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# Use of the fractional Fourier transform in $\pi/2$ converters of laser modes

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***Abstract*.** **The possibility of using the fractional Fourier transform (FrFT) in optical schemes for astigmatic $\pi/2$ converters of Hermite – Gaussian modes to donut Laguerre – Gaussian modes is considered. Several schemes of converters based on the FrFT of the half-integer and irrational orders are presented. The lowest FrFT order than can be used in astigmatic mode converters is found. The properties of converters based on the fractional and ordinary Fourier transforms are compared.**

***Keywords***: *fractional Fourier transform, astigmatic $\pi/2$ converter, Hermite – Gaussian and Laguerre – Gaussian modes.*

## 1. Introduction

Both topics combined in the title of this paper, $\pi/2$ converters of laser modes and the fractional Fourier transform (FrFT) are being actively studied in the last decade. $\pi/2$ converters attract special attention because of the unique possibilities of using donut Laguerre – Gaussian (LG) modes to manipulate microobjects [1, 2], to capture individual atoms and control their motion [3 – 5], and to accelerate electrons [6]. The FrFT is of great interest because it opens up new outlook for the representation and analysis of signals of various types [7 – 9]. The possibility of realisation of the FrFT by optical methods [10], some specific properties of Hermite – Gaussian (HG) modes [11], and a new interpretation of the operation mechanism of $\pi/2$ converters [12, 13] are the means that allow the use of FrFT for the conversion of HG modes to LG modes. The scheme of the $\pi/2$ converter based on the FrFT was first proposed in our paper [14]. In this paper, a systematic analysis of the principles for constructing such devices is given.

## 2. Fractional Fourier transform

To explain the terminology and accepted notation without reference to numerous more detailed papers, we recall the basic concepts of the FrFT and present three basic schemes developed for the FrFT performed with the help of ordinary lenses.

The classical ordinary Fourier transform (FT) is defined as*

$$F(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(x) \exp(-\mathrm{i}x\xi) \mathrm{d}x. \tag{1}$$

The integral in (1) is the transformation of the function $f$ of time $\tau$ or a spatial coordinate $x$ to the function $F(\xi)$ of spectral variables, the angular frequency $\omega$ or the spatial frequency $\xi$, which can be symbolically written in the form $F(\xi) = \mathscr{F}[f(x)]$. One can show that successive transformations (1) give the expressions

$$\begin{aligned} &\mathscr{F}[\mathscr{F}[f(x)]] = f(-x), \\ &\mathscr{F}[\mathscr{F}[\mathscr{F}[f(x)]]] = \mathscr{F}[f(-x)] = F(-\xi), \\ &\mathscr{F}[\mathscr{F}[\mathscr{F}[\mathscr{F}[f(x)]]]] = f(x). \end{aligned} \tag{2}$$

In other words, the cyclicity of the ordinary FT is equal to four. In the notation used below, it is convenient to write relations (1) and (2) as the sequence

$$\begin{aligned} &\mathscr{F}^1[f(x)] = F(\xi),\ \mathscr{F}^2[f(x)] = f(-x), \\ &\mathscr{F}^3[f(x)] = F(-\xi),\ \mathscr{F}^4[f(x)] = f(x), \end{aligned} \tag{3}$$

i.e., the number of FT cycles in (3) can be treated as an order (power) of the ordinary FT.

Similar to the representation of higher-order FTs in (3) as a sequence of powers of $\mathscr{F}^1$, we can also factorise formally the integral in (1), i.e., to decompose $\mathscr{F}^1$ into two (or more) transformations of orders lower than unity. For example,

$$F(\xi) = \mathscr{F}^1[f(x)] = \mathscr{F}^a[\mathscr{F}^b[f(x)]], \tag{4}$$

where $\mathscr{F}^a$ and $\mathscr{F}^b$ are called fractional FTs, and $a$ and $b$ are the orders of the FrFT. The integral representation of the FrFT has the form

* Hereafter, we omit mathematical details concerning limitations imposed on the function $f(x)$.

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$$\mathscr{F}^a[f(x)] = \frac{\exp(\mathrm{i}\psi/2)}{(\mathrm{i}\sin\psi)^{1/2}}$$

$$\times \int_{-\infty}^{+\infty} f(x)\exp\left[\mathrm{i}\pi\,\frac{(x^2+\xi^2)\cos\psi - 2\xi x}{\sin\psi}\right]\mathrm{d}x, \quad (5)$$

where $\psi = a\pi/2$. The ordinary FT (1) is obtained from (5) by substituting $a = 1$. The first of the relations in (2) is obtained by substituting $a = \pm 2$.

There exist several different definitions of the FrFT [15], including the definition based on the analogy with the description of propagation of an optical beam along a graded-index fibre [16]. As for the ordinary FT, there exist several rigorously proven properties for the FrFT as well [17]. Below, we will use two of them, additivity and commutativity:

$$\mathscr{F}^a\mathscr{F}^b = \mathscr{F}^{a+b}, \quad (6)$$

$$\mathscr{F}^a\mathscr{F}^b = \mathscr{F}^b\mathscr{F}^a. \quad (7)$$

Another important property of the FrFT in our case is that, when the order $a$ is an irrational number, the HG function is the only function that can coincide with the eigen FrFT (can be the eigenfuction) [18]:

$$\mathscr{F}^a\left[\exp\left(-\pi x^2\right)H_n\left(x\sqrt{2\pi}\right)\right]$$

$$= \exp\left(-\mathrm{i}\,\frac{an\pi}{2}\right)\exp\left(-\pi\xi^2\right)H_n\left(\xi\sqrt{2\pi}\right), \quad (8)$$

where $H_n$ is the Hermitian polynomial of the $n$th order, and the complex exponential is the eigenvalue of the FrFT. Therefore, in the case of light fields for real $a$, the distributions of the intensity of the initial HG beam and its FrFT coincide.

