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Generation of harmonics of a radiation field ionising hydrogen-like ions in a rapidly rising strong field

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***Abstract*. Approximate scaling dependences of the generation efficiency of the first five harmonics of the exciting field on thier number are determined in the case of Coulomb degeneracy. Analytic dependences of the efficiency of harmonics of a given number on the principal quantum number of the atomic level from which ionisation of an electron occurs are derived. Approximate scaling dependences of the harmonic generation efficiency on the generated harmonic number and the principal quantum number of the ionised atomic level are obtained.**

Keywords: *harmonics generation, hydrogen-like atom.*

1. We pose the problem of determining the efficiency of harmonic generation in a strong field whose intensity satisfies the Bethe condition [1]

$$E \geqslant \frac{I_Z^2}{4Z|e|^3}, \tag{1.1}$$

where

$$I_Z = \frac{Z^2 m_e e^4}{2n^2\hbar^2} \tag{1.2}$$

is the ionisation potential of the atom; Z is the charge of the atomic nucleus of the hydrogen-like atom; m_e is the electron mass; e is the electron charge; and n is the principal quantum number.

In order for the atom ionisation to occur under the Bethe conditions, the ionisation redistribution of the atomic levels should proceed on the atomic time scale. To put it otherwise, to ionise the nth atomic level under the condition (1.1) requires the action of a laser pulse with a very short rise time

$$t = \frac{n^2\hbar^3}{m_e e^4} \approx n^2 2.5\times 10^{-17}\ \text{s}, \tag{1.3}$$

i.e., t should be comparable with the time conventionally referred to as the electron orbiting time corresponding to the quantum state n.

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Having been almost instantly detached from the atom, the ionised electrons retain the memory of their intra-atomic distribution. That is why the plasma electron distribution function in the coordinate system oscillating in the exciting field can be represented in the form

$$f_{nlm} = N_e |a_{nlm}(\boldsymbol{p})|^2, \tag{1.4}$$

where N_e is the electron density; $\boldsymbol{V} = \boldsymbol{p}/m_e$ is the electron velocity; $\boldsymbol{p}$ is the electron momentum; and n, l, m are the quantum numbers of spherical quantisation of electron levels prior to atom irradiation by a rapidly rising high-power laser pulse.

2. Taking into account that the electron velocity distribution after ionisation in a strong field is, first, accompanied by their oscillation in the exciting field and, second, the spread relative to such oscillations is determined by the intra-atomic distribution of electrons prior to thier ionisation, it is reasonable to use Ref. [2], where expressions were obtained for the electron eigenfunctions of a hydrogen-like ion with a nuclear charge $Z|e|$ in the momentum representation:

$$a_{nlm}[p,\theta_p,\phi_p] = \Phi_m[\phi_p]\Theta_{lm}[\theta_p]a_{nl}[p], \tag{2.1}$$

where n is the principal quantum number; l is the orbital quantum number; and m is the projection of the angular momentum in the momentum space. According to Ref. [2],

$$\Phi_m[\phi_p, p] = \frac{1}{\sqrt{2\pi}}\exp(\pm \mathrm{i}m\phi_p), \tag{2.2}$$

$$\Theta_{lm}[\theta_p] = \frac{(2l+1)(l-m)!}{2(l+m)!}P_l^m(\cos\theta_p), \tag{2.3}$$

$$a_{nl}[p] = -\frac{(-\mathrm{i})^l 2^{2l+4}(l!)[n(n-l-1)!]^{1/2}[np/(m_e V_Z)]^l}{(m_e V_Z/n)^{3/2}[(n+1)!]^{1/2}\{[np/(m_e V_Z)]^2+1\}^{l+2}}$$

$$\times C_{n-l-1}^{l+1}\left\{\frac{[np/(m_e V_z)]^2-1}{[np/(m_e V_Z)]^2+1}\right\}, \tag{2.4}$$

where m_e is the electron mass; V_Z is the Coulomb unit of velocity [3]; $P_n^\nu(x)$ is the Legendre polynomial; and $C_n^\nu(x)$ is the Gegenbauer polynomial.

