SOLITONS

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Bright and dark pulses in optical fibres in the vicinity of the zero-dispersion wavelength

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***Abstract*. The influence of the third-order dispersion on the propagation of short pulses in optical fibres is considered. The appearance of coupled nonlinear structures consisting of dark and bright envelope solitons is described. The wavelength range is found in the vicinity of the zero-dispersion wavelength where the effect of the third-order dispersion on the pulse propagation proves to be dominant. It is shown that in this case a nonlinear structure in the form of an embedded soliton appears.**

Keywords: *optical fibre, short optical pulse, dispersion, third-order dispersion, zero-dispersion wavelength, bright soliton, dark soliton, embedded soliton, nonlinear Schrödinger equation, Korteweg – de Vries equation.*

1. Introduction

The broadening of pulses caused by the group-velocity dispersion can be minimised in a natural way by using wavelengths for which the group-velocity dispersion is close to zero. The behaviour of optical pulses in the vicinity of the zero-dispersion wavelength λ_0 was discussed in many papers (see, for example, Refs [1 – 8]), the most attention being paid to the case of the anomalous dispersion of the fibre material. However, if the radiation intensity is substantially lower than the self-focusing threshold, solitons can propagate in the case of the normal dispersion of the fibre material as well [9]. Similar questions were considered for spin waves in magnetics [10, 11].

The smallness of the group-velocity-dispersion term requires the consideration of the next-order dispersion term in the derivation of an equation for the short-pulse envelope. The corresponding generalised nonlinear Schrödinger equation (NSE)

$$\mathrm{i}\frac{\partial \Phi}{\partial \xi} - \frac{1}{2}k''\frac{\partial^2 \Phi}{\partial \tau^2} - \frac{\mathrm{i}}{6}k'''\frac{\partial^3 \Phi}{\partial \tau^3} + \frac{1}{2}\frac{\omega_0 n_2}{c}|\Phi|^2\Phi = 0 \quad (1)$$

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(in a standard notation) can be easily written in the dimensionless form

$$\mathrm{i}\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial t^2} - \mathrm{i}b\frac{\partial^3 u}{\partial t^3} + 2|u|^2 u = 0. \quad (2)$$

We assume in (2) that $a \geqslant 0$ and $b \geqslant 0$. The nonnegative coefficient a corresponds to media with the normal dispersion. On the other hand, the adopted sign of this coefficient provides the absence of the modulation instability [4].

The coefficients a and b in (2) are small near the zero-dispersion wavelength λ_0 and the relation between them can be different. In passing from dimensional equation (1) to dimensionless one (2), a competition between the values of a and $b^{2/3}$ appears naturally, and therefore it is expedient to consider two characteristic regions:

$$a > b^{2/3}, \quad (3)$$

and

$$a < b^{2/3}. \quad (4)$$

The fulfilment of inequality (3) or (4) depends on the closeness of the wavelength λ to the zero-dispersion wavelength λ_0. By using estimates [1] and making some similar estimates, we can easily show that for a single-mode fibre a and $b^{2/3}$ are equal at $\lambda \approx 0.55\lambda_0$. When

$$\lambda < 0.55\lambda_0 \quad (5)$$

inequality (3) is valid. For

$$0.55\lambda_0 < \lambda < \lambda_0 \quad (6)$$

we have inequality (4). The case $\lambda > \lambda_0$ corresponds to the anomalous-dispersion region, which we do not consider here.

The solutions of Eqn (2) are possible both for bright and dark solitons. The influence of the third-order dispersion and the pulse self-steepening on the shape and parameters of a bright soliton was investigated in Refs [12, 13].

The aim of our paper is to describe coupled nonlinear structures by deriving explicit expressions for solitons of both types in cases (3) and (4). We will first derive these expressions for case (3) and then for (4). The latter case requires an essentially different analysis because only small-amplitude pulses can exist in region (6).