The formal analogy between ordinary FT and FrFT can be extended to a plane specified by the Cartesian coordinates $x$ and $\xi$, where a passage from the point $(x, \xi)$ to $(x', \xi')$ is performed by the rotation operator $\mathscr{R}_\psi$:

$$\begin{pmatrix} x' \\ \xi' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \mathscr{R}_\psi\begin{pmatrix} x \\ \xi \end{pmatrix}. \quad (9)$$

The sequence (3) on the $x\xi$ plane for integer orders $a$ corresponds to rotation through angles multiple of $\pi/2$. For irrational orders $a$, rotation is performed through an arbitrary angle $\psi = a\pi/2$ (complex values of $a$ are also possible [19, 20]).

Two simple optical lens systems I and II [10] for performing FrFT are shown in Fig. 1. Their $ABCD$ matrices are

$$T_{\mathrm{FrFT}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\psi & f\sin\psi \\ -f^{-1}\sin\psi & \cos\psi \end{pmatrix}, \quad (10)$$

where $f$ is the FrFT scale expressed in length units. Note that the complex argument in relation (8) is simply the accumulated Gouy phase (AGP) $\theta$ of the beam propagated in any of the schemes I and II. The phase $\theta$ can be fiund from the relation [21]

$$\tan\theta = \frac{\lambda B}{(A + B\rho)\pi w^2} = \frac{\lambda f\sin\psi}{(\cos\psi + \rho f\sin\psi)\pi w^2}, \quad (11)$$

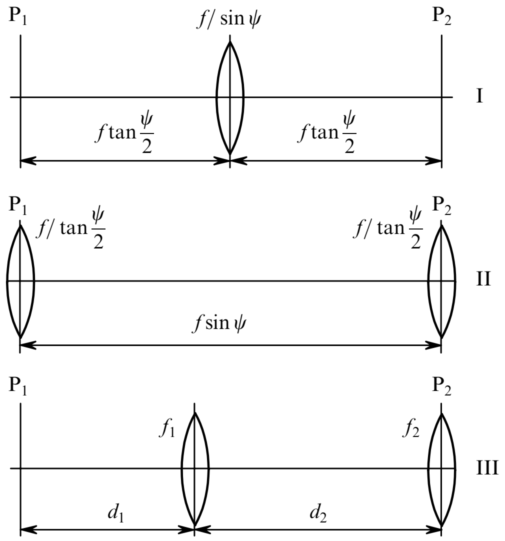


**Figure 1.** Optical schemes of elementary fractional Fourier converters realised by means of lenses.

where $\rho$ and $w$ are the curvature of the beam wave front and its characteristic size, and $\lambda$ is the wavelength. It is clear that the HG mode will be the eigenbeam of fractional FT converters shown in Fig. 1 only when $\rho \equiv 0$ and the Rayleigh length of the beam coincides with $f$.

The asymmetric optical scheme III shown in Fig. 1 can be also used to perform FrFT of the specified order $a \in (0, 1]$ $(0 < \psi \leqslant \pi/2)$. To determine the optical parameters of the elements of the scheme, it is necessary to find conditions at which $\theta$, in the case of propagation of the beam from the input reference plane to the output plane (or in the opposite direction) is equal to $\psi$. Because the Gouy phase is not accumulated after propagation of radiation through a lens (for the $ABCD$ matrix of a lens $B \equiv 0$), the second lens can be temporarily excluded from consideration. As a result, the propagation of the beam from the plane $P_1$ to the plane $P_2$ can be described by the matrix

$$T'_{12} = \begin{pmatrix} 1 - d_2/f_1 & d_1 + d_2 - d_1 d_2/f_1 \\ -1/f_1 & 1 - d_1/f_1 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \quad (12)$$

where $f_1$ is the focal distance (see Fig. 1). The required value of the AGP will be obtained if

$$A' = \cos\psi, \quad B' = f\sin\psi. \quad (13)$$

It follows from (13) that to provide the inequality $d_2 \geqslant 0$ (propagation of the beam in the positive direction along the $z$ axis), $f_1$ should be positive, and $d_1$ and $d_2$ should satisfy the form

$$d_1 = f\tan\psi - \frac{f_1(1 - \cos\psi)}{\cos\psi}, \quad d_2 = f_1(1 - \cos\psi). \quad (14)$$

The condition $d_1 \geqslant 0$ (the absence of an imaginary image in the $P_1$ plane) gives the following limits for $f_1$

$$0 < f_1 \leqslant \frac{f\sin\psi}{1 - \cos\psi} = \frac{f}{\tan(\psi/2)}. \quad (15)$$

The simultaneous fulfilment of inequality (15) and (14) is sufficient for obtaining the equality $\theta = \psi$ during the propagation of the beam from the plane $P_1$ to $P_2$. To obtain the same value of $\theta$ for the propagation of the beam in the opposite direction, it is necessary to add a second lens to the system and set $C = -f^{-1}\sin\psi$ for the complete matrix of the scheme, which leads to the condition

$$f_2 = \frac{f_1 \cos\psi}{f_1 \sin\psi - f}, \tag{16}$$

where $f_2$ is the focal distance. Assuming that $f_1 = f/\sin\psi$ or $f_1 = f/\tan(\psi/2)$ and calculating $d_1$, $d_2$, and $f_2$, we can easily show that symmetric schemes I and II are simply particular cases of the scheme III.