Considering the situation when the plasma electrons are unpolarised, we will employ the electron function averaged over the angles

$$f_{nl}[V] = N_e a_{nl}^2[p]. \quad (2.5)$$

Below, we consider a relatively simple case when it can be assumed, owing to the l degeneracy of the electron energy levels of a hydrogen-like ion, that the states with different values of the orbital quantum number are represented with equal probabilities. Then, we can use the distribution function corresponding to the principal quantum number:

$$f_n[V] = N_e \frac{1}{n^2} \sum_{l=0}^{n-1} a_{nl}^2 [2l+1]. \quad (2.6)$$

We will calculate this function using the following important relationship:

$$\sum_{l=0}^{n-1} (2l+1) 2^{2l} (l!)^2 \frac{n(n-l-1)!}{(n+l)!} (1-x^2)^l [C_{n-l-1}^{l+1}(x)]^2 = n^2 \quad (2.7)$$

which gives

$$f_n[V] = \frac{8}{\pi^2 (V_Z/n)^3 [1 + (nV/V_Z)^2]^4}. \quad (2.8)$$

For $n = 1$, formula (2.8) coincides with the electron velocity distribution function of the ground state of the hydrogen-like ion. The significant difference of expression (2.8) from the ground-state function is that expression (2.8) depends on the principal quantum number. This difference is precisely the reason why the harmonic generation intensity depends on the quantum numbers of excited energy states.

3. Following Refs [4, 5], we will describe the coherent bremsstrahlung generation of harmonics of the field

$$\boldsymbol{E}(t) = \boldsymbol{E} \cos \omega t \quad (3.1)$$

using the Boltzmann kinetic equation with the Landau collision integral. Then, as in Ref. [5], for the collisional contribution to the electric current density we have the following expansion in terms of the odd harmonics of the exciting field:

$$\delta \boldsymbol{j} = \sum_{N=0}^{\infty} \frac{e^2 N_e}{m_e \omega^2} \upsilon^{2N+1} \boldsymbol{E} \cos[(2N+1)\omega t], \quad (3.2)$$

where N belongs to the natural number set. Here, the effective partial nonlinear collision frequencies

$$\upsilon^{2N+1}[n, E] = \upsilon_Z H\left[2N+1, n\frac{V_E}{V_Z}\right] \quad (3.3)$$

are defined by the relationships

$$\upsilon_Z = \frac{16 e^2 e_i^2 N_i \Lambda}{m_e^2 V_Z^3}, \quad (3.4)$$

$$H\left[2N+1, n\frac{V_E}{V_Z}\right] = n^3 A\left[2N+1, \frac{nV_E}{V_Z}\right], \quad (3.5)$$

$$A[2N+1, nx] = \frac{1}{x^3} D\alpha[2N+1, b, x]|_{b=1}, \quad (3.6)$$

$$\alpha[2N+1, b, x] = \frac{\mathrm{d}}{\mathrm{d}b}\left[b \operatorname{arsinh}\left(\frac{x}{b}\right) - \frac{[(x^2+b^2)^{1/2} - b]^{2N}}{(2N+1)x^{2N-1}}\right.$$

$$\left. - \sum_{k=1}^{N} \frac{2b}{2N-2k+1} \frac{[(x^2+b^2)^{1/2} - b]^{2N-2k+1}}{x^{2N-2k+1}}\right], \quad (3.7)$$

$$D = 1 - \frac{\mathrm{d}}{\mathrm{d}b} + \frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}b^2},$$

where $V_E = |e|E/m_e\omega$ is the amplitude of electron oscillation velocity in the field E; e_i is the ion charge; N_i is the ion density; and Λ is the Coulomb logarithm. Expressions (3.3), (3.5), (3.6), and (3.7) are independent of the orbital quantum number and in this sense are simpler than the results of Ref. [5].

We employ the relationship

$$A[2N+1, x] = \frac{1}{x}(\operatorname{arsinh} x + a[2N+1, x]) \quad (3.8)$$

to obtain, bearing in mind that the functions $a[2N+1, x]$ in the limit $x \to \infty$ prove to be of the order of unity, the asymptotic dependence

$$A[2N+1, x] \approx \frac{1}{x} \ln \frac{2nV_E}{V_Z}. \quad (3.9)$$

This leads us to the scaling (cf. Ref. [5])

$$\upsilon^{2N+1}[n, E] = \upsilon_Z \frac{V_Z^3}{V_E^3} \ln \frac{2nV_E}{V_Z}, \quad (3.10)$$

which corresponds to the strong-field limit

$$\frac{nV_E}{V_Z} \gg 1. \quad (3.11)$$

4. For comparatively weak exciting fields, when the condition (3.11) is violated, there arise complicated nonlinear dependences of efficient collision frequencies on the pump field. The aim of our subsequent study is to determine, despite these complications, the approximate (reasonably accurate) scaling dependence of the effective collision frequencies on not only the principal quantum number [as indicated even by relationship (3.5)], but on the number of a harmonic generated.