2. Bright and dark solitons for $a > b^{2/3}$

We seek the solution of Eqn (2) as a combination of solutions for bright (v) and dark (w) solitons:

$$u = \mathrm{e}^{-\mathrm{i}\Phi}(v + \mathrm{i}w) \quad (7)$$

with the phase

$$\Phi = ct - dx, \quad (8)$$

where c and d are constants. We assume that the required real functions v and w depend not on x and t separately but only on their combination

$$X = \alpha t - \beta x, \quad (9)$$

which is naturally called the amplitude phase.

Then we seek localised solutions for v and w. The localisation of the solutions means that they differ from constants only in small vicinities of some points or curves. In our case, the functions v and w are localised over the variable (9) in the vicinity of the point $X = 0$.

By passing in (2) to the differentiation with respect to X (hereafter, denoted by the prime) and substituting (7), we obtain the real system of equations

$$(a + 3bc)\alpha^2 v'' + (d - ac^2 - bc^3)v - \beta w' + (2ac + 3bc^2)\alpha w' - b\alpha^3 w''' - 2(v^2 + w^2)v = 0, \quad (10)$$

$$(a + 3bc)\alpha^2 w'' + (d - ac^2 - bc^3)w + \beta v' - (2ac + 3bc^2)\alpha v' + b\alpha^3 v''' - 2(v^2 + w^2)w = 0, \quad (11)$$

where a and b are small parameters satisfying condition (3). We are trying to obtain a nontrivial solution for system (10), (11) and to take simultaneously into account most completely both nonlinearity and the second- and third-order dispersion. Let us represent the coefficients b, α, and β as the beforehand unknown powers of a small parameter a and assume that the ratio v/w is also proportional to some power of a. By comparing the orders of individual terms in system (10), (11), we find easily that the only consistent possibility in solving this system of equations is that, for $a \to 0$,

$$b \sim a^2, \quad v \sim wa^{1/2}, \quad \beta \sim a^{1/2}, \quad \alpha \sim a^{-1/2}.$$

Then, the principal terms in (11) form the equation

$$w_0'' + 2w_0 - \frac{4}{d}w_0^3 = 0.$$

This equation has an odd localised solution

$$w_0 = \left(\frac{d}{2}\right)^{1/2} \tanh X, \quad (12)$$

which describes a dark soliton.

Similarly, we obtain from (10)

$$v_0 = \frac{3d}{4}\frac{b}{a^{3/2}}\frac{1}{\cosh^2 X}, \quad (13)$$

$$\alpha = \left(\frac{d}{2a}\right)^{1/2}, \quad \beta = \left(\frac{d}{2}\right)^{1/2}\left(2ca^{1/2} + \frac{bd}{a^{3/2}}\right). \quad (14)$$

Expression (13) describes a bright soliton.

The linearisation of Eqns (10) and (11) in the vicinity of the found main parts v_0 and w_0 of solutions,

$$v = v_0 + av_1, \quad w = w_0 + aw_1,$$

leads to a pair of linear inhomogeneous equations for v_1 and w_1. Of the corrections v_1 and w_1, the correction w_1 is a leading one. For this correction, we obtain the equation

$$w_1'' + \left(\frac{6}{\cosh^2 X} - 4\right)w_1 = \frac{2c^2}{d}\left(\frac{d}{2}\right)^{1/2}\tanh X + \left(\frac{6bc}{a^2} + \frac{9b^2 d}{a^4}\right)\left(\frac{d}{2}\right)^{1/2}\frac{\sinh X}{\cosh^3 X} - \frac{63b^2 d}{4a^4}\left(\frac{d}{2}\right)^{1/2}\frac{\sinh X}{\cosh^5 X}. \quad (15)$$

The homogeneous equation corresponding to (15) has the general solution

$$w = c_1\frac{1}{\cosh^2 X} + c_2\left(\sinh 2X + 3\tanh X + \frac{3X}{\cosh^2 X}\right),$$

in which only the first term satisfies the requirement of localisation. Therefore, we obtain for w_1 the equation

$$w_1 = c_1\frac{1}{\cosh^2 X} - \frac{c^2}{2d}\left(\frac{d}{2}\right)^{1/2}\tanh X - \left(\frac{c^2}{2d} + \frac{3bc}{2a^2} + \frac{9b^2 d}{4a^4}\right)\left(\frac{d}{2}\right)^{1/2}\frac{X}{\cosh^2 X} + \frac{21b^2 d}{8a^4}\left(\frac{d}{2}\right)^{1/2}\frac{\sinh X}{\cosh^3 X}, \quad (16)$$

where c_1 is an arbitrary constant.