Because HG laser modes are functions of separable independent variables, all the above said can be easily generalised to two transverse spatial coordinates. Moreover, the orders of the FT and FrFT along the $x$ and $y$ axes can be specified independently [22]. As a result, we have three main 'construction' blocks (Fig. 1), which can be used to design different schemes for $\pi/2$ mode converters.

## 3. Schemes of FrFT $\pi/2$ mode converters

It was shown in Refs [12, 13] that a $\pi/2$ mode converter proposed earlier in Ref. [23] is based on the optical scheme in which two ($a = 2$) and three ($a + 1$) ordinary FTs are performed in planes $xz$ and $yz$, respectively, for a beam propagating along the optical axis $z$ of the converter. The principle of operation of this $\pi/2$ converter in the language of FT can be described as follows.

Any HG eigenmode $u_{nm}^{\text{HG}}$ of the converter rotated at the converter input by 45° with respect to the astigmatism axes ('diagonal' mode) can be expanded into a series in 'normal' (not turned) HG modes $u_{n'm'}^{\text{HG}}(x, y)$ of the same order $N = n + m = n' + m'$:

$$u_{nm}^{\text{HG}}(x, y, 45°) = \sum_{n'm'} a_{n'm'}(45°)\, u_{n'm'}^{\text{HG}}(x, y), \tag{17}$$

where $a_{n'm'}(45°)$ are some real coefficients depending on the rotation angle. Each normal mode accumulates the corresponding Gouy phase after propagation in the converter according to (8). Taking into account that the variables $x$ and $y$ are independent, we can write for the scheme of Ref. [23]

$$\mathscr{F}_x^{\,a}\left[\mathscr{F}_y^{\,a+1}\left[u_{n'm'}^{\text{HG}}(x, y)\right]\right] = \exp\left(-\,\mathrm{i}\,\frac{Na\pi}{2} - \mathrm{i}\,\frac{m'\pi}{2}\right) u_{n'm'}^{\text{HG}}(x, y), \tag{18}$$

which gives at the output of the $\pi/2$ converter the expression

$$\exp\left(-\,\mathrm{i}\,\frac{Na\pi}{2}\right) \sum_{n'm'} (-\mathrm{i})^{m'} a_{n'm'}(45°)\, u_{n'm'}^{\text{HG}}(x, y) = \exp\left(-\,\mathrm{i}\,\frac{Na\pi}{2}\right) u_{pl}^{\text{LG}}(x, y) \tag{19}$$

for the LG mode $u_{pl}^{\text{LG}}$ of the order $N = p + l$.

This result is obtained because the numbers of FTs of the input field in planes $xz$ and $yz$ differ exactly by unity. Note that the order $a$ can be arbitrary, in particular, fractional because it determines only a constant phase shift $Na\pi/2$, which is the same for all expansion terms. Because of this any two elementary schemes (Fig. 1) having the same optical length and composed of cylindrical lenses, provided the FrFT orders in orthogonal planes are equal to $a$ and $a + 1$, are $\pi/2$ converters. Of course, it is convenient in this case to use the same scale along the axes. These are basic and most general recipes for constructing $\pi/2$ converters. It is clear that, by using the additivity and commutativity of the FrFT [see (6) and (7)], the final aim can be achieved differently. To demonstrate the simplicity and flexibility of the procedure for constructing FrFT converters, consider several examples.

*Example 1.* Consider first schemes I and II in Fig. 1 as basic schemes. Let us assume that the power of lenses in scheme I is zero in the $yz$ plane (the generatrices of cylindrical lenses are parallel to the $y$ axis) and $\psi_x = a\pi/2$. Then, we have $\psi_y = (a + 1)\pi/2$ for lenses in scheme II with generatrices parallel to the $x$ axis. Being combined, schemes I and II should have the same optical length $L$. This means that the equality

$$2f\tan\frac{\psi_x}{2} = f\sin\psi_y,$$

should be fulfilled, which , using the relation between $\psi_x$ and $\psi_y$, gives

$$2f\tan\frac{\psi_x}{2} = f\cos\psi_x.$$

The solution of this trigonometric equation is reduced to the solution of a cubic algebraic equation, but because the final result is somewhat cumbersome, we present here only rounded values $a = 0.458013$, $\psi_x = 0.719445$, and $L = 0.752172f$. The scheme corresponding to this parameter is shown in Fig. 2a.

It is clear from this example the main task in the construction of converter schemes is matching of the optical lengths simultaneously with the limitation of the order of the number of FrFTs performed. Therefore, we will consider next examples with minimal comments.

*Example 2.* The combination of scheme I with itself does not result in the construction of a converter, but this is possible for scheme II. In the latter case, the scheme is described by the equation

$$f\sin\psi_x = f\sin\psi_y = f\sin(\psi_x + \pi/2),$$

whose solution yields a very simple result: $a = 1/2$, $\psi_x = \pi/4 = 0.785398$, $L = 0.707107f$.