We consider first the generation of the third and fifth harmonics. In this case, according to expression (3.7), we have

$$\alpha[3, b, x] = \operatorname{arsinh}\left(\frac{x}{b}\right) + \frac{8b}{3x} - \frac{8b^2 + 7x^2}{3x(b^2+x^2)^{1/2}}, \quad (4.1)$$

$$\alpha[5, b, x] = \operatorname{arsinh}\left(\frac{x}{b}\right) + \frac{24b}{5x} + \frac{64b^3}{15x^3}$$

$$- \frac{64b^4 + 104b^2x^2 + 43x^4}{15x^3(b^2+x^2)^{1/2}}. \quad (4.2)$$

What these harmonics have in common, according to expressions (3.6), (3.8), (4.1), and (4.2), is that

$$a[2N+1, x] = \frac{x}{(1+x^2)^{2N+1/2}} P_4[2N+1, x], \quad (4.3)$$

and the difference is only in the polynomials $P_4[2N+1, x]$, which have the form:

$$P_4[3,x] = -\frac{1}{3}\left(3 + 7x^2 + 3x^4\right), \tag{4.4}$$

$$P_4[5,x] = -\frac{1}{15}\left(15 + 35x^2 + 23x^4\right). \tag{4.5}$$

Expressions (3.5), (3.8), and (4.3)–(4.5) completely describe the generation of the third and fifth harmonics.

5. Higher-order harmonics are somewhat more difficult to describe. In particular, for the seventh harmonic it follows from expression (3.7) that

$$\alpha[7,b,x] = \operatorname{arsinh}\left(\frac{x}{b}\right) + \frac{b}{105x^5}\left(720x^4 + 1600x^2b^2 + 1152b^4\right) - \frac{1}{105x^5(b^2+x^2)^{1/2}}\left(337x^6 + 1376x^4b^2 + 2176x^2b^4 + 1152b^6\right). \tag{5.1}$$

Expression (3.8) leads to the following relation:

$$a[7,x] = \frac{1}{x(1+x^2)^{5/2}} R_{10}[7,x] + \frac{1}{x^5} Q_0[7], \tag{5.2}$$

$$Q_0[7] = \frac{1024}{35}, \tag{5.3}$$

$$R_{10}[7,x] = -\frac{1}{105}\left(3072 + 7680x^2 + 5760x^4 + 1065x^6 + 125x^8 + 197x^{10}\right). \tag{5.4}$$

One can see a certain resemblance of seventh harmonic generation to ninth and eleventh harmonic generation. Then, we have:

$$\alpha[9,b,x] = \operatorname{arsinh}\left(\frac{x}{b}\right) + \frac{b}{315x^7}\left(2800x^6 + 11200x^4b^2 + 18816x^2b^4 + 10240b^6\right) - \frac{1}{315x^7(b^2+x^2)^{1/2}}\left(1091x^8 + 6688x^6b^2 + 19328x^4b^4 + 23936x^2b^6 + 10240b^8\right), \tag{5.5}$$

$$\alpha[11,b,x] = \operatorname{arsinh}\left(\frac{x}{b}\right) + \frac{b}{3465x^9}\left(37800x^8 + 235200x^6b^2 + 677376x^4b^4 + 829440x^2b^6 + 358400b^8\right) - \frac{1}{3465x^9(b^2+x^2)^{1/2}}\left(12701x^{10} + 108568x^8b^2 + 492608x^6b^4 + 1047296x^4b^6 + 1008640x^2b^8 + 358400b^{10}\right). \tag{5.6}$$