The function w_1 has the parts that are even and odd in X. The even part $c_1\cosh^{-2} X$ gives a small imaginary addition to the bright soliton, while the odd one determines the correction for the dark soliton (12). Expressions (12) and (13) show that in the case (3), the excitation level of the dark soliton is higher by a factor of $a^{3/2}/b$ than that for the bright soliton. More exactly, the ratio of the amplitudes of the bright and dark solitons for $X = 0$ is a function $g(\lambda)$ describing excitation of the impurity (bright) soliton against the background of the main (dark) soliton. By using expressions from Ref. [1] for the second- and third-order dispersions, we find that $g(\lambda) = 0.74(\lambda_0/\lambda - 1)^{-3/2}$. The dependence $g(\lambda)$ is shown in Fig. 1. The two solitons move in-phase with a finite phase velocity. The velocity of this motion for the amplitude phase is much higher and is of the order of a^{-1}. This follows from the expression

$$X = \left(\frac{d}{2a}\right)^{1/2}\left[t - \left(2ca + \frac{bd}{a}\right)x\right],$$

which is obtained from (9) and (14).

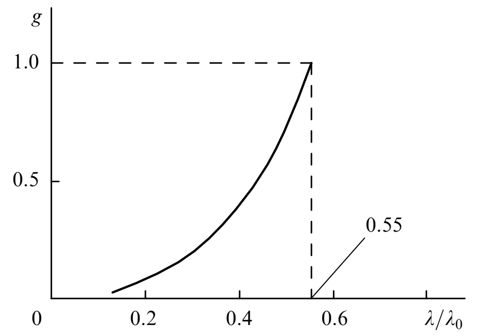

Figure 1. Excitation coefficient for a bright pulse against the dark-pulse background in the case of a weak third-order dispersion.

The solution obtained in the approximation used here contains arbitrary coefficients c and d in (8) and following expressions, as well as an arbitrary constant c_1 in (16).

3. Small-amplitude approximation

The analysis of system of equations (10), (11) under condition (4), i.e., in the vicinity of the zero-dispersion wavelength, shows that localised solutions of the type (12) and (13) can no longer be separated. Analysis shows that pulses under study have very low amplitudes in region (6).

Let us now use the substitution

$$u = [u_0 + U(x,t)]\exp\left[2iu_0^2 x + i\varphi(x,t)\right], \tag{17}$$

where $u_0 = \text{const}$. The form of (17) corresponds to the exact solution of the NSE. By substituting (17) into (2), we obtain equations

$$\frac{\partial U}{\partial x} - 2a\frac{\partial U}{\partial t}\frac{\partial \varphi}{\partial t} - a(u_0+U)\frac{\partial^2\varphi}{\partial t^2} + 3b\frac{\partial U}{\partial t}\left(\frac{\partial\varphi}{\partial t}\right)^2$$

$$-b\frac{\partial^3 U}{\partial t^3} + 3b(u_0+U)\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2} = 0,$$

$$(u_0+U)\frac{\partial\varphi}{\partial x} + a\frac{\partial^2 U}{\partial t^2} - a(u_0+U)\left(\frac{\partial\varphi}{\partial t}\right)^2 - 4u_0^2 U$$

$$-6u_0U^2 - 2U^3 - 3b\frac{\partial^2 U}{\partial t^2}\frac{\partial\varphi}{\partial t} + b(u_0+U)\left(\frac{\partial\varphi}{\partial t}\right)^3$$

$$-3b\frac{\partial U}{\partial t}\frac{\partial^2\varphi}{\partial t^2} - b(u_0+U)\frac{\partial^3\varphi}{\partial t^3} = 0.$$