*Example 3.* Let us now combine two identical schemes I placed in succession in the $xz$ plane and one scheme II in the $yz$ plane (Fig. 2c). This can be done because of the FrFT additivity (6). Then, the FrFT order for each elementary scheme I is $a' = a/2$, and its is described by the equation

$$4\tan\frac{\psi_x}{4} = \sin\left(\psi_x + \frac{\pi}{2}\right).$$

The solution to this equation has the form: $a = 0.467335$, $\psi_x = 0.734088$, $L = 0.742442f$.

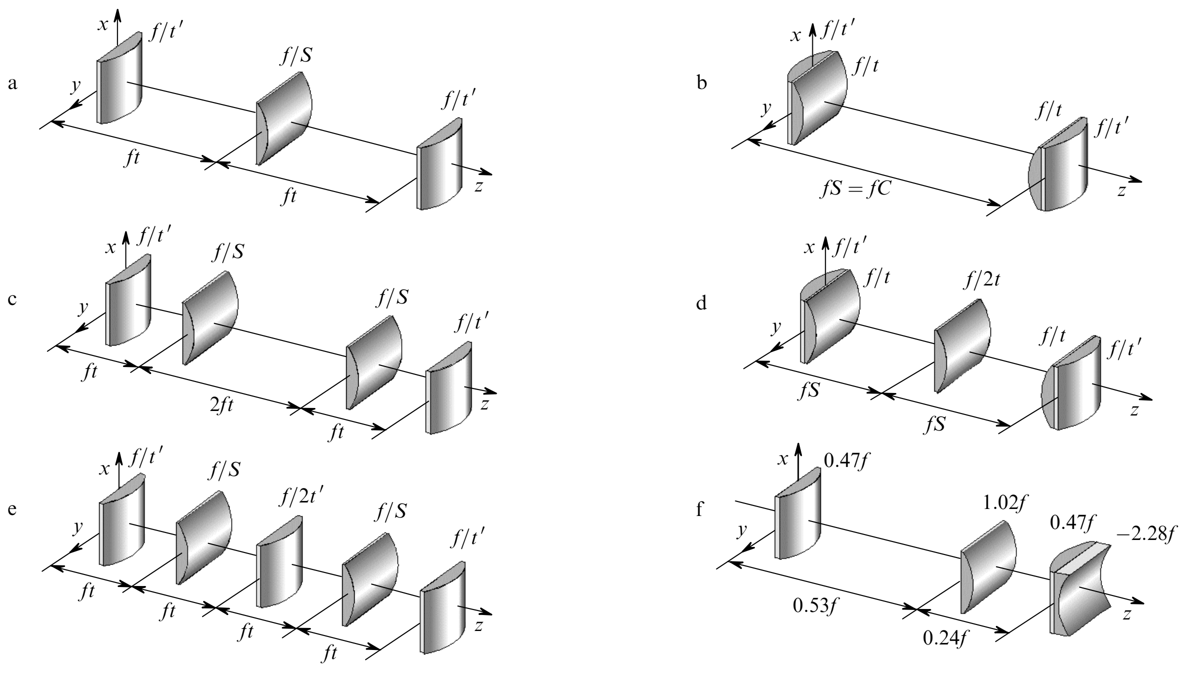


**Figure 2.** Variants of the use of the FrFT to construct $\pi/2$ converters of HG modes to LG modes. The parameters of optical schemes: $S = \sin\psi_x$, $t = \tan(\psi_x/2)$, $t' = \tan(\psi_x/2 + \pi/4)$ (a); $S = \sin(\pi/4)$, $t = \tan(\pi/8)$, $t' = \tan(3\pi/8)$ (b); $S = \sin(\psi_x/2)$, $t = \tan(\psi_x/4)$, $t' = \tan(\psi_x/2 + \pi/4)$ (c); $S = \sin(\psi_x/2)$, $t = \tan(\psi_x/4)$, $t' = \tan(\psi_x/2 + \pi/4)$ (d); $S = \sin(\psi_x/2)$, $t = \tan(\psi_x/4)$, $t' = \tan(\psi_x/4 + \pi/8)$ (e).

It is obvious that instead of identical schemes I, schemes with different orders $a'$ and $a''$, so that $a' + a'' = a$, could be used. In this case, we should solve the equation

$$2\left(\tan\frac{\psi_x'}{2} + \tan\frac{\psi_x''}{2}\right) = \sin\left(\psi_x + \frac{\pi}{2}\right),$$

where one of the angles in the left-hand side can be chosen arbitrarily within some limits (both tangents should be positive). It is also clear that, because of the FrFT commutativity (7), schemes corresponding to FrFTs with orders $a'$ and $a''$ can be placed one after another on the $z$ axis in any succession.

*Example 4*. In example 2, we used two schemes II with the orthogonal orientation of lenses. Consider now two schemes II placed in succession in one plane and one scheme II placed in another plane (Fig. 2d). The matching condition for the optical lengths gives the equation

$$2\sin\frac{\psi_x}{2} = \sin\left(\psi_x + \frac{\pi}{2}\right),$$

whose solution has the form: $a = 0.477127$, $\psi_x = 0.749469$, $L = 0.888318f$.

