Quantum Electronics **33** (9) 777 – 797 (2003)

INVITED PAPER

PACS numbers: 03.50.De; 14.70.Bh
DOI: 10.1070/QE2003v033n09ABEH002500

Stopped light: Towards plane-wave-free electrodynamics

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Contents

***Abstract*. A spatially infinite plane wave, as one of the possible solutions of Maxwell's equations, is a mathematically correct but physically empty image. It is pointed out that physically realised solutions in the form of nonplane waves are not strictly transverse waves outside a material medium and are characterised by three polarisation components, the superluminal phase and subluminal group velocities, the dispersion, the necessary presence of the fragments of a standing wave ('stopped light'), and the existence of a mass-like quantity, which can be defined as a finite observable (but not immanent) inertial and gravitational photon rest mass. This mass cannot be distinguished in a number of thought ('gedanken') experiments from the rest mass in a standard treatment.**

Keywords: *standing wave, photon rest mass.*

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Received 25 September 2002
Kvantovaya Elektronika **33** (9) 777 – 797 (2003)
Translated by M.N. Sapozhnikov

1. Introduction

Electromagnetic waves in vacuum are strictly transverse and have only two possible polarisations, their phase and group velocities are identical and equal to c, and they have no dispersion. This means in the language of quantum mechanics that the rest mass of photons represented by such waves is zero [1]. These absolutely indisputable fundamental concepts of classical electrodynamics are obviously demonstrated, for example, upon a consideration of a plane electromagnetic wave.

At the same time, there exist a number of quite obvious phenomena in physically realisable fields that contradict, at the first glance, to these fundamental concepts, although these phenomena are considered below strictly within the framework of Maxwell's electrodynamics without the use of any additional hypotheses. These phenomena include the superluminal and subluminal propagation and dispersion of electromagnetic waves outside a material medium, the necessary presence of the longitudinal components and the third component of polarisation in any electromagnetic fields in free space, which violate a strictly transverse nature of the electromagnetic wave, as well as the presence of an immobile component in the form of the fragments of a standing wave ('stopped light'). The latter results in the existence of a finite mass-like quantity, which can be interpreted as the rest mass of photons filling the modes of a free space. Although it is impossible to distinguish this mass-like quantity in thought experiments from the nonzero inertial and gravitational rest mass of a heavy particle in a standard treatment, this quantity is not, of course, an immanent characteristic of a photon as, for example, proton or electron rest masses. Therefore, as distinct from the real rest mass, we will call this quantity the *observable photon rest mass*.

The starting point of our analysis is the often-ignored fact of a physical emptiness of the concept of an ideal spatially infinite plane wave. The abandonment of the concept of a plane wave as a physical reality transforms all the solutions of electrodynamic equations to spatially limited solutions, which therefore satisfy the known corollaries of the Fourier theorem, which are treated as the 'classical uncertainty relation'. The interrelated properties of physically realisable wave fields in a free space listed above follow directly in essence from the latter statement.

Thus, the consequence of the spatial restriction of the fields over the transverse coordinates with respect to the main propagation direction of the wave is the necessary presence of the fragments of a standing wave in the wave. It is this so-called stopped light that gives rise to the nonzero observable inertial and gravitational photon rest mass.

For analysis presented below, the advice of L.I. Mandelstam concerning the approach to the study of complicated electromagnetic phenomena is exclusively productive. He said: 'I believe that the following approach is proper: to consider a simple case of the propagation of electromagnetic waves, which can be really accurately and rigorously studied, – to investigate this case and understand what is behind it' [2].

Following this advice, the main attention in this paper is paid to the most expressive and physically clear example of a spatial restriction of the wave fields – the fields located in an infinitely deep potential well for photons. To this limiting case of spatially restricted wave fields, the fields of the modes of an ideal hollow metal waveguide correspond, which are used below as a basic model for phenomena under study. To construct the basic model, the principal differences between the fields of other physically realisable nonplane waves and the idealised field of a plane wave are preliminary considered. Several typical examples of the waves in a free space are considered, such as the radiation field of a harmonic electric dipole – a test object in the sense of Mandelstam's advice, etc. It is found that these fields, as almost all other fields that can be represented in the form of analytic solutions, can be topologically continuously transformed to the fields of the modes of hollow metal waveguides. This is an additional motivation for using the waveguide modes as a basic model for studying the problems considered here.

A number of thought kinematic and dynamic experiments show that the behaviour of the observable inertial and gravitational photon rest mass in a waveguide is indistinguishable from that of usual heavy particles under similar conditions.

The facts studied in the paper, which reveal more clearly some laws of classical electrodynamics, are also of heuristic interest because they make it possible to observe the continuos transformation of photons of the massless wave field to quanta having the nonzero inertial and gravitational rest mass. It is found that a typical massless field is capable, under certain conditions, of producing the field of massive particles.

1.1 A plane wave as a trivial but physically nonexistent solution of electrodynamic equations

'... and God divided the light from the darkness' (Genesis 1:4)

As pointed out above, all the above statements do not assume the revision of the foundations of Maxwell's electrodynamics, i.e., the equations relating the field vectors $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, and $\boldsymbol{B}$ to the volume charge density ρ and the conduction current density $\boldsymbol{j}$:

$$\mathrm{rot}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.1}$$

$$\mathrm{rot}\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j}, \tag{1.2}$$

$$\mathrm{div}\boldsymbol{D} = \rho, \tag{1.3}$$

$$\mathrm{div}\boldsymbol{B} = 0, \tag{1.4}$$

$$\boldsymbol{D} = \varepsilon_0 \varepsilon \boldsymbol{E}, \tag{1.5}$$

$$\boldsymbol{B} = \mu_0 \mu \boldsymbol{H}, \tag{1.6}$$

where ε_0 and μ_0 are the electric and magnetic constants of vacuum, respectively, and ε and μ are permitivity and permeability for a substance; and t is time. Below, it is assumed that $\varepsilon = \mu = 1$ in most cases.

A spatially infinite plane wave, which is invariable over the transverse spatial coordinates x and y and is propagating with the frequency ω and the wave vector $\boldsymbol{k}$ ($|\boldsymbol{k}| = \omega(\varepsilon\mu)^{1/2}/c$) along the coordinate z

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0 \cos(\omega t - kz), \tag{1.7}$$

$$H(z, t) = H_0 \cos(\omega t - kz) \tag{1.8}$$

(E_0 = const and H_0 = const are amplitude factors) is one of the possible solutions of the wave equations

$$\nabla^2 E - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 E}{\partial t^2} = 0, \tag{1.9}$$

$$\nabla^2 H - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 H}{\partial t^2} = 0, \tag{1.10}$$

which are direct corollaries of Maxwell's equations in the absence of field sources ($\rho = 0$ and $j = 0$).

Of course, a plane wave (1.7), (1.8) is a perfect mathematical tool of Maxwell's electrodynamics. However, it is well known that an attempt to assign the physical meaning to this idealised concept leads at once to the conclusion that such a solution of the wave equation is trivial and a real single plane wave cannot exist. It is sufficient to refer to a direct corollary of the integral Poynting theorem according to which the propagation of a plane wave with the finite field vectors should be accompanied by transfer of an infinitely large integrated energy flux and, vice versa, when the energy flux is finite, the amplitudes of the field vectors should be inevitably zero. It is clear that both these cases are physically unacceptable.

Certainly, this concerns only a single isolated plane wave. It is obvious that a superposition of many plane waves can represent any real wave field and, vice versa, the latter can be always expanded in a system of plane waves with different wave vectors. However, such partial plane waves do not exist independently beyond their superposition representing a real wave field, which is, naturally, not a plane wave.

Therefore, it is clear that any physically realised wave fields are spatially limited and differ to some extent from the concept of a perfect plane wave.

Note, however, that the epigraph implies that a fundamental fact of the finiteness of the space occupied by any electromagnetic field was understood even in quite ancient times. Indeed, one of the main acts during the first creation day was *the spatial localisation of the primary electromagnetic field.*

1.2 The observable photon rest mass as a result of the presence of fragments of a standing wave in the field. The momentum defect

We should discuss especially the fact of the inevitable presence of the fragments of a standing wave (stopped light) in any real wave field. The term stopped light sounds as an absolute oxymoron because an electromagnetic wave exists only during its propagation. Nevertheless, people could 'stop' light long ago. 'If you place a light source at night between two smooth mirrors separated by a distance of cubit, you will see the infinite reflections of light, each next reflection being smaller than the previous one. And in this way they go away to infinity, as if each mirror contains an infinite number of other mirrors'. This description of a device, which is now known as an open optical resonator, belongs to Leonardo da Vinci and is presented in a notebook, which was called later 'The Atlantic Codex'.

Here, it is also pertinent to recall the answer of Schrödinger to the question that he formulated himself in paper [3]: 'Whether the statements that a plane wave cannot be transformed to rest and, hence, the photon rest mass is zero are absolutely rigorous or only approximate? The answer is that, when an attempt is made to stop a plane wave, its field decreases to zero in the limit, i.e., the plane wave ceases to exist'.

The spatial localisation of any physical wave field requires the refinement of the statement that the photon rest mass M is equal to zero [4]. Indeed, this standard statement directly follows, strictly speaking, from the form of the Hamiltonian for a free particle:

$$E^2 = (Mc^2)^2 + (c|p|)^2, \tag{1.11}$$

where $E = \hbar\omega$ is the total photon energy, if it is assumed, as usual, that the photon momentum $|p|$ is $\hbar\omega/c$. This is undoubtedly valid only for photons described by an infinite plane wave, which, as mentioned above, does not have any real physical meaning.

The presence of the fragments of a standing wave in real wave fields drastically changes the situation. The partial waves forming such a fragment have the counterpropagating components of the wave vectors Δk_1 and $\Delta k_2 = -\Delta k_1$ with equal moduli, whose vector sum is $\Delta k_1 + \Delta k_2 = 0$. Therefore, the quantity

$$\delta|p| = 2\hbar|\Delta k_1|, \tag{1.12}$$

is subtracted from the modulus of the total photon momentum represented by a real electromagnetic wave. This quantity can be called *the momentum defect*, and expression (1.11) for a photon belonging to the field containing the fragment of a standing wave should be now written as

$$E^2 = (Mc^2)^2 + \left(c\hbar\frac{\omega}{c}\right)^2 - (c\delta|p|)^2. \tag{1.13}$$

It is obvious from this modified expression that there exists the nonzero mass-like quantity

$$M = \frac{\delta|p|}{c}, \tag{1.14}$$

to which, as will be shown below, one can assign the properties of the nonzero observable photon rest mass in a standard treatment.

If the momentum defect becomes equal to $\delta|p| = \hbar\omega/c$ (a perfect standing wave), then the photon momentum proves to be completely exhausted ($|p| = 0$), and the light is fully stopped, while the observable photon rest mass M becomes equal to $\hbar\omega/c^2$.

All the above said is close to two statements: 'Any real light flux has the nonzero intrinsic mass. Only an infinite plane wave, i.e., the flux of strictly collinear photons has the intrinsic mass equal to zero. However, such a light flux can be never produced because any real light flux is spatially restricted, i.e., it is not an infinite plane wave' [5] and 'A system of two photons will have a zero mass only when they are propagating in the same direction' [6]. Both these statements were formulated only for light fluxes but not for a single photon. The question arises of whether these statements can be applied to the case of a standing wave containing a *single* photon in a mode, to which a mass should be assigned? It seems that the answer should be undoubtedly positive, as the answer to the question about the exclusive possibility of observing interference from the same single photon.

It is important to note that so far no experimental methods are available for distinguishing the physical quantity M from the corresponding quantity for usual massive particles. At the same time, as mentioned above, this nonzero observable photon rest mass (unlike, for example, the proton and electron rest masses) is not an immanent characteristic of the photon, but, always remaining finite, varies depending on the structure of the wave representing the photon.

It is appropriate here to make a comment concerning the contemporary concept of a photon itself. It is reasonably stated [7] that 'a photon, as the elementary particle of an optical field, does not have a reasonable and clear definition... Nevertheless, it helps... to predict qualitatively the results of new experimental situations. Generally, the abandonment of the axiomatic approach facilitates progress at a certain stage of studies.' Without going into details of this problem we will follow here such an approach.

Another comment concerns the terms used to denote the mass. Of course, following the exhaustive analysis [6, 8, 9], it would be desirable to avoid the use of the term 'rest mass' and call it the observable mass of the stopped light corresponding to the rest energy of the light. However, this can only encumber the text of the paper, which has a predominantly methodological character. An additional excuse can be the words of Einstein: "'It is better not to introduce any masses except the 'rest mass'" [10].

1.3 Post-Maxwell electrodynamics admitting the existence of a finite photon rest mass

As pointed out above, we perform our analysis within the framework of Maxwell's electrodynamics. Nevertheless, we will mention here the attempts to modify Maxwell's equations (1.1)–(1.6) in order to take the finite photon rest mass into account. These attempts were made many times by such physicists as Proca [11], Schrödinger [12], de Broglie [13], Feynman [14], et al.

For example, the Maxwell–Proca equations [11] assuming the presence of the nonzero photon rest mass contain in (1.2) and (1.3), along with the field sources ρ and $\boldsymbol{j}$, the additional terms

$$\operatorname{rot}\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j} - \frac{1}{\mu_0}\frac{\boldsymbol{A}}{\Lambda^2}, \tag{1.15}$$

$$\operatorname{div}\boldsymbol{D} = \rho - \varepsilon_0 \frac{\varphi}{\Lambda^2}, \tag{1.16}$$

where

$$\Lambda = \frac{2\pi\hbar}{Mc} \tag{1.17}$$

is the Compton wavelength of a photon with the rest mass M; $\boldsymbol{A}$ and φ are the vector and scalar potentials, respectively. All these post-Maxwell versions are not the object of our analysis.

