmetry was violated in favour of electrons and protons. A similar assumption can be made concerning the origin of life with the left chirality of proteins. Such assumptions have been discussed in the literature.

The self-organisation processes proceeding in nonlinear systems have been already discussed over about twenty years [95–101]. A special term, synergetics, was even coined, which unifies a variety of self-organisation processes. However, this term combines processes only by their superficial properties. In our opinion, the mechanism inherent in all self-organisation processes is stimulated transitions.

The study of global problems of the society development attracts increasing current interest. One of the main objects of discussion in paper [101] is the increase in the Earth population. Based on the long-standing facts, the author [101] obtained the seemingly paradoxical formula, which explains a comparatively slow increase in the Earth population in ancient times and its rapid increase in the 20th century, and also predicts a rather strong slowing down of

this process in the 21st century. It is likely that this and other processes of the society development can be better understood if the formal description of their dynamics is based not on the Earth population as a whole and on its separate parts but on the corresponding order parameter. The squared modulus of the order parameter can be related to the Earth population, while its phase will reflect various public relations. The stimulated processes of the formation of the order parameter of the society as whole and of its separate parts should be undoubtedly taken into account.

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