For all other combinations of schemes #I and #II, including those in which FTs with integer orders are used, either a greater number of lenses or a greater optical length are required. One of such schemes is considered in the next example.

*Example 5*. The combination of two schemes I with two schemed II (Fig. 2e) is described by the equation

$$4\tan\frac{\psi_x}{4} = 2\sin\left(\frac{\psi_x}{2} + \frac{\pi}{4}\right)$$

which is satisfied for $a = 1.171433$, $\psi_x = 1.840083$, and $L = 1.9819f$. The values of these parameters are approximately twice that for any preceding examples.

All the schemes presented above are only a particular case of combinations that can be obtained by using elementary scheme III. These schemes use no more than three cylindrical lenses with different focal distances. Lenses with the focal distance $f/(2t')$ employed in schemes in Fig. 2d, e can be replaced by pairs of lenses with the focal distance $f/t'$. It is obvious that schemes considered in examples 1 and 2 are more convenient from the practical point of view because they have a smaller size and use a smaller number of lenses. Nevertheless, it is useful to consider a more general case of the construction of a $\pi/2$ converter based on scheme III. For this purpose, it is necessary to solve the system of equations

$$\begin{aligned}
d_{1x} &= f\tan\psi_x - f_{1x}\frac{1-\cos\psi_x}{\cos\psi_x},\\
d_{2x} &= f_{1x}(1-\cos\psi_x),\\
f_{2x} &= \frac{f_{1x}\cos\psi_x}{f_{1x}\sin\psi_x - f},\\
d_{1y} &= f\tan\psi_y - f_{1y}\frac{1-\cos\psi_y}{\cos\psi_y},\\
d_{2y} &= f_{1y}(1-\cos\psi_y),\\
f_{2y} &= \frac{f_{1y}\cos\psi_y}{f_{1y}\sin\psi_y - f},\\
\psi_y &= \psi_x + \pi/2,\\
d_{1x} + d_{2x} &= d_{1y} + d_{2y},
\end{aligned} \tag{20}$$

under the condition that

$$0 < f_{1x} \leqslant \frac{f \sin\psi_x}{1-\cos\psi_x}, \ \ 0 < f_{1y} \leqslant \frac{f \sin\psi_y}{1-\cos\psi_y}.$$

By presetting the value of $\psi_x$, we have eight equations with nine unknowns. By using $f_{1x}$ as a free parameter, we obtain that the system of equations will be compatible if $f_{1y}$ is found from the condition

$$f_{1y} = \frac{f - f_{1x}\sin\psi_x(1-\cos\psi_x)^2}{\cos\psi_x(1+\sin\psi_x)^2}. \tag{21}$$

In this case, the values of $f_{1x}$ should satisfy the inequalities

$$0 < f_{1x} \leqslant f \min\left\{\frac{\sin\psi_x}{1-\cos\psi_x};\ \frac{\sin\psi_x - \cos\psi_x^2}{(1-\cos\psi_x)^2}\right\}. \tag{22}$$

The boundaries for $f_{1x}$ are shown at the left in Fig. 3. The corresponding boundaries for $f_{1y}$, which are obtained by substituting (22) into (21), are shown at the right in Fig. 3. By using (20)–(22), consider one more example of the π/2 converter scheme.

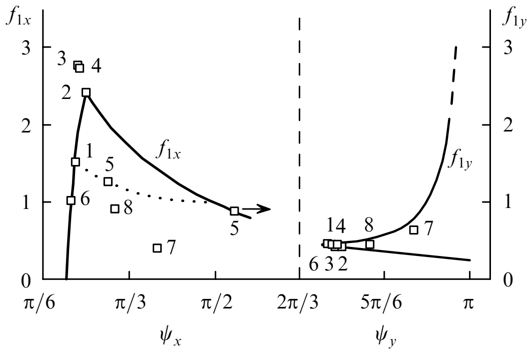


**Figure 3.** Admissible regions of values of the parameters of FrFT π/2 converters. Squares are the values of $f_{1x}(\psi_x)$ and $f_{1y}(\psi_y)$ for schemes in Fig. 2; the numbers 1–6 of squares correspond to the numbers of examples in the text; the numbers 7 and 8 correspond to some rather arbitrary converters composed of schemes III.

*Example 6*. Consider the combination of schemes III and II. The latter allows the minimisation of the number of different lenses. In this case, scheme II is scheme III for which the condition $d_{1y} = 0$ is additionally satisfied or, which is the same, $f_{1y} = f_{2y}$. The combined scheme and its parameters are presented in Fig. 2f.

The parameters of π/2 converters considered in examples are summarised in Table 1. The converter considered in the second example has the smallest length $L = 0.70710f$, which is approximately shorter by a factor of 5.7 than the converter length in [23].

The limitation imposed by inequality (22) shows that it is impossible to construct a FrFT π/2 converter with the order $a < 0.424121$, i.e., $\psi_x \geqslant 0.666239$. In this case, the upper limit in (22) corresponds to the condition $f_{1x} = f_{2x}$ (scheme II). The dashed curve in Fig. 3 determines the limitation imposed on the use of scheme I in the converter: scheme I can be used only if $\pi/2 \geqslant \psi_x \geqslant 0.719445$. The lower limit for I in this inequality exactly corresponds to the scheme of example I. The values of $f_{1x}(\psi_x)$ and $f_{1y}(\psi_y)$ for this and other schemes considered above are shown by squares in Fig. 3, the number of squares corresponding to the numbers of examples. Note that points $f_{1x}(\psi_x)$ for examples 3 and 4 lie beyond the limits specified by relation (22) because the corresponding schemes are composed of pairs of elementary schemes I and II, whereas inequality (22) was obtained for a simple combination of schemes I, II, and III in pairs. For the same reason, the abscissas of points for the scheme in example 5 are separated only by π/4 because this scheme is composed of a sequence of two identical converters, which can be called π/4 converters.