The various theoretical, observational, and experimental estimates of the upper limit of the photon rest mass based to some extent on post-Maxwell concepts are also of interest and are presented in Table 1.

1.4 Summary comments

Therefore, the abandonment of the physically empty concept of a perfect plane wave leads to the inevitable presence of the fragments of a standing wave (stopped light) in any fields, resulting in the compensation of the counterpropagating components of the momentum of a photon represented by such a nonplane wave. This compensation results in the appearance of the momentum-defect term in the Hamiltonian, which gives rise to a mass-like quantity to which the properties of the finite observable photon rest mass can be assigned.

Table 1. Estimates of the upper limit of the photon rest mass.

M/g	Mc^2/eV	Λ/m	Year, references	Estimate method
10^{-44}	6×10^{-12}	2×10^{5}	1940, [15]	Absence of a colour effect during the eclipse of binary stars
10^{-47}	6×10^{-15}	2×10^{8}	1943, [16]	Possible deviation of the Earth magnetic field from the Gaussian law
10^{-47}	6×10^{-15}	2×10^{8}	1971, [17]	Verification of the Coulomb law (geometry of concentric spheres)
2×10^{-43}	10^{-10}	10^{4}	1971, [14]	Laboratory measurements of light dispersion
8×10^{-49}	5×10^{-16}	2×10^{9}	1975, [18]	Measurements of the Jupiter magnetic field
10^{-48}	6×10^{-16}	2×10^{9}	1980, [19]	Summary data base
2×10^{-66}	10^{-33}	10^{27}	1984, [20]	The smallest mass measured from the uncertainty relation for the Universe age equal to 5×10^{17} s
10^{-45}	5×10^{-13}	2.5×10^{6}	1992, [21]	Low-temperature zero-test of the Ampere law
2×10^{-48}	10^{-15}	2×10^{9}	1994, [22]	Measurements of the Earth magnetic field
$2\times10^{-50}-4\times10^{-49}$	$10^{-17}-2\times10^{-16}$	$10^{11}-6\times10^{9}$	1998, [23]	Effect of the cosmic vector potential (experiment with a torsion balance)
10^{-44}	6×10^{-12}	2×10^{5}	1999, [24]	Dispersion of radiation from cosmic gamma ray bursts
10^{-51}	7×10^{-19}	2×10^{12}	2003, [25]	Effect of the cosmic vector potential (experiment with a torsion pendulum)

2. Details of the structure of the field of nonplane waves

2.1 The field of a harmonic electric dipole oscillator

In any textbook on electrodynamics, an exact solution can be found for the vectors of the radiation field of a harmonic electric dipole in the form of its three nonzero components in the spherical coordinate system (R, θ, φ) with the polar axis coinciding with the direction of the dipole moment vector. The meridional component of the electric vector $\boldsymbol{E}$ has the form

$$E_\theta = E_0 \sin\theta[1 + \mathrm{i}kR - (kR)^2]\exp(\mathrm{i}\Phi), \tag{2.1}$$

the radial component $\boldsymbol{E}$ is

$$E_R = 2E_0 \cos\theta(1 + \mathrm{i}kR)\exp(\mathrm{i}\Phi) \tag{2.2}$$

and the azimuthal component of the magnetic induction $\boldsymbol{B}$ tangential to the parallels of the coordinate system has the form

$$B_\varphi = \mathrm{i}E_0 \frac{n}{c} kR \sin\theta[1 + \mathrm{i}kR]\exp(\mathrm{i}\Phi), \tag{2.3}$$

where $E_0 = p_0/4\pi\varepsilon_0\varepsilon R^3$; p_0 is the dipole moment modulus; $\Phi = \omega t - kR$ is the phase; $k = \omega n/c$ is the wave number; $n = (\varepsilon\mu)^{1/2}$ is the refractive index of the medium. Fig. 1 shows the known structure of the electric lines of force for the field of a harmonic oscillator in the meridional plane of the spherical coordinate system.

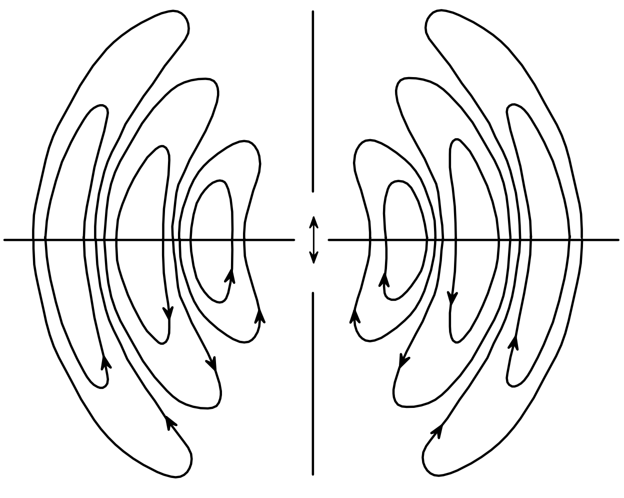

Figure 1.

It is important to note that field components (2.1)–(2.3) for $R = \mathrm{const}$ have a character of a standing wave propagating along the meridional line ($R = \mathrm{const}$, $\varphi = \mathrm{const}$) of the coordinate system ($\sim\cos\theta\exp\mathrm{i}\omega t$ or $\sim\sin\theta\exp\mathrm{i}\omega t$). In essence, these dependences are equivalent to the meridional eigenfunctions (Legendre functions) of the wave equation in spherical coordinates [26], which play the same role in the field structure as, for example, the sinusoidal–cosinusoidal transverse eigenfunctions of the problem on a rectangular cross-section waveguide.

Expressions (2.1)–(2.3) have the following trigonometric form

$$E_\theta = E_0 \sin\theta[(1 - k^2R^2)\cos\Phi - kR\sin\Phi], \tag{2.4}$$

$$E_R = 2E_0 \cos\theta[\cos\Phi - kR\sin\Phi], \tag{2.5}$$

$$B_\varphi = -E_0 \frac{n}{c} kR \sin\theta[\sin\Phi + kR\cos\Phi]. \tag{2.6}$$

If, as usual, in the far-field (wave) zone, where $kR \gg 1$, only the terms with higher powers of (kR) are retained, then

$$E_\theta \approx -E_0(kR)^2 \sin\theta\cos\Phi, \tag{2.7}$$

$$E_R \approx 2E_0(kR)\cos\theta\sin\Phi, \tag{2.8}$$

$$B_\varphi \approx -E_0 \frac{n}{c}(kR)^2 \sin\theta\cos\Phi = E_\theta \frac{n}{c}. \tag{2.9}$$

Because $|E_R|/|E_\theta| \ll 1$, one can neglect the radial component E_R, which leads to the popular approximation of a purely transverse spherical wave with the mutually orthogonal E_θ, B_φ and k.

2.2 The superluminal phase velocity of a wave in the field of a harmonic dipole oscillator

The asymptotics expressed by relations (2.7)–(2.9) and the inequality $|E_R|/|E_\theta| \ll 1$, which is convenient for numerous practical problems, masks in fact the detailed structure of the field of a harmonic dipole because, strictly speaking, $(kR)^{-1} \neq 0$ and $E_R \neq 0$ even in the far-field wave zone where $kR \gg 1$. The presence of a small but finite radial component E_R of the electric vector directly indicates that the wave is not purely transverse (so that it should be denoted in a standard notation as the TM wave rather than the TEM wave) and that the wave vector $\boldsymbol{k}$ has the meridional component k_θ, thereby losing the radial directivity inherent in a spherical wave.

Therefore, because $E_R \neq 0$, the square of the wave vector $k^2 = k_R^2 + k_\theta^2 + k_\varphi^2$ has, along with the radial component $k_R \neq 0$, also the component $k_\theta \neq 0$ (for $k_\varphi = 0$). As a result, the radial phase velocity of the wave

$$\upsilon_{\mathrm{ph}} = \frac{\omega}{k_{\mathrm{R}}} = \frac{c/n}{[1 - (k_\theta/k)^2]^{1/2}} > \frac{c}{n} \tag{2.10}$$

proves to be superluminal in a free space (when the refractive index of the medium $n = 1$). Because E_R and, hence, k_θ depend on the meridional angle θ, the phase velocity υ_{ph} also depends on θ, which again indicates to the deviation of the phase front of the wave from a perfect spherical shape.

2.3 Energy fluxes in the field of a harmonic dipole oscillator

Along with a standard expression

$$P_R = E_\theta \frac{B_\varphi}{\mu_0\mu} = E_\theta^2 \left(\frac{\varepsilon\varepsilon_0}{\mu_0\mu}\right)^{1/2}$$

$$= E_0^2 \left(\frac{\varepsilon_0\varepsilon}{\mu_0\mu}\right)^{1/2} (kR)^4 \sin^2\theta\cos^2\Phi \tag{2.11}$$

for the radial component of the Poynting vector, in the far-field zone approximation $[(kR)^{-1} \to 0]$ there exists although a small but nonzero meridional component with the instantaneous value

$$P_\theta = -E_R \frac{B_\varphi}{\mu_0\mu} = -\frac{E_0^2}{2}\left(\frac{\varepsilon_0\varepsilon}{\mu_0\mu}\right)^{1/2}(kR)^3 \sin 2\theta \sin 2\Phi. \quad (2.12)$$

This meridional component of the Poynting vector (unlike the radial component) changes its sign four times during a period, so that its average value over the period is zero, i.e., a constant transfer of energy along the meridian is absent, as expected. However, alternating pulsations of P_θ caused by the phase shift by $\pi/2$ between the electric (E_R) and magnetic (B_φ) components once more confirms the existence of the standing wave along the meridian of the spherical coordinate system.

The presence of the pulsed meridional component of the Poynting vector results in the deviation of the trajectory of the electromagnetic energy flux from a strictly radial direction. The hodograph of the Poynting vector in the meridional plane has the form of a winding curve, with the amplitude of the transverse deviation from a strict radial direction being proportional to $\sin 2\theta$. Due to such a deviation from the rectilinear propagation of energy, the radial velocity of the energy transport proves to be lower than the speed of light.

2.4 The subluminal group velocity of a wave in the field of a harmonic dipole oscillator

The local value of the radial group velocity at the point (t, R, θ, φ)

$$v_{\mathrm{gr}}(R,\theta) = \frac{P_R}{w(R,\theta)} \quad (2.13)$$

can be determined as the ratio of the radial component P_R of the Poynting vector to the total field energy density

$$w(R,\theta) = \frac{\varepsilon_0\varepsilon}{2}(E_\theta^2 + E_R^2) + \frac{\mu_0\mu}{2}H_\varphi^2$$

$$= \frac{\varepsilon_0\varepsilon}{2}(E_\theta^2 + E_R^2) + \frac{B_\varphi^2}{2\mu_0\mu} = \varepsilon_0\varepsilon E_0^2 (kR)^4 \quad (2.14)$$

$$\times\left[\sin^2\theta\cos^2\Phi + \frac{2}{(kR)^2}\cos^2\theta\sin^2\Phi\right],$$

i.e.

$$v_{\mathrm{gr}}(R,\theta) = \frac{c/n}{1 + \dfrac{2}{(kR)^2}\dfrac{\tan^2\Phi}{\tan^2\theta}} \leqslant \frac{c}{n}. \quad (2.15)$$

The local group velocity (2.15) is not constant, oscillating upon variation of the phase Φ from $v_{\mathrm{gr}} = c/n$ at $\Phi = 0, \pi$ to a full stop ($v_{\mathrm{gr}} = 0$) at $\Phi = \pi/2,\ 3\pi/2$. In addition, it varies from $v_{\mathrm{gr}} = 0$ in the direction of the polar axis at $\theta = 0$ to the coincidence with the speed of light $v_{\mathrm{gr}} = c/n$ in the equatorial plane at $\theta = \pi/2$. All this takes place even in the far-field zone, where $kR \gg 1$, however, due to the smallness of the parameter $(kR)^{-2}$ these effects become significant only near the values of $\Phi = \pi/2,\ 3\pi/2$ and (or) $\theta = 0$.