In this case, according to (3.8),

$$a[9,x] = \frac{1}{x^7(1+x^2)^{5/2}} R_{12}[9,x] + \frac{1}{x^7} Q_2[9,x]\,, \tag{5.7}$$

$$a[11,x] = \frac{1}{x^9(1+x^2)^{5/2}} R_{14}[11,x] + \frac{1}{x^9} Q_4[11,x]\,, \tag{5.8}$$

$$R_{12}[9,x] = -\frac{1}{315}\left(81920 + 254976x^2 + 279040x^4 + 119680x^6 + 12795x^8 - 265x^{10} + 671x^{12}\right), \tag{5.9}$$

$$Q_2[9,x] = \frac{1024(80+49x^2)}{315}, \tag{5.10}$$

$$R_{14}[11,x] = -\frac{1}{3465}\left(5734400 + 20971520x^2 + 29147136x^4 + 18749440x^6 + 5236480x^8 + 375945x^{10} - 12715x^{12} + 8081x^{14}\right), \tag{5.11}$$

$$Q_4[11,x] = \frac{4096(1400 + 1620x^2 + 441x^4)}{3465}. \tag{5.12}$$

These expressions describe analytically nonlinear effective partial permittivities which appear upon the generation of harmonics when a plasma is produced due to ionisation of ions of the hydrogen-like atoms by l-degenerate electrons.

6. In the strong-field limit, the expressions obtained above are reduced, as mentioned in Section 5, to a relatively simple dependence of the harmonic intensity, which is common for all harmonics, on the exciting field intensity. We show that these expressions additionally possess one more relatively simple and approximately common scaling property, which is revealed if we use the expression

$$G[2N+1,x] = (2N+1)^3 A[2N+1, x(2N+1)]. \tag{6.1}$$

Indeed, expression (6.1) describes an approximate unified scaling, which corresponds both to the dependence on the principal quantum number and the exciting field intensity owing to the argument $x = nV_E/V_Z$ and to the dependence on the harmonic number by virtue of the argument $(2N+1)nV_E/V_Z$ as well as to the dependence on the harmonic number of the form $(2N+1)^3$. This statement follows from the approximate similarity of the curves plotted in Fig. 1. This similarity enables determining an approximate dependence for the effective collision frequency, which describes the generation of the $(2N+1)$th harmonic

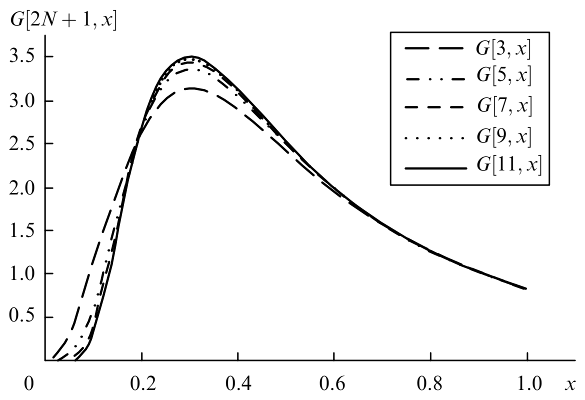

Figure 1. Dependence of the scaling function $G[2N+1,x]$ on the harmonic number $(2N+1)$, on the excited ion quantum state n, and the electron oscillation velocity V_E in the pump field.

$$\upsilon^{2n+1}[n,E] = \frac{\upsilon_Z n^3}{(2N+1)^3} A\left[2N+1, \frac{nV_E}{V_Z}(2N+1)\right].$$

The expressions obtained above allow us to determine, for linearly polarised exciting radiation, the generation efficiency of linearly polarised harmonic waves as the ratio of their energy densities

$$\eta^{2N+1}[n,E] = \left[\frac{(2N+1)\upsilon^{2N+1}[n,E]}{4N(N+1)\omega}\right]^2$$

$$= \left(\frac{\upsilon_Z}{\omega}\right)^2 \left[\frac{n^3 G[2N+1,(nV_E/V_Z)(2N+1)]}{4N(N+1)(2N+1)^2}\right]^2.$$

Since $G_{\max} \approx 3.5$, the maximum values of generation efficiency for different harmonics are approximately related as $\upsilon_Z^2 n^6/(16\omega^2 N^4)$.

Therefore, the efficiency of harmonic generation is proportional to the sixth power of the principal quantum number. This statement is valid when the electrons are ionised from the same energy level in the case of Coulomb l degeneracy.

Note that the main result of our paper consists in the demonstration of a strong dependence of harmonic generation efficiency on the principal quantum number of the energy level from which the electrons are ionised in the case of fast above-barrier Bethe ionisation. The result obtained is exact for the model of a hydrogen-like atom. Another result of our paper is the derivation of the approximate scaling of the dependence of the maximum generation efficiency on the generated harmonic number.

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