**Table 1.**

| Example number | Basic schemes* | Lowest FrFT order | Total number of lenses | Number of different lenses | $L/f$ |
|---|---|---|---|---|---|
| 1 | I + II | 0.458013 | 3 | 2 | 0.752172 |
| 2 | II + II | 0.5 | 4 | 2 | 0.707107 |
| 3 | 2I + II | 0.467335 | 4 | 2 | 0.742442 |
| 4 | 2II + II | 0.477127 | 5 | 3 or 2** | 0.888318 |
| 5 | 2I + 2II | 1.171433 | 5 | 3 or 2** | 1.981899 |
| 6 | 2III + II | 0.444444 | 4 | 3 | 0.766044 |

* Scheme notation as in Fig. 1.
** The number of different schemes can be reduced by combining two identical lenses.

## 4. Some properties of FrFT π/2 converters

Methods of matrix *ABCD* optics allow us to analyse the operation of a π/2 converter for any deviation of an input beam from the eigenbeam of the system and to compare converters based on the ordinary and fractional FTs. Taking into account the form of matrices (10) and the condition $\psi_y = \psi_x + \pi/2$ for the input beam, for which the wavefront radius and curvature are specified as $w_{x\,\rm in} = w_0 M_x$, $w_{y\,\rm in} = w_0 M_y$ ($w_0 = (\lambda f/\pi)^{1/2}$ is the size of the eigenbeam of the converter) and $\rho_{x\,\rm in} = \rho_{y\,\rm in} = \rho$, respectively, we have at the converter output

$$w_x = w_0\,\frac{\left[M_x^4(f\rho S + C)^2 + S^2\right]^{1/2}}{M_x},$$

$$w_y = w_0\,\frac{\left[M_y^4(f\rho C - S)^2 + C^2\right]^{1/2}}{M_y},$$

$$\rho_x = \frac{1}{f}\,\frac{M_x^4(f\rho S + C)(f\rho C - S) + CS}{M_x^4(f\rho S + C)^2 + S^2}, \tag{23}$$

$$\rho_y = \frac{1}{f}\,\frac{M_y^4(S - f\rho C)(f\rho S + C) - CS}{M_y^4(f\rho C - S)^2 + C^2},$$

$$\tan\theta_x = \frac{S}{(f\rho S + C)M_x^2}, \quad \tan\theta_y = \frac{C}{(f\rho C - S)M_y^2},$$

where $C = \cos\psi_x$ and $S = \sin\psi_x$. The main condition for the conversion of the improper HG mode to the LG mode is the inequality $\theta_x - \theta_y = \pm\pi/2$, which, taking into account (23), yields the equation

$$(\rho_{\rm in} f)^2 + \frac{2\rho_{\rm in} f}{\tan 2\psi_x} - \frac{M_x^2 M_y^2 - 1}{M_x^2 M_y^2} = 0, \tag{24}$$

Equation (24) has real roots for $M_x M_y \geqslant \sin 2\psi_x$. From equations (23) and (24), the following conclusions can be made:

(i) In a particular case of $M_x M_y = 1$ and $\rho = 0$, all converters based on the ordinary or fractional FT convert the HG mode to the LG mode similarly because the difference of AGPs is always $\Delta\theta = \pi/2$ and $w_{x\,\mathrm{out}} = w_{y\,\mathrm{out}}$, $\rho_{x\,\mathrm{out}} = \rho_{y\,\mathrm{out}}$. The only difference is in the increase (or decrease) of the input beam size and the absolute value of the wavefront curvature at the converter output. In this case, the values of $M_x$ and $M_y = 1/M_x$ can be selected for any $\psi_x$ (apart from obvious $M_{x,y} = 1$) so that $w_{x\,\mathrm{out}} = w_{y\,\mathrm{out}} = w_0$. The case of a converter based on the ordinary FT, when $\psi_x = \pi/2$, is also not excluded, although $M_x \to \infty$ and $M_y \to 0$ (or vice versa) when $\psi_x$ approaches $\pi/2$. The input (the HG mode with $n = 0$, $m = 3$) and output (the LG mode with $p = 0$, $l = 3$) distributions of radiation in the case of conversion of the eigenmode and deformed mode ($M_x = 1.5$, $M_y = 1/M_x$) by the FrFT converter (see the scheme in Fig. 2b) are shown in Fig. 4. In the case of the eigenmode (Fig. 4a), the size and the wave front of the input and output beams coincide. For the deformed mode (Fig. 4b), the size of the output beam is $w_{x,y} = 1.16w_0$ and the wavefront curvature is $\rho_{x,y} = -0.67/f$ (it is represented in a grey scale with a linear variation from 0 to $2\pi$).