Now we introduce new independent variables

$$\tau = \varepsilon(t - Cx), \quad \xi = \varepsilon^3 x, \quad C = \text{const} \tag{18}$$

containing a small amplitude parameter ε and will seek the amplitude and phase in the form of series

$$U = U_0(\tau,\xi)\varepsilon^2 + U_1(\tau,\xi)\varepsilon^4 + \ldots,$$

$$\varphi = \varphi_0(\tau,\xi)\varepsilon + \varphi_1(\tau,\xi)\varepsilon^3 + \ldots .$$

The functions U and φ are even and odd over ε, respectively, which is easily confirmed by substituting them into the system of equations. In the leading order over ε, we obtain

$$au_0\frac{\partial\varphi_0}{\partial\tau} + CU_0 = 0,$$

$$C\frac{\partial\varphi_0}{\partial\tau} + 4u_0U_0 = 0.$$

For this system to be consistent, we should set

$$C = 2u_0a^{1/2}. \tag{19}$$

By substituting (19) into the previous equation, we obtain

$$\frac{\partial\varphi_0}{\partial\tau} = -\frac{2}{a^{1/2}}U_0. \tag{20}$$

The equations in the next-order approximation have the form

$$a^{1/2}\frac{\partial\varphi_1}{\partial\tau} + 2U_1 = \frac{1}{2u_0}\left[\frac{\partial\varphi_0}{\partial\xi} - 6U_0^2 + \left(\frac{2b}{a^{1/2}} + \frac{a}{u_0}\right)\frac{\partial^2 U_0}{\partial\tau^2}\right], \tag{21}$$

$$a^{1/2}\frac{\partial^2\varphi_1}{\partial\tau^2} + 2\frac{\partial U_1}{\partial\tau} = \frac{1}{a^{1/2}u_0}\left[\frac{\partial U_0}{\partial\xi} + \frac{12u_0b}{a}U_0\frac{\partial U_0}{\partial\tau}\right.$$

$$\left. + 6a^{1/2}U_0\frac{\partial U_0}{\partial\tau} - b\frac{\partial^3 U_0}{\partial\tau^3}\right]. \tag{22}$$

By differentiating (21) with respect to τ and using (20), we obtain a new condition for the compatibility of Eqns (21) and (22). This condition is the Korteweg–de Vries equation

$$\frac{\partial U_0}{\partial\xi} + 6\left(\frac{u_0b}{a} + a^{1/2}\right)U_0\frac{\partial U_0}{\partial\tau} - \left(b + \frac{a^{3/2}}{4u_0}\right)\frac{\partial^3 U_0}{\partial\tau^3} = 0. \tag{23}$$

By writing the exact soliton solution of Eqn (23), returning to old variables (18) and taking (19) into account, we find

$$U_0 = -12\beta^2\cosh^{-2}Q,$$

where

$$Q = \varepsilon\beta B\left[t - 2u_0a^{1/2}x + \frac{24}{a}\left(u_0b + a^{3/2}\right)\beta^2\varepsilon^2 x\right]; \tag{24}$$

$$B = \left(\frac{24u_0}{a}\,\frac{u_0b + a^{3/2}}{4u_0b + a^{3/2}}\right)^{1/2}; \tag{25}$$

and β is an arbitrary parameter contained in the solution of the Korteweg–de Vries equation. The main part of the phase φ_0 is found from (20). The arbitrary coefficient β can be then included into the small parameter ε, which is also arbitrary so far. Finally, by expanding the phase factor $\exp(i\varphi_0\varepsilon)$ into a series, we obtain again the expression similar to (7):

$$u = \exp(2iu_0^2x)(v + iw), \tag{26}$$

$$v = u_0\left[1 - \frac{12\varepsilon^2}{u_0\cosh^2 Q} - \frac{288\varepsilon^2}{aB^2}\left(1 - \frac{1}{\cosh^2 Q}\right) + O(\varepsilon^3)\right], \tag{27}$$

$$w = \varepsilon u_0 \left[\frac{24}{a^{1/2} B} \tanh Q + O(\varepsilon^3) \right]. \quad (28)$$

Expression (26) is a sum of expressions (27) and (28) for bright and dark solitons. They differ from the corresponding expressions for the case (5) in that now a bright soliton is the main term, while a dark soliton is the correction to it, and bright soliton (27) is located on a pedestal of height u_0 (u_0 is arbitrary).