Additional information on the behaviour of the local group velocity is contained in its value averaged over the period

$$\bar{v}_{\mathrm{gr}}(R,\theta) = \frac{\bar{P}_R}{\bar{w}(R,\theta)} = \frac{c/n}{1 + \dfrac{2}{(kR)^2\tan^2\theta}} \leqslant \frac{c}{n}, \quad (2.16)$$

which is equal to the ratio of the radial component of the Poynting vector averaged over the period

$$\bar{P}_R = \frac{E_0^2}{2}\left(\frac{\varepsilon_0\varepsilon}{\mu_0\mu}\right)^{1/2}(kR)^4\sin^2\theta \quad (2.17)$$

to the electromagnetic energy density averaged over the period

$$\bar{w}(R,\theta) = \varepsilon_0\varepsilon\frac{E_0^2}{2}(kR)^4\left[\sin^2\theta + \frac{2}{(kR)^2}\cos^2\theta\right]. \quad (2.18)$$

The integral group velocity, as the total transport velocity of the dipole radiation energy can be determined from the ratio

$$v_{\mathrm{gr}}^* = \frac{P_R^*}{w^*} \quad (2.19)$$

of the total flux of the radial component of the Poynting vector

$$P_R^* = 2\pi\int_0^\pi P_R R^2 \sin\theta \mathrm{d}\theta = \frac{8\pi}{3}E_0^2\left(\frac{\varepsilon_0\varepsilon}{\mu_0\mu}\right)^{1/2}(kR)^4 R^2\cos^2\Phi \quad (2.20)$$

to the total electromagnetic energy in a spherical layer of radius R

$$w^* = 2\pi\int_0^\pi wR^2\sin\theta\mathrm{d}\theta = \frac{8\pi}{3}E_0^2\varepsilon_0\varepsilon(kR)^4R^2$$

$$\times\left[\cos^2\Phi + \frac{\sin^2\Phi}{(kR)^2}\right], \quad (2.21)$$

i.e.

$$v_{\mathrm{gr}}^* = \frac{c/n}{1 + \dfrac{\tan^2\Phi}{(kR)^2}} \leqslant \frac{c}{n}. \quad (2.22)$$

Let us introduce the integral group velocity averaged over the period

$$\bar{v}_{\mathrm{gr}}^* = \frac{\bar{P}_R^*}{\bar{w}^*} \quad (2.23)$$

as the ratio of the total flux of the radial component of the Poynting vector averaged over the period

$$\bar{P}_R^* = \frac{4\pi}{3}E_0^2\left(\frac{\varepsilon_0\varepsilon}{\mu_0\mu}\right)^{1/2}(kR)^4R^2 \quad (2.24)$$

to the total electromagnetic energy in a spherical layer averaged over the period

$$\bar{w}^* = \frac{4\pi}{3}E_0^2\varepsilon_0\varepsilon(kR)^4R^2[1 + (kR)^{-2}]. \quad (2.25)$$

We obtain

$$\bar{v}_{\rm gr}^{*} = \frac{c/n}{1 + (kR)^{-2}} \leqslant \frac{c}{n} . \tag{2.26}$$

According to both above definitions, the group velocity proves to be subluminal. The curves describing the dependences of $\bar{v}_{\rm gr}(R\theta)$ on the angle θ (2.16) and of $v_{\rm gr}^{*}$ on the phase Φ (2.22) have narrow holes where the group velocity vanishes and the radial movement of the wave stops. The widths of these holes are

$$\Delta\theta \approx \frac{2}{kR} \ll 1 , \tag{2.27}$$

and

$$\Delta\Phi \approx \frac{2}{kR} \ll 1 , \tag{2.28}$$

respectively, and their decrease inversely proportional to the distance R. Therefore, at a considerable distance from the emitting dipole, the group velocities determined by these two different methods only slightly differ from c/n, remaining subluminal [except narrow intervals $\Delta\theta$ (2.27) and $\Delta\Phi$ (2.28)].

2.5 Dispersion of a wave in the field of a harmonic dipole

The behaviour of the phase and group velocities considered above is determined by the parameter $kR = \omega nR/c$, i.e., by the wave frequency ω at a given R. In other words, there exists normal frequency dispersion of phase and group velocities of a wave outside a material medium.

To crown it all, it is possible to assert that the properties of nonplane electromagnetic waves mentioned in Introduction are found in the structure of the radiation field of a harmonic electric dipole oscillator, which is considered as a typical example of a wave field in free space.

2.6 Topological unity of the structure of the fields of a harmonic oscillator and a waveguide mode

As pointed out in Introduction, the main properties of real nonplane waves, which differ from the notions neglecting the physical emptiness of the concept of a plane wave, are most distinctly manifested from the physical point of view in the case of the mode fields in an ideal metal waveguide, which are used below as a basic model. All these properties have been also found for the radiation field of a harmonic dipole oscillator, however, their mathematical description is less descriptive than for the basic model considered below. First of all, this concerns the absence of an exact mathematical expression for the nonzero observable rest mass of photons occupying a radiation mode of the oscillator. This mass is produced by the stopped light, which is found in the radiation field in the form of a standing wave along the meridional coordinate ($R = \text{const}$, $\varphi = \text{const}$).

To find the way for passing from the radiation field of a harmonic dipole to the basic model of the waveguide-mode fields, which is studied below, and to reveal the topological unity of these fields, it is useful to show how these fields can be transformed to each other upon the topologically continuous displacement of boundary conditions.

It is found that the radiation field of a harmonic electric dipole in a free space can be represented as the result of a continuous transformation of the TM_{01} mode of a circular cylindrical waveguide (Fig. 2a) upon continuous transformation of the waveguide to two circular conical waveguides (Figs 2b, c show the consecutive stages of this transformation, but for simplicity, without the exact reproduction of the shape of the lines of force in Fig. 1; metal boundaries are shown by solid thick straight lines). The common axis of the waveguides is directed along the vector of the exciting dipole moment, while the apexes of the cones coincide with the point of position of the latter. The boundary conditions are transformed by increasing gradu-

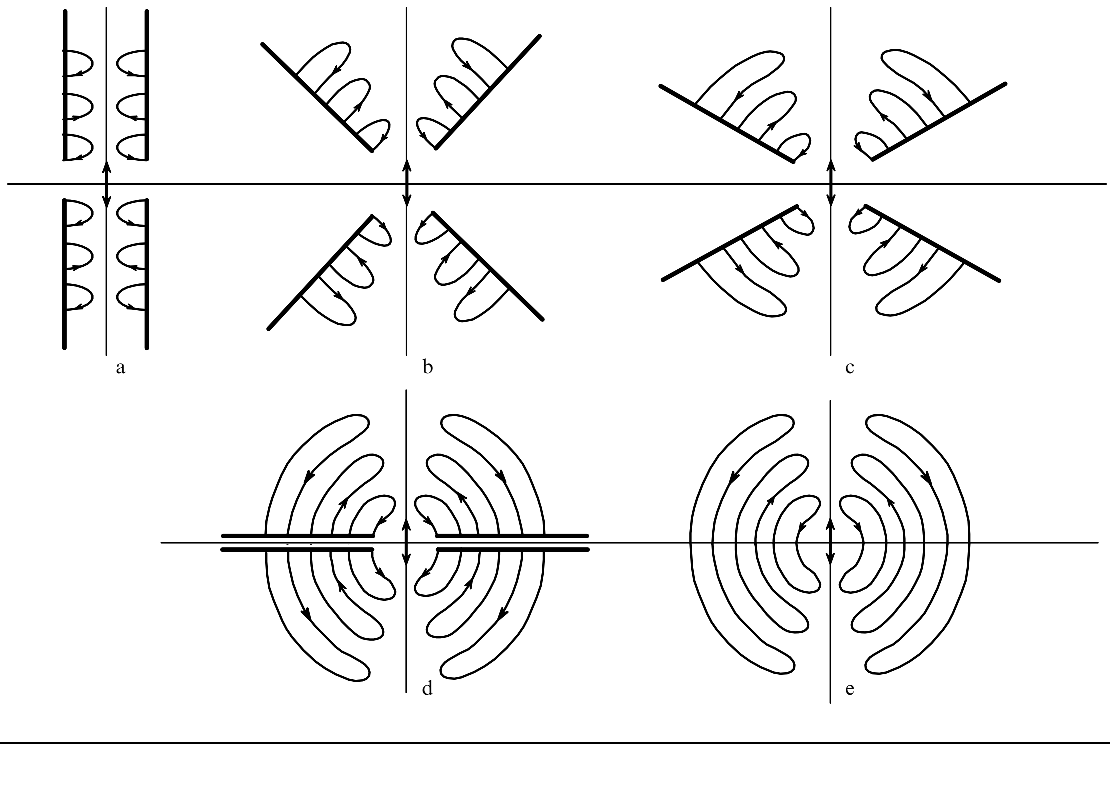

Figure 2.

ally the cone angles and are completed by a full reproduction of the dipole field when the angles of the cones achieve 180° and the generatrices of both cones prove to be coincident (Fig. 2d). In this case, the wave field fills the entire infinite space, which is divided by a metal boundary over the equatorial plane of the spherical coordinate system in which the dipole field is written. It is clear that such a division does not distort the structure of the dipole field because the tangential component of the electric vector in the equatorial plane is zero, and equatorial metal planes (former walls of the waveguides) can be removed at all after the transformation completion (Fig. 2e). It is obvious that the reverse sequence of transformations is also possible.

Therefore, we can say within the framework of the picture outlined above that the field of a harmonic dipole is nothing but the field of two coaxial circular conical waveguides with cone angles equal to 180° and vice versa.

2.7 Other simple examples of nonplane waves reduced to the transformed fields of waveguide modes

The simplest and descriptive method to avoid the idealised concept of a spatially infinite plane wave is to restrict its propagation, for example, by reflecting the wave from a plane mirror at an angle of ϑ. In this case, the wave field acquires all the properties discussed above: it becomes limited, occupying only the half-space from one side of the mirror. It is no longer represented by a single plane wave but is a result of the interference between the incident and reflected waves. The immobile component (stopped light) appears in the wave field in the form of a standing wave directed normally to the mirror surface with the step $\pi c/\omega \sin\vartheta$ and the component running parallel to the mirror at the phase velocity

$$v_{\mathrm{ph}} = c/\sin\vartheta > c \tag{2.29}$$

and the velocity of the electromagnetic energy transport

$$v_{\mathrm{gr}} = c\sin\vartheta < c. \tag{2.30}$$

Talking about the relation to the basic model considered below, note that the interference field coincides in this case with the field of the mode of a planar metal waveguide when one of the waveguide walls is removed to infinity.

This example allows one to find easily the photon momentum defect introduced in section 1.2. Each of the interfering waves (incident and reflected) makes the same contribution to the photon energy. The vector of the photon momentum consists of four components belonging to the normal and tangential components. For the incident wave,

$$p_{1n} = \frac{\hbar\omega}{2c}\cos\vartheta,\ \ p_{1t} = \frac{\hbar\omega}{2c}\sin\vartheta; \tag{2.31}$$

and for the reflected wave,

$$p_{2n} = -\frac{\hbar\omega}{2c}\cos\vartheta,\ \ p_{2t} = \frac{\hbar\omega}{2c}\sin\vartheta. \tag{2.32}$$

The total normal and tangential components are

$$p_n = p_{1n} + p_{2n} = 0, \tag{2.33}$$

$$p_t = p_{1t} + p_{2t} = \frac{\hbar\omega}{c}\sin\vartheta, \tag{2.34}$$

respectively.

The corollary following from the Hamiltonian (see section 1.2)

$$(Mc^2)^2 = (\hbar\omega)^2 - (\hbar\omega\sin\vartheta)^2 = (\hbar\omega\cos\vartheta)^2 \tag{2.35}$$

leads to a finite value of the observable rest mass of a photon represented by the interference wave field under study:

$$M_\vartheta = \frac{\hbar\omega}{c^2}\cos\vartheta > 0. \tag{2.36}$$

The observable photon rest mass is produced due a mutual compensation of the normal components of the momentum vector and is equal numerically to the quantity (divided by c), which can be identified with the momentum defect δp introduced above:

$$M_\vartheta = \frac{\delta p}{c} = \frac{1}{c}(p_{1n} - p_{2n}) = \frac{\hbar\omega}{c^2}\cos\vartheta > 0. \tag{2.37}$$

One can see that, depending on the angle of incidence ϑ of the wave on the mirror, the photon energy can be transformed to its observable rest mass in the range from zero to $\hbar\omega$.

For $\vartheta = \pi/2$ (as in the rest of the cases of the realisation of a standing wave), the light is completely stopped and the photon momentum modulus is zero.

Another example is Fraunhofer diffraction from a slit, which forms the field with a standing wave behind the slit in the direction perpendicular to the axis of propagation of the diffracted wave, having the electric-vector zeroes on the surfaces that come from the slit at the angles depending on the ratio of the wavelength to the slit width. The structure of this field remains invariable if perfect metal boundaries are made coincident with these surfaces. Such metal boundaries form a waveguide with the one-dimensional restriction over the transverse coordinate and with the cross section increasing with distance from the slit. The field is characterised by the superluminal phase and subluminal group velocities, the frequency dispersion, and the standing wave (stopped light) in the transverse direction. The stopped light can produce the nonzero observable rest mass of a photon belonging to the mode of the diffraction field.

2.8 Summary comments

It follows from the above discussion that the electromagnetic field that can be physically realised in a free space, which is the solution of the wave equation, has the properties of a spatially restricted nonplane wave, which are listed in Introduction. In these fields, the surfaces can be found where the electric vector vanishes. The disposition of perfect metal boundaries on these surfaces does not perturb the solution inside the latter and transforms the initially arbitrary electrodynamic problem to the internal problem with metal boundary conditions. Its solution can be treated as the field of a mode of a perfect waveguide with a complicated variable cross section. This circumstance serves as the methodologic motivation and ground of the possibility of using the basic waveguide model for studying the most general properties of nonplane electromagnetic waves.

3. Basic waveguide model: the potential well for photons

The basic model proposed in Refs [27, 28] is a limiting case of the transverse spatial restriction of an electromagnetic wave, when the field is located in the potential well for photons with the two-dimensional restriction of its infinite depth over transverse coordinates and the infinite propagation over the third longitudinal coordinate. This extremely simple model, which is not overloaded with redundant calculations, makes it possible to investigate clearly the above properties of the electromagnetic wave, especially as the solutions for this model are well known and coincide with those for the fields of the modes of an ideal hollow metal waveguide. Because the waveguide is used here only as a model, this term means hereafter the potential well described above, while the question of to what extent ideal metal boundary conditions can be realised in different frequency regions is not considered.