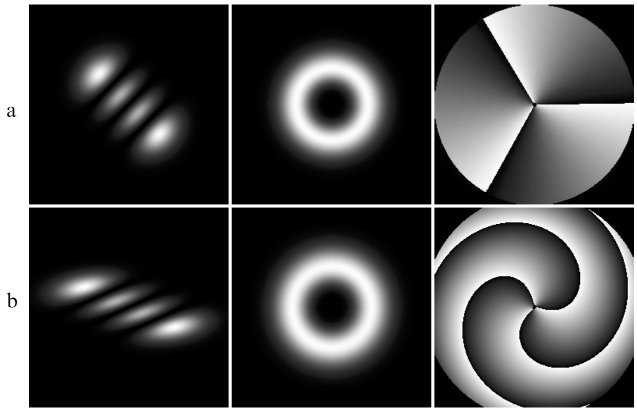


**Figure 4.** Conversion of the eigenmode (a) and the deformed (extension $M_x = 1.5$, compression $M_y = 1/1.5$) (b) $u_{03}^{\mathrm{HG}}$ mode to the $u_{03}^{\mathrm{LG}}$ mode by the FrFT $\pi/2$ converter of the order $a = 1/2$. The first column shows the beam at the converter input; the second and third columns are the radiation distribution and the wave front, respectively, at the converter output.

(ii) If $M_x = M_y = M \neq 1$, then for any $\psi_x$ and $M > (2SC)^{1/2}$, we can find from (24) two values of the input-beam curvature $\rho_{\mathrm{in1,in2}}$ at which $\Delta\theta = \pi/2$. In this case, $\rho_x = \rho_y = 0$, $w_{x,y} \neq w_0$, and $w_x w_y = 1$. In the case of a converter with the half-integer FrFT order ($a = 1/2$, example 2 from section 3), we have $\rho_{\mathrm{in1}} = -\rho_{\mathrm{in2}}$. For FrFT orders different from half-integers, the required values of the wavefront curvature $\rho_{\mathrm{in1,in2}}$ differ in absolute values (Fig. 5). For converters based on the ordinary FT ($a = 2$ [23] or $a = 1$ [12]), the conversion of the beam to the LG mode with $\rho_x = \rho_y = 0$ is possible only when $\rho_{\mathrm{in}} = 0$ or in the case of a saddle-like wave front at the converter input [13]. In the latter case, as for the FrFT converter, the HG mode is converted to the elliptical LG mode.

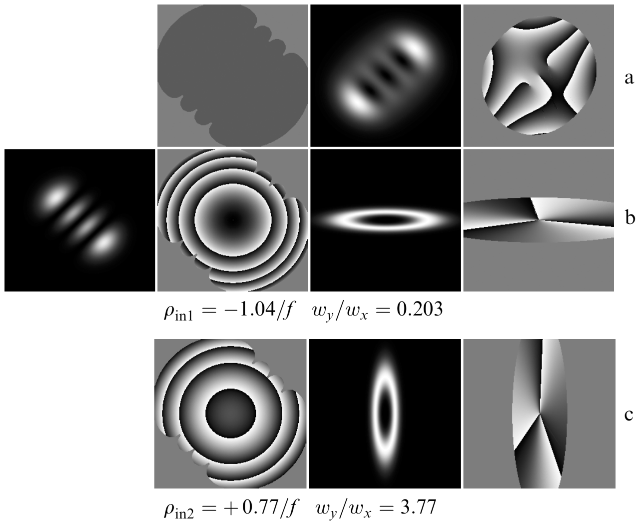


**Figure 5.** Conversion of the improper beam (the $u_{03}^{\mathrm{HG}}$ mode at $M_{x,y} = M = 1.5$) by the FrFT converter of the order $a = 0.458013$ without the wavefront correction (a) and with its correction by adding the negative (b) and positive (c) curvature. The first and third columns are the beam intensity distributions at the converter input and output, respectively; the second and fourth columns are their corresponding wave fronts.

A comparison of converters based on the ordinary and fractional FTs shows that the latter are more sensitive to the deviations of the input-beam parameters from the eigenbeam parameters. Thus, in the case of a random simultaneous variation in the parameters $w_{x,y}$ and $\rho$ of the input beam, the results of calculations presented in Fig. 6 show that the scatter in the size, the wavefront curvature of the beam and differences of AGPs at the output of converters for the FT order $a = 1$ (Fig. 6a) are smaller than for $a = 1/2$ (Fig. 6b).

## 5. Conclusions

We have considered the possibility of using the FrFT in optical schemes for astigmatic $\pi/2$ converters of HG modes to donut LG modes and presented some converters based on the FrFTs of the half-integer and irrational orders. We have found that there exists the FrFT of the lowest order ($a = 0.424121$), which can be used to obtain the AGP difference in the orthogonal planes $xz$ and $yz$ equal to $\pi/2$. Our calculations show that $\pi/2$ converters in which the fractional or ordinary FTs are used has as whole similar properties with respect to the input-beam perturbations (deviations from the eigenmode parameters). However, FrFT converters prove to be more sensitive to simultaneous random variations in the size and wavefront curvature of the input beam. Although the difference in the sensitivity seems not very large, it can be useful in some cases. In any case, the use of the FrFT of the half-integer order ($a = 1/2$) allows the construction of the astigmatic $\pi/2$ converter with the optical length shorter than 1/5 of the length of the converter in Ref. [23].