Expressions (26)–(28) are not related to the assumption about the smallness of coefficients in (2). They are valid under condition (4) down to the boundary $\lambda = 0.55\lambda_0$ of region (6).

Finally, expressions (24), (27), and (28) are simplified under condition (4):

$$Q = \varepsilon \left(\frac{6u_0}{a} \right)^{1/2} \left(t - 2u_0 a^{1/2} x + \frac{24 u_0 b}{a} \varepsilon^2 x \right),$$

$$v = u_0 - \frac{12\varepsilon^2}{\cosh^2 Q} - 48\varepsilon^2 \tanh^2 Q,$$

$$w = 4\varepsilon (6u_0)^{1/2} \tanh Q.$$

The dependences of the amplitudes of bright and dark solitons on the phase variable Q are demonstrated qualitatively in Fig. 2.

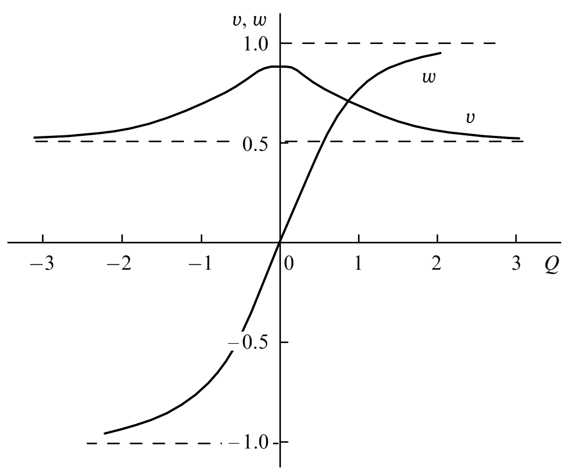

Figure 2. Bright and dark pulses in the case of a dominant third-order dispersion. Calculations were performed for $u_0 = 1$ and $\varepsilon = 0.1$.

We can find the relations between the parameter ε and coefficients in Eqn (2). Thus, by calculating the right-hand sides in Eqns (21) and (22) in the case (4) and estimating the correction U_1, we obtain

$$\varepsilon^2 \leqslant 0.1 \frac{a^{3/2}}{b}. \quad (29)$$

Inequality (29) refines the conditions under which the propagation of an embedded soliton on a fixed pedestal can be observed.

4. Conclusions

We have described analytically the influence of the third-order dispersion on the propagation of a short pulse in an optical fibre. The dynamics of the pulse envelope is described by the generalised NSE (1), and the effect under study is manifested in the change in the soliton solution of the standard NSE due to the term containing the third derivative. The estimates of the coefficient at the third derivative based on results obtained in Ref. [1] show that the influence of the third-order dispersion should not be neglected already at wavelengths above 0.67 μm, and the third-order dispersion increases upon approaching the wavelength $\lambda_0 = 1.27$ μm.

The formation of the impurity structure substantially depends on the relation between the second- and third-order dispersions. In the case (3), when the wavelength still substantially differs from the zero-dispersion wavelength (5) and the third-order dispersion can be treated as a perturbing effect, a bright pulse with the amplitude of the order of $b/a^{3/2}$ appears against the background of a dark soliton of the standard NSE, as well as additional dark and bright pulses with lower amplitudes. The distortions of a standard pulse are described explicitly by expressions (13) and (16).

The nonlinear dynamics of the short-pulse envelope near the zero-dispersion wavelength in the case (4) is qualitatively different. A nonlinear structure appears in the form of an embedded soliton due to the dominant influence of the third-order dispersion [14]. A distinct feature of this structure is that a bright soliton propagates on a pedestal of a finite height and obeyes the Korteweg–de Vries equation. The bright soliton is described by expressions (26)–(28); in particular, it has much steeper fronts compared to those for the NSE soliton. The pulse evolution in this case also occurs due to the propagation of coupled dark and bright solitons.

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