A hollow metal waveguide, which is a customary tool of modern experimental physics and applied radio engineering, was not such a trivial device even about a century ago. In 1893, even such a penetrating physicist as Heaviside denied the possibility of propagation of electromagnetic waves through tubes [29]. However, only four years later Rayleigh publishes the study 'On the passage of electric waves through tubes or the vibration of dielectric cylinders' [30].

By the way, a variety of spatial periodic structures can be used as model objects (see, for example, Ref. [31]): from single crystals to artificial metal and dielectric microwave gratings, which were used for the development of the so-called photonic crystals.

3.1 The field of modes of a hollow metal waveguide

It is known [32] that the mode field of a metal waveguide of an arbitrary cross section consists of two travelling waves for the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) vectors

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{e}(x, y)\exp[\mathrm{i}(\omega t - kz)], \tag{3.1}$$

$$\boldsymbol{H}(x, y, z, t) = \boldsymbol{h}(x, y)\exp[\mathrm{i}(\omega t - kz)] \tag{3.2}$$

with the transverse eigenfunctions $\boldsymbol{e}(x,\ y)$ and $\boldsymbol{h}(x,\ y)$ and the eigenvalues ω_{nm} (critical frequencies) with the integer subscripts n and m, where

$$k = \frac{\omega}{c}\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{1/2} \tag{3.3}$$

is the propagation constant and x and y are the transverse coordinates. The existence of the eigenfunctions $\boldsymbol{e}$ and $\boldsymbol{h}$ indicates to the presence of a standing wave in the waveguide cross section.

The phase velocity of the wave along the longitudinal axis z exceeds the speed of light,

$$\upsilon_{\mathrm{ph}} = \frac{\omega}{k} = c\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{-1/2} > c, \tag{3.4}$$

and the group velocity of the electromagnetic energy transport, which is equal to the velocity of the longitudinal propagation of photons through the waveguide, is lower than c,

$$\upsilon_{\mathrm{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{1/2} < c. \tag{3.5}$$

Both υ_{ph} and υ_{gr} depend explicitly on the frequency ω, confirming the existence of normal dispersion.

The eigenvalues ω_{mn} and propagation constants k are related to the wave frequency ω by the dispersion relation

$$\omega^2 = \omega_{nm}^2 + (ck)^2, \tag{3.6}$$

while the transverse components of the eigenfunctions e_x, e_y, h_x, and h_y are expressed in terms of the spatial derivatives of the longitudinal components e_z and h_z as

$$e_x = -\mathrm{i}\left(\frac{c}{\omega_{nm}}\right)^2\left(k\frac{\partial e_z}{\partial x} + \mu_0\omega\frac{\partial h_z}{\partial y}\right), \tag{3.7}$$

$$e_y = -\mathrm{i}\left(\frac{c}{\omega_{nm}}\right)^2\left(k\frac{\partial e_z}{\partial y} - \mu_0\omega\frac{\partial h_z}{\partial x}\right), \tag{3.8}$$

$$h_x = -\mathrm{i}\left(\frac{c}{\omega_{nm}}\right)^2\left(k\frac{\partial h_z}{\partial x} - \varepsilon_0\omega\frac{\partial e_z}{\partial y}\right), \tag{3.9}$$

$$h_y = -\mathrm{i}\left(\frac{c}{\omega_{nm}}\right)^2\left(k\frac{\partial h_z}{\partial y} + \varepsilon_0\omega\frac{\partial e_z}{\partial x}\right). \tag{3.10}$$

In turn, the longitudinal components of the vectors e_z and h_z are the solutions of the equations

$$\nabla_{xy}^2 e_z + \left(\frac{\omega_{nm}}{c}\right)^2 e_z = 0, \tag{3.11}$$

and

$$\nabla_{xy}^2 h_z + \left(\frac{\omega_{nm}}{c}\right)^2 h_z = 0 \tag{3.12}$$

with metal boundary conditions (∇_{xy}^2 is the Laplace operator over the transverse coordinates).

Depending on the direction of the polarisation vector, two types of solutions exist: the TM type, when $h_z = 0$ and the TE type, when $e_z = 0$, to which the system (3.7)–(3.10) with one-term right-hand sides corresponds.

The longitudinal Poynting vector directed along the z axis is

$$\begin{aligned} P = {} & \left(\frac{c\omega}{\omega_{nm}^2}\right)^2\left[1 + \left(\frac{\upsilon_{\mathrm{gr}}}{c}\right)^2\right]\left(\frac{\partial e_z}{\partial x}\frac{\partial h_z}{\partial y} - \frac{\partial e_z}{\partial y}\frac{\partial h_z}{\partial x}\right) \\ & + \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\left(\frac{c\omega}{\omega_{nm}^2}\right)^2\frac{\upsilon_{\mathrm{gr}}}{c}\left[\left(\frac{\partial e_z}{\partial x}\right)^2 + \left(\frac{\partial e_z}{\partial y}\right)^2\right] \\ & + \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}\left(\frac{c\omega}{\omega_{nm}^2}\right)^2\frac{\upsilon_{\mathrm{gr}}}{c}\left[\left(\frac{\partial h_z}{\partial x}\right)^2 + \left(\frac{\partial h_z}{\partial y}\right)^2\right]; \end{aligned} \tag{3.13}$$

for the TM and TE polarisations

$$\begin{aligned} P_{\mathrm{TM}} = {} & \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\left(\frac{c}{\omega_{nm}}\right)^2\frac{\upsilon_{\mathrm{gr}}/c}{1 - (\upsilon_{\mathrm{gr}}/c)^2} \\ & \times\left[\left(\frac{\partial e_z}{\partial x}\right)^2 + \left(\frac{\partial e_z}{\partial y}\right)^2\right], \end{aligned} \tag{3.14}$$

$$P_{\mathrm{TE}} = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}\left(\frac{c}{\omega_{nm}}\right)^2\frac{\upsilon_{\mathrm{gr}}/c}{1 - (\upsilon_{\mathrm{gr}}/c)^2}$$

$$\times\left[\left(\frac{\partial h_z}{\partial x}\right)^2+\left(\frac{\partial h_z}{\partial y}\right)^2\right]. \quad (3.15)$$

As in the previous examples for the real wave fields (even more clearly), the presence of the standing wave, the longitudinal components of the field vectors, dispersion, the superluminal and subluminal propagation of light, etc. is well demonstrated here.

3.2 The de Broglie analogy: the waveguide dispersion equation versus the Hamiltonian for a free particle

The structural similarity of expressions (1.11) and (3.6), which has been pointed out many times, becomes especially clear after the multiplication each term of the dispersion relation (3.6) on the square of Plank's constant $\hbar$ which gives

$$(\hbar\omega)^2=(\hbar\omega_{nm})^2+(c\hbar k)^2. \quad (3.16)$$

One can juxtapose the total energies E and $\hbar\omega$, the energy Mc^2 corresponding to the particle rest mass and the energy $\hbar\omega_{nm}$ of a quantum of the critical frequency, and finally the the momentum modulus $|p|$ and the propagation constant $\hbar k$.

Pointing out this analogy, de Broglie [32] notes: 'Everything occurs so as if a photon has its own mass, which is determined by the waveguide shape and the eigenvalue considered; in a given waveguide, a photon can have a series of own masses'. And further on: 'Let us put aside these considerations, which distract our attention from the subject'. Feynman [33], making similar comparisons, follows de Broglie, saying only: 'It is interesting, isn't it?'

Meanwhile, note in connection with this analogy that the nature cannot be so wasteful for really incidental coincidences to occur quite often in it. Each time when apparently incidental analogies and coincidences are encountered, one should try to understand what is behind them. It is sufficient to recall here the result of the reflection about the apparently 'groundless' coincidence of the inertial and heavy masses of a body.

The assumption about the physical meaning of the analogy, according to which the observable rest mass of a photon in a waveguide is

$$M_{nm}=\frac{\hbar\omega_{nm}}{c^2}, \quad (3.17)$$

requires the answer to two questions: (i) Whether the product $\hbar\omega_{nm}$ is a physically real energy (for example, the energy accumulated in some observable process, which can be transformed to other types of energy) and (ii) what is a source of this energy (the energy origin)?

Note here that the critical wavelength λ_{nm} in the waveguide is simply equal to the Compton wavelength Λ (1.17)

$$\lambda_{nm}=\frac{2\pi c}{\omega_{nm}}=\frac{2\pi\hbar}{Mc}=\Lambda \quad (3.18)$$

appearing in the Maxwell–Proca equations.

3.3 The time evolution of the mode field upon a change in the waveguide cross section

To answer the questions posed above, we should consider the evolution of the mode field with changing the cross section of a waveguide of infinite length caused by the movement of the waveguides walls parallel to themselves and uniformly over the entire waveguide length [34].

If the cross section changes continuously and sufficiently slow, so that both its shape and the field structure of the observed mode are retained, then the behaviour of the wave is determined by the fact that the propagation constant is invariable ($k=\text{const}$), i.e., by the invariance of the phase interval.

This means that the phase difference of the wave on a fixed finite piece of the waveguide remains invariable when its cross section is varied as described above. According to (3.6), it follows from the invariance $k=\text{const}$ that

$$(ck)^2=\omega_0^2-\omega_{nm0}^2=\omega^2-\omega_{nm}^2=\text{const}, \quad (3.19)$$

where ω and ω_{nm} are the current wave frequency and critical mode frequency, respectively, and ω_0 and ω_{nm0} are their initial values, respectively.

A special case of the phase invariance relation (3.19) is the initial waveguide with infinitely separated walls, i.e., the free space with $\omega_{nm0}=0$:

$$\omega^2=\omega_0^2+\omega_{nm}^2. \quad (3.20)$$

Such a situation can be considered as a starting one for the construction of a waveguide with a finite cross section from the free space by drawing together initially infinitely separated waveguide walls. According to (3.20), the initial field with the frequency ω_0 gives rise to the wave in the frequency ω in the waveguide.

Of special attention is the limiting case of a vanishingly low frequency of the initial field ($\omega_0\to 0$), when this field is in fact purely static and

$$\omega=\omega_{nm}. \quad (3.21)$$

In this limiting case, the *static* initial field produces the *wave* field with the critical frequency in the waveguide.

One can easily see that, because free space can be considered as the limiting case of a waveguide with infinitely separated walls, the zero initial frequency means the critical frequency for free space. Indeed, the static field with $\omega_0\to 0$ has the main property of the critical-frequency field, namely, the inability to propagate. In other words, in this latter transformation, the field frequency remains equal to the critical frequency.

In the opposite case, when the cross section of a critical waveguide is increased from a finite size to infinite free space, the wave field degrades to a purely static field.

The next fundamental invariant of the process of waveguide evolution is the photon occupation number of the mode $N=\text{const}$, whose invariability is quite reasonable in the absence of dissipation and nonlinearity. Along with this, the energy of the mode field

$$W'=\hbar\omega(N+1/2) \quad (3.22)$$

depends on the frequency ω (3.19). The term 1/2 in (3.22) is the contribution from zero vacuum fluctuations.

The total energy of the electromagnetic field with a given transverse structure and critical frequency is

$$W=\hbar\omega(N+1). \quad (3.23)$$

Here, the sum $1=1/2+1/2$ is the total energy of zero fluctuations, which are inevitably present in the identical counterpropagating modes even when the second of these modes does not contain real photons.

Therefore, the field energy W changes during the waveguide evolution in accordance with the adiabatic Manly–Row invariant for the ratio of the oscillator energy to its frequency:

$$\frac{W}{\omega}=\frac{W_0}{\omega_0}=\hbar(N+1), \tag{3.24}$$

where W_0 is the initial energy for $\omega=\omega_0$.

The energy conservation law requires that an increase in the field energy should occur due to the work of an external force, which displaces the waveguide walls and reduces its cross section. The source of this external force is unimportant within the framework of our treatment, only the fact itself of accumulating of the energy in the waveguide mode being significant. It is obvious that in the reverse process of reducing the mode field energy with increasing waveguide cross section, this energy is returned back to the external source. We will show below by a simple example that a force applied from inside to the waveguide walls in these processes and counteracting to the external force is the light pressure of the mode field.

Especially instructive is the case (3.21) of the waveguide construction from free space with the zero critical frequency $\omega_{nm0}=0$ which is occupied by a static field with $\omega_0\to 0$. The wave field produced in this case has the critical frequency $\omega=\omega_{nm}$ corresponding to the achieved final cross section of the waveguide with the quantum energy (3.24)

$$\hbar\omega_{nm}=\frac{W}{N+1}, \tag{3.25}$$

where the right-hand side is the work of the external force spent to produce a photon.

It is essential that this process does not require any real initial photons in the mode. When $N=0$, the seeding role is played by the always present energy $\hbar\omega_0(1/2+1/2)=\hbar\omega_0$ of the zero fluctuations of vacuum, which have, moreover, the zero frequency $\omega_0\to 0$ (something that is absolutely imperceptible like the Cheshire cat smile!)

Returning to the definition of the observable photon rest mass M_{nm} in the waveguide (3.17) and to the questions posed at the end of section 3.2, we note that the quantity $M_{nm}c^2$ is equal to the work of an external force producing the standing-wave field of one quantum with the critical frequency. The energy that is equivalent to the observable mass M_{nm} is taken from the external force source and can be transformed to the energy of the external force during the reverse process of increasing waveguide cross section until its transformation to free space. Therefore, the definition (3.17) is not purely formal but it has a real physical meaning.