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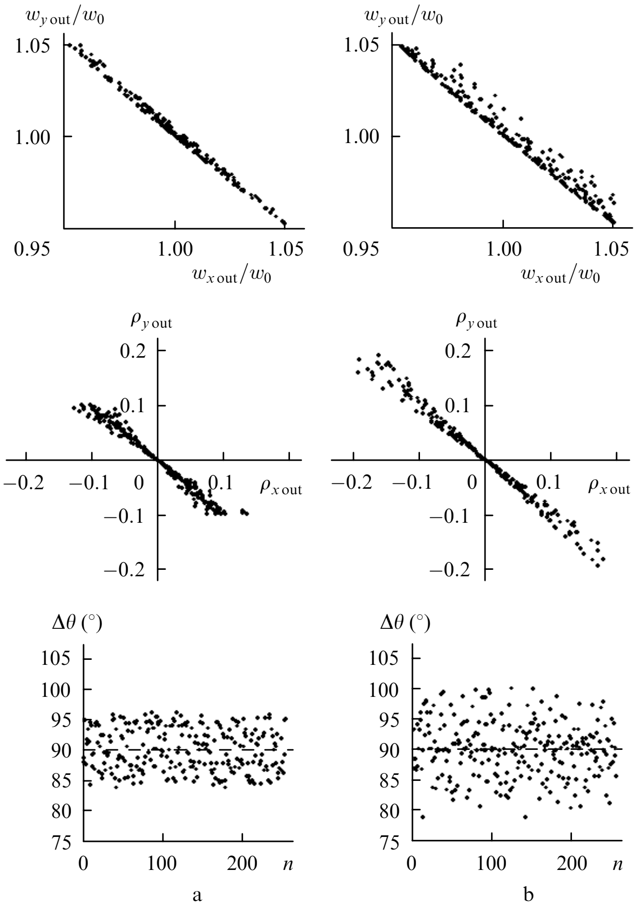


**Figure 6.** Calculated scatter in the size and wavefront curvature of the beam and the AGP difference at the outputs of converters based on the ordinary (a) and fractional FT of the order $a = 1/2$ (b). The size and the wavefront curvature of the input beam lie within the limits $M_{x,y} \in [0.9, 1.1]$ and $\rho_{x\,\mathrm{in},\,y\,\mathrm{in}} \in [-0.1, +0.1]$. The number of simultaneous random variations is 256; $n$ is the variation number.

## References


1. O'Neil A.T., Courtial J. *Opt. Commun.*, **193**, 45 (2001).
2. MacDonald M.P., Paterson L., Armstrong G., Arlt J., Bryant P.E., Sibbett W., Dholakia K. *Tech. Dig. Intern. Conf. on Advanced Laser Technologies (ALT-02)* (Adelboden, Switzerland, 2002) p. 26.
3. Kuga T., Torii Y., Shiokawa N., Hirano T., Shimizu Y., Sasada H. *Phys. Rev. Lett.*, **78**, 4713 (1997).
4. Wright E.M., Arlt J., Dholakia K. *Phys. Rev. A*, **63**, 013608 (2001).
5. Rhodes D.P., Lancaster G.P.T., Livesey J., McGloin D., Arlt J., Dholakia K. *Opt. Commun.*, **214**, 247 (2002).
6. Varin C., Piche M. *Appl. Phys. B*, **74**, S83 (2002).
7. Ozaktas H.M., Mendlovic D. *Opt. Lett.*, **19**, 1678 (1994).
8. Alieva T., Bastiaans M.J. *Opt. Lett.*, **24**, 1206 (1999).
9. Wei H., Xue X. *Opt. Lett.*, **26**, 1746 (2001).
10. Lohmann A.W. *J. Opt. Soc. Am. A*, **10**, 2181 (1993).
11. Bultheel A., Martinez H. *Katholieke Universiteit Leuven, Report TW337* (April 2002).
12. Malyutin A.A. *Proc. SPIE Int. Soc. Opt. Eng*, **4900**, 908 (2002).
13. Malyutin A.A. *Kvantovaya Elektron.*, **33**, 235 (2003) [*Quantum Electron.*, **33**, 235 (2003)].
14. Malyutin A.A. *Kvantovaya Elektron.*, **33**, 1015 (2003) [*Quantum Electron.*, **33**, 1015 (2003)].
15. Ozaktas H.M., Zalevsky Z., Kutay M.A. *The Fractional Fourier Transform* (Chichester: Wiley, 2001).
16. Ozaktas H.M., Mendlovic D. *Opt. Commun.*, **101**, 163 (1993).
17. Alieva T., Bastiaans M.J., Calvo M.L. *Recent Research Developments in Optics*, **1**, 105 (2001).
18. James D.F.V., Agarwal G.S. *Opt. Commun.*, **126**, 207 (1996).
19. Shih Ch.-Ch. *Opt. Lett.*, **20**, 1178 (1995).
20. Wang C., Lu B. *Opt. Commun.*, **203**, 61 (2002).
21. Erden M.F., Ozaktas H.M. *J. Opt. Soc. Am. A*, **14**, 2190 (1997).
22. Sahin A., Ozaktas H.M., Mendlovic D. *Opt. Commun.*, **120**, 134 (1995).
23. Beijersbergen M.W., Allen L., van der Veen H.E.L.O., Woerdman J.P. *Opt. Commun.*, **96**, 123 (1993).