3.4 The mechanism of energy transformation during the movement of waveguide walls

It is useful to study the details of the energy-transformation processes during the movement of the waveguide walls for a simple planar waveguide [34] formed by two parallel metal planes separated by the distance a. In this case, the critical frequency of polarisation modes TE_{n0} ($n=1, 2, ..., m=0$) and TM_{n0} ($n=0, 1, 2, ..., m=0$) is

$$\omega_{n0}=\pi cn/a. \tag{3.26}$$

The field of the planar waveguide mode can be represented as the result of the interference of two partial plane waves incident on the waveguide walls at an angle of ϑ satisfying the relation

$$\cos\vartheta=\omega_{n0}/\omega. \tag{3.27}$$

The variation in the parameters of these waves upon approaching (or removing) the waveguide walls with respect to the symmetry plane of the waveguide at the velocity $c\beta$ consists in the relativistic changes in the frequency (caused by the Doppler effect) and in the angle of reflection in each reflection event ($\beta>0$ and $\beta<0$ upon approaching or removing, respectively):

$$\omega'=\omega\frac{1+2\beta\cos\vartheta+\beta^2}{1-\beta^2}, \tag{3.28}$$

$$\cos\vartheta'=\cos\vartheta\frac{1+2\beta/\cos\vartheta+\beta^2}{1+2\beta\cos\vartheta+\beta^2}. \tag{3.29}$$

It follows from this that, during the passage of a partial wave from one waveguide wall to another, the waveguide gap a increases by

$$\Delta a=-2a\beta\frac{1+2\beta\cos\vartheta+\beta^2}{(1+3\beta)\cos\vartheta+\beta(3+\beta^2)}, \tag{3.30}$$

while the corresponding increments of the frequency and the angle of incidence are

$$\Delta\omega=\omega\beta\frac{\beta(1+\cos^2\vartheta)+2\cos\vartheta}{1-\beta^2}, \tag{3.31}$$

and

$$\Delta(\cos\vartheta)=\frac{2\beta\sin^2\vartheta}{1+2\beta\cos\vartheta+\beta^2}, \tag{3.32}$$

respectively.

The passage from finite differences to differentials leads to the system of differential equations governing variations in the frequency and the angle of incidence depending on a:

$$\frac{d\omega}{da}=-\frac{\omega}{2a}\frac{\beta(1+\cos^2\vartheta)+2\cos\vartheta}{1-\beta^2}\times\frac{3\beta+\cos\vartheta+\beta(\beta^2+3\cos\vartheta)}{1+2\beta\cos\vartheta+\beta^2}, \tag{3.33}$$

$$\frac{d\vartheta}{da}=\frac{\sin\vartheta}{a}\frac{3\beta+\cos\vartheta+\beta(\beta^2+3\cos\vartheta)}{(1+2\beta\cos\vartheta+\beta^2)}. \tag{3.34}$$

The integration of this system over many reflection events for $\beta\ll 1$ gives the following relations for the frequency and the angle of reflection:

$$\omega=\omega_0\frac{\sin\vartheta_0}{\sin\vartheta}=\omega_0\frac{a_0}{a}\frac{\cos\vartheta_0}{\cos\vartheta}, \tag{3.35}$$

$$\tan\vartheta=\frac{a}{a_0}\tan\vartheta_0, \tag{3.36}$$

where the zero subscript denotes initial values.

It should be emphasised that the frequency and the angle of incidence depend on a even for $\beta \to 0$. During a slow motion with $\beta \ll 1$, the condition (3.27) remains valid, indicating that the structure of the field of the mode observed is invariable – the radiation frequency and the angle of incidence are continuously tuned to the stationary interference pattern of the mode. And vice versa, the interference pattern disappears at large values of β, which is accompanied by scattering of photons from one mode to another.

By excluding the angle of reflection ϑ from system (3.35), (3.36), we obtain the invariant (3.19), which was derived earlier for the general case.

For a planar waveguide under study, we can directly calculate the work performed against the light pressure force of the mode field

$$F_\omega = \frac{W}{a}\cos^2\vartheta, \quad (3.37)$$

where W is the mode field energy, and the work performed during a change in the gap by Δa is

$$\Delta W = -\frac{W\cos^2\vartheta}{a}\Delta a. \quad (3.38)$$

The passage from finite differences to differentials taking into account invariant (3.19) leads to the differential equation

$$\frac{\mathrm{d}W}{\mathrm{d}a} = -\frac{W/a}{1 + (a/a_0)^2[(\omega_0/\omega_{n00})^2 - 1]}. \quad (3.39)$$

The integral from this equation, as expected, coincides with (3.24). Here, ω_{n00} is the initial value of the critical frequency (3.26) for $a = a_0$.

Thus, a simple example of a planar waveguide explains both the mechanism of operation of the external force against the light pressure force of the filed mode during the waveguide construction from infinite free space and the general conclusions of section 3.3.

Let us make a brief comment about the TM_{00} mode of a planar waveguide with the zero critical frequency ω_{00} for any value of the gap a. The polarisation of this wave is such that the electric vector is always perpendicular to the waveguide walls and, hence, the light pressure is absent, resulting in both the zero work of an external force and the zero critical frequency, in complete accordance with the above discussion.

3.5 Energy transformation upon varying the cross section of a generalised cylindrical waveguide of an arbitrary cross section

A clear result obtained in section 3.4 for a simple planar waveguide can be extended (although less descriptive) to the general case of a cylindrical waveguide of an arbitrary cross section [35], for which the dimensionless scalar function $\psi(q_1, q_2)$, which is the solution of the equation of type (3.11), (3.12)

$$\nabla_q^2\psi + (\omega_{nm}/c)^2\psi = 0, \quad (3.40)$$

determines the dependence of the mode fields on the transverse orthogonal coordinates q_1 and q_2. Here,

$$\nabla_q^2 \equiv \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1}{h_2}\frac{\partial}{\partial q_2}\right)\right] \quad (3.41)$$

is the Laplace operator over the transverse coordinates, and the Lame coefficients h_1 and h_2 satisfy the conditions

$$\frac{\partial}{\partial z}\left(\frac{h_1}{h_2}\right) = 0, \quad \frac{\partial}{\partial z}(h_1 h_2) = 0. \quad (3.42)$$

The components of the fields are:
for the TM polarisation

$$E_z = E_0\psi\exp(\mathrm{i}\Phi), \quad (3.43)$$

$$\boldsymbol{E}_q = E_0\frac{c^2 k}{\omega_{nm}^2}(\nabla_q\psi)\exp(\mathrm{i}\Phi), \quad (3.44)$$

$$\boldsymbol{H}_q = E_0\varepsilon_0\frac{c^2\omega}{\omega_{nm}^2}[(\nabla_q\psi)\boldsymbol{z}_0]\exp(\mathrm{i}\Phi), \quad (3.45)$$

$$H_z = 0; \quad (3.46)$$

and for the TE polarisation

$$H_z = E_0\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\psi\exp(\mathrm{i}\Phi), \quad (3.47)$$

$$\boldsymbol{H}_q = E_0\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\frac{c^2 k}{\omega_{nm}^2}(\nabla_q\psi)\exp(\mathrm{i}\Phi), \quad (3.48)$$

$$\boldsymbol{E}_q = E_0\frac{c\omega}{\omega_{nm}^2}[(\nabla_q\psi)\boldsymbol{z}_0]\exp(\mathrm{i}\Phi), \quad (3.49)$$

$$E_z = 0, \quad (3.50)$$

where $\boldsymbol{E}_q$ and $\boldsymbol{H}_q$ are the vector components of the electric and magnetic fields in the cross section plane; ∇_q is the operator over the transverse coordinates; the gradient $(\nabla_q\psi)$ forms the vector product with the unit vector $\boldsymbol{z}_0$. The metal boundary conditions on the contour S limiting the waveguide cross section have the form

$$\psi|_L = 0, \quad (\nabla_q\psi)|_L \perp \boldsymbol{L}, \text{ for TM polarisation}, \quad (3.51)$$

$$\frac{\partial\psi}{\partial\boldsymbol{n}}|_L = 0, \quad (\nabla_q\psi)|_L \parallel \boldsymbol{L}, \text{ for TE polarisation}, \quad (3.52)$$

where $\boldsymbol{n}$ is the normal to the contour L and $\boldsymbol{L}$ is its directrix.

The time-averaged force of light pressure on the walls of a waveguide of unit length, calculated via the component of the Maxwell tension tensor taken with the opposite sign, is equal to the integral over the contour L:

$$f_{\mathrm{TM}} = E_0^2\frac{c^2\varepsilon_0}{4\omega_{nm}^2}\int_L(\nabla_q\psi)^2\mathrm{d}l, \quad (3.53)$$

$$f_{\mathrm{TE}} = E_0^2\frac{c^2\varepsilon_0}{4\omega_{nm}^2}\int_L[(\omega_{nm}\psi/c)^2 - (\nabla_q\psi)^2]\mathrm{d}l. \quad (3.54)$$

It follows from this that the total force of light pressure on the walls of a waveguide of an arbitrary length is proportional to the total field energy W

$$F_\omega = \frac{1}{2}\left(\frac{\omega_{nm}}{\omega}\right)^2 W\frac{L}{S}. \quad (3.55)$$

Here, we took into account that the field energy per waveguide unit length is expressed in terms of the integral over the cross section S:

$$W^* = E_0^2 \frac{\varepsilon_0}{2} \left(\frac{\omega}{\omega_{nm}} \right)^2 \int_S \psi^2 \mathrm{d}s. \tag{3.56}$$

The elementary work performed by moving the contour L against the light pressure force F_ω upon a change in the waveguide cross section S is

$$\mathrm{d}W = -F_\omega \mathrm{d}n, \tag{3.57}$$

where $\mathrm{d}n$ is the differential of movement along the normal to the contour L.

If the shape of the contour L does not change with varying cross section, then the dimensionless function $\psi(q_1, q_2)$ also does not change, and the product

$$\omega_{nm}^2 S = (\omega_{nm}^2)_0 S_0 = \mathrm{const}, \tag{3.58}$$

where the zero subscript denotes the initial values. Taking into account that $\mathrm{d}S = L\mathrm{d}n$, we obtain

$$\mathrm{d}\omega_{nm}^2 = -\omega_{nm}^2 \frac{\mathrm{d}S}{S}. \tag{3.59}$$

Thus, expression (3.59), taking into account the invariant (3.19), gives the equation for the accumulation of the total mode energy during a change in the waveguide cross section:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega_{nm}^2} = \frac{W/2}{\omega_{nm}^2 - (\omega_{nm}^2)_0 + \omega_0^2}, \tag{3.60}$$

whose integral coincides with the Manly–Row relation (3.24), with all the conclusions of sections 3.3 and 3.4 following from this.

3.6 Summary comments

Thus, the observable photon rest mass M_{nm} (3.17) in a waveguide is equivalent to the energy of a quantum of critical frequency ω_{nm}, which is equal to the work performed by an external force against the mode field force during the waveguide formation when the light pressure is 'raked' from infinite space into the limited volume of the waveguide. Therefore, the quantity M_{nm} (3.17) is not formal but has a certain physical meaning.

4. The observable photon rest mass in the co-moving coordinate system

The properties of the observable photon rest mass can be revealed in a purely kinematic experiment by observing the travelling wave of a waveguide mode from a co-moving inertial coordinate system moving along the z axis of the waveguide at the velocity $c\beta = v_{\mathrm{gr}}$ [36]. [The latter is not forbidden by the laws of the special relativity theory (SRT) because $v_{\mathrm{gr}} < c$ (3.5).] The result of such a relativistic transformation is obviously the wave stopping and the possibility of the direct observation of the stopped light.

4.1 Relativistic kinematics of the waveguide-mode field

What happens upon such a transformation of coordinates to the wave frequency and propagation constant in a waveguide, which were initially specified in the laboratory coordinate system?

The standard relativistic derivation of the Doppler formula is based on the postulate of invariance of the wave phase $\Phi = \omega t - kz$ in different inertial coordinate systems. By substituting the values of t and z

$$t = \frac{t' + \beta z'/c}{(1-\beta^2)^{1/2}}, \quad z = \frac{z' + c\beta t'}{(1-\beta^2)^{1/2}}, \tag{4.1}$$

which are expressed in terms of the Lorentz-transformed primed coordinates of the co-moving system, into the relation for the phase Φ, we can easily obtain the expression for the phase in the primed system:

$$\Phi = \omega \frac{1 - c\beta k/\omega}{(1-\beta^2)^{1/2}} t' - k \frac{1 - \beta\omega/ck}{(1-\beta^2)^{1/2}} z'. \tag{4.2}$$

Here, the factors in front of t' and $z'$are the values of the frequency ω' and the propagation constant k' in the co-moving system, and the second terms in the numerators are transformed with the help of the dispersion relation (3.6) to give

$$\omega' = \omega \frac{1 - \beta[1 - (\omega_{nm}/\omega)^2]^{1/2}}{(1-\beta^2)^{1/2}} = \omega \frac{1 - \beta(v_{\mathrm{gr}}/c)}{(1-\beta^2)^{1/2}}, \tag{4.3}$$

$$k' = k \frac{1 - \beta/[1 - (\omega_{nm}/\omega)^2]^{1/2}}{(1-\beta^2)^{1/2}} = k \frac{1 - \beta(c/v_{\mathrm{gr}})}{(1-\beta^2)^{1/2}}. \tag{4.4}$$

Expression (4.3) is the Doppler effect formula for a waveguide.

It follows from (4.3) and (4.4) that, if the movement velocity $c\beta$ of the primed system coincides with the group velocity v_{gr} of the wave in the laboratory frame, then

$$\omega' = \omega_{nm} = \omega(1-\beta^2)^{1/2}, \tag{4.5}$$

$$k' = 0, \quad v'_{\mathrm{gr}} = 0. \tag{4.6}$$

The latter means that the wave *stops* in the primed coordinate system and the transport of the electromagnetic energy ceases, the travelling component of the wave disappears at all, and only the transverse standing component is retained at the frequency $\omega' = \omega_{nm}$ (4.5) corresponding to the second-order transverse Doppler effect. This *stopped* light wave accumulates the *resting* electromagnetic energy $N\hbar\omega_{nm}$ or, in other words, it contains N *stopped* photons, the energy of each of the photons being equal to that of the photon of the critical frequency of the waveguide mode. According to the principle of equivalence, this energy of the *resting* photon corresponds to the rest mass (3.17) coinciding with the nonzero observable photon rest mass.

As the movement velocity of the primed coordinate system further increases ($c\beta > v_{\mathrm{gr}}$), the direction of the wave propagation changes to the opposite ($k' < 0$) and the wave frequency ω' increases (Fig. 3).

4.2 Relativistic transformations of the field vectors and the Poynting vector

The change in the wave propagation direction to the opposite and the wave standstill are also confirmed by the

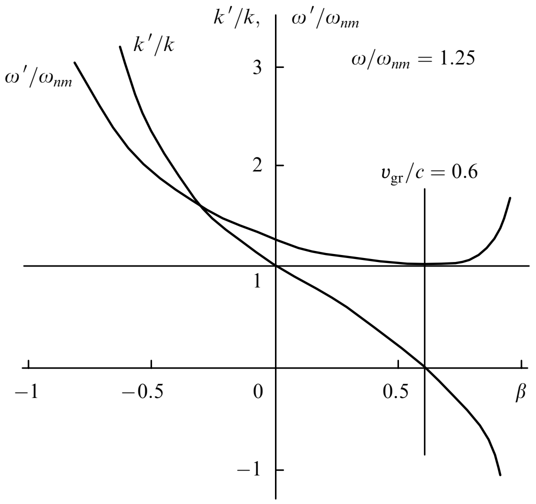

Figure 3.

change of sign and the vanishing of the Poynting vector in the primed coordinate system.

The field vectors are transformed on passing to the co-moving primed coordinate system, the longitudinal components of the field vectors remaining invariable ($e'_z = e_z$ and $h'_z = h_z$), while the transverse components are transformed by the known rules

$$e'_x = \frac{e_x - (\mu_0/\varepsilon_0)^{1/2}\beta h_y}{(1-\beta^2)^{1/2}}, \tag{4.7}$$

$$e'_y = \frac{e_y + (\mu_0/\varepsilon_0)^{1/2}\beta h_x}{(1-\beta^2)^{1/2}}, \tag{4.8}$$

$$h'_x = \frac{h_x + (\varepsilon_0/\mu_0)^{1/2}\beta e_y}{(1-\beta^2)^{1/2}}, \tag{4.9}$$

$$h'_y = \frac{h_y - (\varepsilon_0/\mu_0)^{1/2}\beta e_x}{(1-\beta^2)^{1/2}}. \tag{4.10}$$

It follows from this, taking into account (3.7)–(3.10) and (3.14), (3.15), that the longitudinal component of the Poynting vector for each of the polarisations is

$$P'_{\mathrm{TM,TE}} = P_{\mathrm{TM,TE}}\frac{(1-\beta v_{\mathrm{gr}}/c)(1-c\beta/v_{\mathrm{gr}})}{(1-\beta^2)^{1/2}}. \tag{4.11}$$

Therefore, when the group velocity of the wave in the laboratory coordinate system and the velocity of an observer moving together with the primed system are identical, the longitudinal Poynting vector vanishes, i.e., the transport of the electromagnetic energy ceases and photons are *stopped*. As the velocity of the observer further increases, the energy flux changes its sign and begins to propagate in the opposite direction, which corresponds to the change in the sign of the wave vector pointed out above.

4.3 Summary comments

In addition to the above conclusions about the mass-like behaviour of photons in real electromagnetic fields, which is caused by the presence of the fragments of standing waves (stopped light) in them, we can also conclude that the nonzero observable photon rest mass in a waveguide is revealed upon its purely kinematic stopping due to the relativistic transformation of coordinates. Such a relativistic kinematic photon stopping, resulting in the detection of their rest mass, can be undoubtedly found in any fields outside a material medium, because, as pointed out above, $v_{\mathrm{gr}} < c$ in these fields [for example, in the field of the wave reflected from a plane mirror (section 2.7) during the movement of an observer along the mirror surface at the velocity (2.30)].

5. Dynamic experiments with photons with a finite observable rest mass

The characteristic properties of the quantity defined as the inertial and gravitational rest mass of a body are revealed when the body is subjected to the action of one or another force producing acceleration, i.e., in a thought dynamic experiment with photons belonging to the modes of the basic waveguide model [28].

5.1 Longitudinal acceleration of a photon in a waveguide

It follows from dispersion expression (3.6) and relation (3.5) for the group velocity that the photon energy in a waveguide is

$$\hbar\omega = \frac{M_{nm}c^2}{[1-(v_{\mathrm{gr}}/c)^2]^{1/2}}, \tag{5.1}$$

and the product of the propagation constant by Planck's constant corresponding to the photon momentum is

$$\hbar k = \frac{M_{nm}v_{\mathrm{gr}}}{[1-(v_{\mathrm{gr}}/c^2]^{1/2}}. \tag{5.2}$$

Both these expressions obviously coincide with the corresponding relativistic formulas for the energy and momentum of a massive particle.

The only force capable of accelerating a photon along a waveguide can be probably the gravity force, which will be considered below. Here, it is sufficient to find in the expression for the time derivative of $\hbar k$ (5.2) [37]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\hbar k) = \frac{M_{nm}}{[1-(v_{\mathrm{gr}}/c)^2]^{3/2}}\frac{\mathrm{d}v_{\mathrm{gr}}}{\mathrm{d}t} \tag{5.3}$$

the coefficient at the longitudinal acceleration $\mathrm{d}v_{\mathrm{gr}}/\mathrm{d}t$. This coefficient

$$M_{\rightarrow} = \frac{M_{nm}}{[1-(v_{\mathrm{gr}}/c)^2]^{3/2}} \tag{5.4}$$

has the form and meaning of the so-called longitudinal mass of a particle.

5.2 Transverse acceleration of a photon in a waveguide

The transverse (normal to the waveguide axis) acceleration A of a photon acquires the physical meaning only if we study the movement of the field and of the waveguide holding the field, treated as whole, caused by the action of the force $F_{\uparrow}$ normal to the axis. The difference ΔF of the forces of light pressure of the mode field with the only photon on the opposite walls of the waveguide counteracts to this force [37]. Then, the so-called transverse observable photon mass in the waveguide (of course, neglecting the mass of the waveguide walls) can be defined as

$$M_{\uparrow} = \frac{F_{\uparrow}}{A} = \frac{\Delta F}{A}\,. \tag{5.5}$$

During the transverse movement of the waveguide at the variable velocity $V = At$ (for simplicity, the acceleration A is assumed constant), the wave frequency changes according to the law (3.19), in which the critical frequency ω_{nm} depends on the instantaneous value of the velocity V due to the relativistic contraction of the transverse size of the waveguide. In a simple case of a planar waveguide with the walls separated by the distance a,

$$\omega_{nm} = \frac{\omega_{nm0}}{[1-(V/c)^2]^{1/2}} \tag{5.6}$$

and the frequency changes (in fact, due to the Doppler effect on the moving walls) as

$$\omega = \omega_0\left[1 + \left(\frac{\omega_{nm0}}{\omega_0}\right)^2 \frac{(V/c)^2}{1-(V/c)^2}\right]^{1/2} \tag{5.7}$$

(the zero subscripts denote the initial values).

At the instant $t > 0$, when the velocity $V = At$, a photon incident on the wall at an angle of ϑ imparts the momentum

$$p' = \frac{2\hbar\omega}{c}\frac{\cos\vartheta \pm V/c}{1-(V/c)^2} \tag{5.8}$$

to the wall after reflection. After the next reflection from the opposite wall, which occurs at the instant $t + \Delta t$, when the velocity acquires the increment $\Delta V = A\Delta t$, the momentum

$$p'' = \frac{2\hbar\omega}{c} \times \frac{\cos\vartheta \pm V/c \mp \Delta(V/c)[1 \pm 2(V/c)\cos\vartheta + (V/c)^2]}{1-(V+\Delta V)^2/c^2} \tag{5.9}$$

is imparted to the waveguide in the opposite direction. (Here, the frequencies and angles are given in the immobile coordinate system, set for $t = 0$ and $V = 0$; the upper signs refer to the first reflection from the wall, which moves toward the photon, the lower signs refer to reflection from a co-propagating wall.)

After two successive reflections of the photon, the waveguide acquires the difference momentum

$$\Delta p = \pm(p' - p'') = \frac{2\hbar\omega}{c} \times \frac{1 \mp (\cos\vartheta \pm V/c)\Delta(V/c)/[1-(V/c)^2]}{1-(V+\Delta V)^2/c^2} \times \frac{\Delta V}{c} \approx \frac{2\hbar\omega}{c}\frac{\Delta V}{c} \tag{5.10}$$

directed toward the acceleration vector. In the second approximate equality in (5.10) for the case of the non-relativistic movement of the waveguide, the terms quadratic in V/c and $\Delta V/c$, which depend on the succession of reflection events, are omitted. The increment of the velocity during the time between two reflections is

$$\Delta V \approx \frac{aA}{c\cos\vartheta}, \tag{5.11}$$

and the momentum imparted to the waveguide in one reflection is

$$\Delta p^* = \frac{\Delta p}{2} = \frac{\hbar\omega}{c^3}\frac{aA}{\cos\vartheta}\,. \tag{5.12}$$

If N is the photon occupation number of the waveguide mode, then

$$J = \frac{cN}{a}\cos\vartheta \tag{5.13}$$

is the total photon flux incident on the waveguide walls, and

$$\Delta p^* J = \frac{\hbar\omega}{c^2} A \frac{N}{[1-(v_{\mathrm{gr}}/c)^2]^{1/2}} \tag{5.14}$$

is the light pressure force of the mode photons, which counterbalances the external force $F_{\uparrow}$ in (5.5) accelerating the waveguide. By equating $F_{\uparrow}$ to the force (5.14) for the waveguide containing a single photon in the mode ($N = 1$), we obtain the expression

$$M_{\uparrow} = \frac{M_{nm}}{[1-(v_{\mathrm{gr}}/c)^2]^{1/2}} \tag{5.15}$$

for the so-called transverse observable photon mass in the waveguide, which coincides with the standard SRT formula.

5.3 Centripetal acceleration of a photon in a curvilinear waveguide

To observe and measure the photon mass, we can follow a simple recipe of Feynman [38]: ‘The mass is a quantitative measure of inertia. It can be measured by simply tying a subject to a rope and rotating it at a certain velocity by measuring the force required to hold the subject. In this way, one can measure the mass of any subject’. In other words, it is necessary to measure the reaction force of the walls of a waveguide curved along the arc of a circle to the wave of the specified mode propagating through the waveguide [39].

For simplicity, we can set the central angle of the arc equal to 2π, i.e., to consider in fact a ring resonator of a rectangular cross section with sizes a and b, the radii of cylindrical walls R_2 and $R_1 = R_2 - a$, and with two counter-propagating travelling waves with the same frequency ω with the field components [32] in the cylindrical coordinate system (R, φ, x), which form the standing wave.

For the TM polarisation,

$$E_R = -A_0\frac{\tau}{\rho}D'_m(\rho R)\sin(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi)], \tag{5.16}$$

$$E_{\varphi} = A_0\frac{m\tau}{\rho^2 R}D_m(\rho R)\sin(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi + \pi/2)], \tag{5.17}$$

$$E_x = A_0 D_m(\rho R)\cos(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi)], \tag{5.18}$$

$$H_x = 0, \tag{5.19}$$

where

$$D_m(\rho R) \equiv N_m(\rho R_2)J_m(\rho R) - J_m(\rho R_2)N_m(\rho R); \quad (5.20)$$

$\rho = \rho_{nm} = \alpha_{nm}R_1$ is found from the roots α_{nm} of the characteristic equation

$$D_m(\rho_{nm}R_1) = 0, \quad m = 0, 1, 2, ..., \quad n = 1, 2, ... \quad (5.21)$$

For the TE polarisation,

$$H_R = A_0\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\frac{\tau}{\rho}G'_m(\rho R)\cos(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi)], \quad (5.22)$$

$$H_\varphi = -A_0\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\frac{m\tau}{\rho^2 R}$$

$$\times G_m(\rho R)\cos(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi + \pi/2)], \quad (5.23)$$

$$H_x = A_0\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}G_m(\rho R)\sin(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi)], \quad (5.24)$$

$$E_R = -A_0\frac{\omega m}{c\rho^2 R}G_m(\rho R)\sin(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi)], \quad (5.25)$$

$$E_\varphi = A_0\frac{\omega}{c\rho}G'_m(\rho R)\sin(\tau x)\exp[\mathrm{i}(\omega t \mp m\varphi + \pi/2)], \quad (5.26)$$

$$E_x = 0, \quad (5.27)$$

where

$$G_m(\rho R) \equiv N'_m(\rho R_1)J_m(\rho R) - J'_m(\rho R_1)N_m(\rho R); \quad (5.28)$$

$\rho = \rho_{nm} = \alpha_{nm}/R_1$ is found from the roots α_{nm} of the characteristic equation

$$G'_m(\rho_{nm}R_1) = 0, \quad m = 0, 1, 2, ..., \quad n = 1, 2, ...; \quad (5.29)$$

J_m is the Bessel function of the first kind; N_m is the m-order Neumann function (primes denote the differentiation over the argument); and A_0 is the amplitude factor.

The resonance frequencies of the waves of both polarisations are determined from the expression

$$\omega^2 = \omega_{nml}^2 = (c\rho_{nm})^2 + (c\tau)^2 = \omega_{n0l}^2 + c^2(\rho_{nm}^2 - \rho_{n0}^2), \quad (5.30)$$

where

$$\tau = \pi l/b; \quad l = 0, 1, 2, \ldots . \quad (5.31)$$

The radial component of the vector of light pressure on the cylindrical boundary with $R = \text{const}$ is equal to the corresponding component T_{11} of the Maxwell tension tensor with the opposite sign

$$Q_R = -T_{11} = -\frac{\varepsilon_0}{2}(E_R^2 - E_\varphi^2 - E_x^2)$$

$$-\frac{\mu_0}{2}(H_R^2 - H_\varphi^2 - H_x^2). \quad (5.32)$$

It follows from this, taking into account (5.16)–(5.19) and (5.22)–(5.27) that the radial components of the light pressure vector for the TM and TE polarisations are

$$Q_R^{\mathrm{TM}} = -\frac{\varepsilon_0}{2}E_R^2 + \frac{\mu_0}{2}H_\varphi^2 \quad (5.33)$$

and

$$Q_R^{\mathrm{TE}} = -\frac{\varepsilon_0}{2}E_R^2 + \frac{\mu_0}{2}(H_\varphi^2 + H_x^2) \quad (5.34)$$

respectively, and the time-averaged total force of light pressure on the entire cylindrical surface of radius R_1 or R_2 is

$$F_R = \int_0^{2\pi/\omega} R\omega \mathrm{d}t \int_0^b Q_R \mathrm{d}x. \quad (5.35)$$

Therefore, the differences of the forces of light pressure on the opposite cylindrical walls for the TM and TE polarisations are

$$\Delta F_{\mathrm{TM}} = \frac{B}{R_2}\left[1 - \frac{R_2}{R_1}\frac{J_m^2(\rho R_2)}{J_m^2(\rho R_1)}\right] \quad (5.36)$$

and

$$\Delta F_{\mathrm{TE}} = \frac{B}{R_2}\left[1 - \left(\frac{m}{\rho R_2}\right)^2 - \frac{R_2}{R_1}\frac{J'^2_m(\rho R_2)}{J'^2_m(\rho R_1)}\right.$$

$$\left.\times\left(1 - \frac{m^2}{(\rho R_1)^2}\right)\right], \quad (5.37)$$

where

$$B \equiv \frac{\varepsilon_0 b}{\pi\rho^2}A_0^2 \quad (5.38)$$

has the dimensionality of energy and appears as a factor in the total energy of the mode fields

$$W_{\mathrm{TM}} = B\left(\frac{\omega}{c\rho}\right)^2\left[1 - \frac{J_m^2(\rho R_2)}{J_m^2(\rho R_1)}\right], \quad (5.39)$$

$$W_{\mathrm{TE}} = B\left(\frac{\omega}{c\rho}\right)^2\left\{1 - \left[\frac{m}{\rho R_2}\right]^2\right.$$

$$\left.-\frac{J'^2_m(\rho R_2)}{J'^2_m(\rho R_1)}\left[1 - \frac{m^2}{(\rho R_1)^2}\right]\right\}. \quad (5.40)$$

Therefore, the difference ΔE of light pressure forces (5.36), (5.37) can be expressed in terms of the total energy W of the mode field

$$\Delta F = \frac{W}{R_2}\left(\frac{c\rho}{\omega}\right)^2 f, \quad (5.41)$$

where the factor f for the TM and TE polarisations is different:

$$f_{\mathrm{TM}} = \frac{R_2}{R_1}\left(1 - \frac{1 - R_1/R_2}{1 - J_m^2(\rho R_2)/J_m^2(\rho R_1)}\right), \quad (5.42)$$

$$f_{\rm TE} = \frac{R_2}{R_1}\left(1 - \frac{1 - R_1/R_2}{1 - \dfrac{J'^2_m(\rho R_2)}{J'^2_m(\rho R_1)} \dfrac{1 - (m/\rho R_1)^2}{1 - (m/\rho R_2)^2}}\right). \quad (5.43)$$

Returning to the recipe of Feynman, we should take into account that the entire mass in this simple mechanical analogue is concentrated at the rope end, whereas the field energy in a toroidal resonator is distributed within a ring with radial coordinates differing by a, which was taken into account with the help of the factor f in (5.41). Therefore, a direct comparison with the experiment of Feynman can be performed only for the mode in which the field is concentrated closely to the cylindrical surface of a larger radius R_2. This requirement is fulfilled for modes with multiply periodic fields over the circle and a small number of zeroes along the radius

$$m \gg 1 \text{ and } n \sim 1, \quad (5.44)$$

when we can assume for simplicity that $R_1 \ll R_2$, which does not affect noticeably the result. Then, expressions (5.42) and (5.43) take the approximate form and have almost coincident roots [40]:

for the TM polarisation

$$J_m(\rho_{1m}R_2) = 0, \ \rho_{1m} \approx \frac{m}{R_2}\left(1 + 1.86m^{-2/3}\right) \approx \frac{m}{R_2}, \quad (5.45)$$

and for the TE polarisation

$$J'_m(\rho_{1m}R_2) = 0, \ \rho_{1m} \approx \frac{m}{R_2}\left(1 + 0.81m^{-2/3}\right) \approx \frac{m}{R_2}. \quad (5.46)$$

For both polarisations, the difference between light pressure forces is

$$\Delta F \approx \frac{W}{R_2}\left(\frac{cm}{\omega R_2}\right)^2, \quad (5.47)$$

where the frequency determined from (5.30), taking into account the above approximations, is

$$\omega \approx \frac{\omega_{n0l}}{\left[1 - (c/\omega)^2(\rho^2_{1m} - \rho^2_{10})\right]^{1/2}} \approx \frac{\omega_{n0l}}{\left[1 - (cm/\omega R_2)^2\right]^{1/2}}. \quad (5.48)$$

Let us assume now that the mode contains a single photon, so that $W = \hbar\omega$. Then, taking into account the expression for the group velocity of the wave in a toroidal waveguide

$$v_{\rm gr} = \frac{c^2 m}{\omega R} \quad (5.49)$$

the expression

$$\Delta F = \frac{\hbar\omega_{n0l}/c^2}{\left[1 - (v_{\rm gr}/c)^2\right]^{1/2}} \frac{v^2_{\rm gr}}{R_2} = \frac{M_{n0l}}{\left[1 - (v_{\rm gr}/c)^2\right]^{1/2}} \frac{v^2_{\rm gr}}{R_2} = \frac{M_\uparrow v^2_{\rm gr}}{R_2} \quad (5.50)$$

for the difference of light pressure forces [where $M_{n0l} = \hbar\omega_{n0l}/c^2$ is the observable photon rest mass and $M_\uparrow$ is the transverse photon mass (5.15)] obviously coincide with the standard formula for the centripetal acceleration of a body.

5.4 Gravitational acceleration of a heavy photon. A desktop 'black hole'

The properties of the gravitational mass in the quantity M_{nm} (3.17) can be found by observing the propagation of a wave through a waveguide mounted vertically in the gravitational field [4]. According to (3.17), the value of M_{nm} is proportional to the critical frequency

$$\omega_{nm} = \frac{c\alpha_{nm}}{a}, \quad (5.51)$$

where a is the size of the transverse section of the waveguide and α_{nm} is the root of the characteristic equation, i.e., the mathematical invariant independent of the size. If the waveguide is located in the gravitational field with the potential $\Psi < 0$, $|\Psi| \ll c^2$, then the speed of light is

$$c \approx c_\infty(1 + 2\Psi/c^2), \quad (5.52)$$

and the transverse size (as any linear dimensions of bodies)

$$a \approx a_\infty(1 + 2\Psi/c^2) \quad (5.53)$$

depend on Ψ (hereafter, the subscript ∞ denote values at infinity, where the potential is normalised as $\Psi_\infty = 0$) [41]. It follows from this that the critical frequency, and hence, the observable photon rest mass in the waveguide also depend on the gravitational potential

$$M_{nm} = \frac{\hbar\omega_{nm}}{c^2} \approx \frac{M_{nm\infty}}{(1 + \Psi/c^2)(1 + 2\Psi/c^2)} \approx M_{nm\infty}(1 - 3\Psi/c^2), \quad (5.54)$$

where

$$M_{nm\infty} = \frac{\hbar\alpha_{nm}}{c_\infty a_\infty}. \quad (5.55)$$

Note here that dependence (5.55), obtained by taking into account the influence of the gravitational field on the quantities entering the definition of the observable rest mass M_{nm} (3.17), does not differ from the standard dependence of the mass of a usual heavy body on the gravitational potential [41], which is also obvious from the dimensionality considerations.

Returning to the process of the wave propagation in the vertical waveguide, note that the propagation constant k also varies in the gravitational field

$$k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{nm}}{c}\right)^2 \approx k^2_\infty\left[1 - 2\frac{\Psi}{c^2}\left(1 + \frac{2}{1 - \omega^2_{nm\infty}/\omega^2_\infty}\right)\right]. \quad (5.56)$$

This is explained by the fact that both the critical frequency

$$\omega_{nm} \approx \omega_{nm\infty}(1 + \Psi/c^2), \quad (5.57)$$

and the wave frequency

$$\omega = \omega_\infty (1 - \Psi/c^2) \tag{5.58}$$

depend on the potential. It is important to note that, although the second term in (5.56) can be not small even when $|\Psi| \ll c^2$, near the critical conditions, $\omega_{\cdot\infty} \to \omega_{nm\infty}$, the propagation constant k does not vanish nevertheless because $\Psi < 0$, and $k_\infty \neq 0$ due to the normalisation.

The observation of a decrease in the propagation constant k during the propagation of the wave upward through the waveguide from a point with the potential Ψ, in which $k = k^*$, to a point with a larger potential $\Psi + \Delta\Psi$ ($\Delta\Psi > 0$) reveals the phenomenon of the gravitational stop of the wave:

$$k^2 \approx (k^*)^2 \left[1 - 2\frac{\Delta\Psi}{c^2}\left(1 + \frac{2}{1 - \omega_{nm}^2/\omega^2}\right)\right] \tag{5.59}$$

(ω and ω_{nm} are taken at the point with the potential Ψ).

Indeed, near the critical regime, when $1-(\omega_{nm}/\omega)^2 \ll 1$, despite the fact that $\Delta\Psi \ll c^2$, the propagation constant vanishes when

$$\Delta\Psi \approx \frac{c^2}{4}\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right] \tag{5.60}$$

and the propagation of the wave upward through the waveguide ceases. Then, together with the sign of $\pm\sqrt{k^2}$, the direction of the wave propagation changes to the opposite – downward the waveguide.

This means in the language of the mechanical analogy that a body thrown upward with the kinetic energy that is small compared to the rest energy (due to the smallness of k^* near the critical regime), i.e., in the nonrelativistic case, stops in the upper point of the trajectory, having exhausted its initial momentum, and then falls back.

Using another analogy, we can say that this situation reproduces in a certain sense a desktop 'black hole', which holds the photons of the waveguide mode.

The vertical distance H to the turning point behind which the wave propagates downward can be obtained from condition (5.60). Thus, if

$$\Delta\Psi = gz \tag{5.61}$$

(g is the gravitational acceleration and z is the vertical coordinate), then the height of the turning point of the wave is

$$H \approx \frac{c^2}{4g}\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right] = \frac{v_{\rm gr}^2}{4g}. \tag{5.62}$$

One can easily see that the value of H is half the lift of a heavy body thrown at the initial velocity $v_{\rm gr}$, calculated from the laws of Newton mechanics. The reason for this discrepancy is the same as that appearing upon comparison of the estimates of the gravitational deviation of a light beam made using the laws of the general relativity theory and the Newton gravitation theory taking into account the principle of equivalence (see, for example, Ref. [41]).

Indeed, the reversal of the wave in the vertical waveguide is in fact nothing but one of the possible realisations of the process of gravitational deviation of a light beam separated into individual parts due to multiple successive reflections from the waveguide walls. Thus, in a planar waveguide for $g = 0$, a light beam circumscribes a zigzag broken straight line with a constant step and apexes on the waveguide walls (Fig. 4). Using the known method for sweeping a beam reflected from plane mirrors (which are the planar waveguide walls), the zigzag can be represented by an inclined straight line. The gravitational field with $g > 0$ bends this initially linear beam, transforming it into an arc with a maximum corresponding to the gravitational height H (5.62). This arc is in essence the beam sweep inside the waveguide, which is shown by the broken line with a step decreasing with height. A similar arc, with a two times slower variation of the angle calculated according the laws of Newton mechanics, taking into account the principle of equivalence, is shown by the dashed line with a maximum of height $2H$.

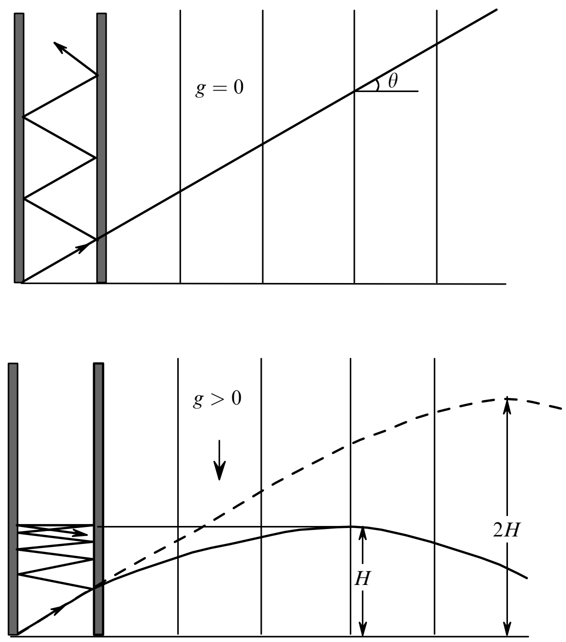

Figure 4.

5.5 An optical resonator with a gravitational mirror

The reversal of the wave propagation in the vertical waveguide can be interpreted as refraction reflection from a gravitational mirror mounted at the height $z = H$. If the vertical waveguide is supplemented with a usual reflecting mirror mounted at its base at $z = 0$, then such a waveguide of height exceeding H and open from above forms an optical resonator [42] with the resonance condition

$$\int_0^H k(z){\rm d}z = \pi q \; (q = 1, 2, ...). \tag{5.63}$$

In a particular case (5.61), when

$$k(z) = k^*(1 - zH)^{1/2}, \tag{5.64}$$

the resonance frequency measures at $z = 0$ is

$$\omega_q = \left[\omega_{nm}^2 + \left(\frac{3\pi c q}{2H}\right)^2\right]^{1/2} = \omega_{nm}\left[1 + \left(\frac{6\pi g q}{c\omega_{nm}}\right)^2\right]^{1/2}. \tag{5.65}$$

5.6 Photon weighing

To weigh a photon directly, it is sufficient to place the gravitational resonator described in section 5.5 with a single photon in the mode on the balance platform and to find the light pressure force acting downward on the lower mirror, which is imparted to the balance. It is important that the force directed upward after reflection from the upper gravitational mirror can be neglected because this force is imparted directly to the gravitational source, for example, the Earth rather than to the balance. As a result, the light pressure force of a single photon acting on the bottom of the gravitational resonator is equal to the photon weight.

The round-trip transit time (upward – downward) for a photon in the resonator is equal to the doubled time of energy transport to the height H at the velocity $\upsilon_{\rm gr}$

$$t_H = 2\int_0^H \frac{\omega}{c^2 k}\,{\rm d}z, \tag{5.66}$$

where the integrand is determined taking into account (5.59). In particular, if the potential is described by expression (5.61), then

$$t_H = \frac{2}{\upsilon_{\rm gr}}\int_0^H \frac{{\rm d}z}{(1-z/H)^{1/2}} = \frac{4H}{\upsilon_{\rm gr}} = \frac{\upsilon_{\rm gr}}{g}. \tag{5.67}$$

In each reflection event, the photon imparts the momentum

$$p_q = 2\frac{\hbar\omega_q}{c}\left[1-\left(\frac{\omega_{nm}}{\omega_q}\right)^2\right]^{1/2}, \tag{5.68}$$

to the balance platform, and the photon weight, as the ratio of p_q to t_H, is

$$f_g = \frac{p_q}{t_H} = 2\frac{\hbar\omega_q}{c^2}g = 2M_{nm}g. \tag{5.69}$$

Note that the photon weight f_g is twice as large as the standard Newton weight. The reason is the relativistic velocity of the photon propagation between the waveguide walls, unlike the nonrelativistic velocity of a heavy ball.

Here, it is necessary to make a digression [42] in order to consider the problem of weighing a usual heavy particle (ball), which is not, however, at rest on the balance platform but periodically bounces off it at a relativistic velocity. In the general case of the absolutely elastic 'reflection' of a ball, the trajectory of its movement during each fall on the balance platform is a cyclic succession of the arc-like fragments resting on the horizontal plane, and the ball weight can be determined from the expression

$$f_{\rm g} = \frac{2p_z(z=0)}{\Delta t}, \tag{5.70}$$

where $p_z(z=0)$ is the vertical component of the momentum at the reflection point $z=0$ and Δt is the time interval between two successive 'reflections' (bounces).

The relativistic equation specifying one cycle of the trajectory of a ball with the mass M^*, the velocity $\boldsymbol{\upsilon}$, and the momentum $\boldsymbol{p}$ in the gravitational field with the potential Ψ has the form [41]

$$\frac{{\rm d}\boldsymbol{p}}{{\rm d}t} = M^*\left[-\left(1+\frac{\upsilon^2}{c^2}\right){\rm grad}\Psi + \frac{\boldsymbol{\upsilon}}{c}\left(\frac{\boldsymbol{\upsilon}}{c}{\rm grad}\Psi\right)\right]. \tag{5.71}$$

In particular, if ${\rm grad}\Psi = g$, vector equation (5.71) is decomposed into the vertical and horizontal projections

$$\frac{{\rm d}p_z}{{\rm d}z} = -M^*g\left[1+\left(\frac{\upsilon_x}{c}\right)^2\right], \tag{5.72}$$

$$\frac{{\rm d}p_x}{{\rm d}t} = M^*g\frac{\upsilon_x\upsilon_z}{c^2}, \tag{5.73}$$

where the subscripts z and x denote the vertical and horizontal components of the vectors, respectively. The solutions for the two limiting cases are substantially different.

In the case of a strictly vertical movement, $\upsilon_x = 0$ and the arc-like trajectory degenerates to a segment of the vertical straight line, which is characterised by values

$$p_z = -M^*gt + p_z(z=0),\quad p_x = 0,\quad \Delta t = 2\frac{p_z(z=0)}{M^*g}, \tag{5.74}$$

while the ball weight (5.70)

$$f_{\rm g} = M^*g \tag{5.75}$$

coincides with the Newton value.

In the opposite case of a very sloping, almost horizontal trajectory, when we can assume that $\upsilon_z \ll c$ and $\upsilon_x \approx \upsilon$, we have

$$p_z = -M^*g\left[1+\left(\frac{\upsilon}{c}\right)^2\right]t + p_z(z=0),$$
$$p_x = {\rm const},\quad \Delta t = 2\frac{p_z(z=0)}{M^*g[1+(\upsilon/c)^2]} \tag{5.76}$$

and, hence, the ball weight (5.70) is

$$f_{\rm g} = M^*g[1+(\upsilon/c)^2], \tag{5.77}$$

which gives, for the relativistic velocity $\upsilon \to c$, the value

$$f_{\rm g} \to 2M^*g, \tag{5.78}$$

which is twice as large as the Newton value. For a moderately inclined trajectory, there exist the intermediate values of the relativistic ball weight, so that, generally speaking,

$$M^*g \leqslant f_{\rm g} \leqslant 2M^*g. \tag{5.79}$$

All this is valid both for a usual relativistic heavy particle and for the vertical movement of a photon in a waveguide considered above, where in the case close to critical conditions, the photon trajectory approaches the horizontal trajectory, which explains result (5.69).

5.7 The wave equation versus the Schrödinger equation

A characteristic feature of the above consideration is, in particular, the demonstration of the creation of the field of massive particles by a typical massless wave field. It is interesting that the genetic relation between the wave equation and the Schrödinger equation can be also revealed (of course, without pretending to derive the latter), as well as the transformation of the massless wave equation (1.9) or (1.10) to the equation of the Schrödinger type, containing the mass explicitly but without its introduction as a priory quantity.

The nonrelativistic character of the Schrödinger equation requires the fundamental dispersion equation (3.6) for a wave in a waveguide to be also written in the nonrelativistic approximation close to critical conditions:

$$\omega \approx \omega_{nm}\left[1+\frac{1}{2}\left(\frac{ck}{\omega_{nm}}\right)^2\right], \tag{5.80}$$

which corresponds in the language of the mechanical analogy to the smallness of the kinetic energy compared to the rest energy.

One can now easily see that the differentiation of the phase Φ of a travelling wave suggests that the differential operators can be written as

$$\frac{\partial}{\partial t}\to \mathrm{i}\omega \quad \text{and} \quad \frac{\partial^2}{\partial z^2}\to -k^2. \tag{5.81}$$

Then, the approximate dispersion equation (5.80) for the wave can be rewritten in the form of two operator equations

$$\frac{\partial^2}{\partial z^2} = -2\frac{\omega_{nm}}{c^2}(\omega-\omega_{nm}), \tag{5.82}$$

$$\frac{1}{\mathrm{i}}\frac{\partial}{\partial t} = \omega_{nm}\left[1-\frac{1}{2}\left(\frac{c}{\omega_{nm}}\right)^2\frac{\partial^2}{\partial z^2}\right]. \tag{5.83}$$

These operator equations can refer to any of the six components of the field vectors, for example, to H_x:

$$\frac{\partial^2 H_x}{\partial z^2} + 2\frac{M_{nm}}{\hbar^2}(E-\hbar\omega_{nm})H_x = 0, \tag{5.84}$$

$$\frac{\hbar}{\mathrm{i}}\frac{\partial H_x}{\partial t} = -\frac{\hbar^2}{2M_{nm}}\frac{\partial^2 H_x}{\partial z^2} + \hbar\omega_{nm}H_x. \tag{5.85}$$

Here, $E=\hbar\omega$ is the total energy and $M_{nm}=\hbar\omega_{nm}/c^2$ is the observable rest mass.

The above consideration shows that the energy of a photon with the critical frequency $\hbar\omega_{nm}$ is in fact the potential energy U accumulated in the waveguide upon the compression of the mode field from infinite free space. Then, the replacement $\hbar\omega_{nm}=U$ transforms (5.84) and (5.85) to the the familiar form

$$\frac{\partial^2 H_x}{\partial z^2} + 2\frac{M_{nm}}{\hbar^2}(E-U)H_x = 0, \tag{5.86}$$

$$\frac{\hbar}{\mathrm{i}}\frac{\partial H_x}{\partial t} = -\frac{\hbar^2}{2M_{nm}}\frac{\partial^2 H_x}{\partial z^2} + UH_x. \tag{5.87}$$

Of course, these one-dimensional equations of the Schrödinger type do not contain the Laplacian over the transverse coordinates because the integration over these coordinates have been already performed, which resulted in a priory appearance of the observable rest mass M_{nm} in (5.85) and (5.87).

5.8 Summary comments

The cycle of the thought dynamical experiments with the basic model of a photon in the waveguide mode have demonstrated that neither of these experiments reveal any features that would demonstrate the difference of the mass-like quantity M_{nm} (3.17), which is equivalent to the energy of a quantum with the critical frequency, from the inertial and gravitational rest mass of a body in its standard sense.

It is interesting to compare the possible values of the photon rest mass observed in some particular situations with the data presented in Table 1 (section 1.3). Thus, for a microwave photon in a standard waveguide, $M_{nm}c^2\sim 10^{-5}$ eV, for an optical photon in a glass fibre, $M_{nm}c^2\sim 0.1$ eV, and $M_{nm}c^2\sim 10^{-5}$ eV for the fundamental mode of a laser beam of diameter ~ 1 cm. These values are many orders of magnitude greater than those presented in Table 1, which is explained by the extremely strict spatial restriction imposed on the fields compared to the conditions used in section 1.3.

6. Conclusions

The main content of the analysis performed in this paper consists in a chain of the interrelated and following from each other statements:

(i) Plane electromagnetic waves do not exist in nature; all the physically realised fields are nonplane and spatially restricted.

(ii) Strictly transverse physical wave fields do not exist; any physically realised nonplane wave field has three polarisation states.

(iii) Nonplane waves realised in nature possess outside the material medium the frequency dispersion, the superluminal phase and subluminal group velocities.

(iv) To photons, represented by physically realisable nonplane fields, a finite observable inertial and gravitational rest mass can be assigned, which is not, however, their immanent characteristic, but depends on the field structure. Attempts to propose the experiment for detecting the difference between the behaviour of a photon having a finite observable rest mass and belonging to a nonplane wave field and the behaviour of usual massive bodies under similar conditions have failed.

(v) The observable photon rest mass is caused by the presence of a standing component in the form of the fragments of a standing wave (stopped light) in any physically realised nonplane wave field, which produce the momentum defect in the Hamiltonian.

(vi) All the above statements are in fact the direct corollaries of the 'classical uncertainty relation', i.e., are the result of the application of the Fourier theorem to spatially restricted nonplane wave fields. (Note here that a standard expression for the density function of the radiative modes in free space (the radiative oscillators of the field) is also a direct corollary of the 'classical uncertainty relation' applied to nonplane wave fields, which are the only fields that can be realised physically [43].)

(vii) It is curious heuristically that a typical massless photon field can is capable of producing, under certain conditions, the field of quanta having a finite observable rest mass, which is indistinguishable from the standard rest mass.

Of course, the problem of the nonplane electromagnetic waves and finite observable photon rest mass has not been comprehensively studied in this paper. Recall, however, the words Einstein has said once: 'All these fifty years of persistent reflections have not drawn me nearer to the answer what the light quanta are. Of course, now anyone thinks that he knows the answer, but he is deceiving himself'.

Acknowledgements. The work was partially supported by the US CRDF-RF Ministry of Education Award VZ-010-0.

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