Quantum Electronics **34** (4) 299 – 306 (2004)

PHYSICAL FOUNDATIONS OF QUANTUM ELECTRONICS

PACS numbers: 42.25.Bs
DOI: 10.1070/QE2004v034n04ABEH002670

On the distortion of a wave packet propagating in an amplifying medium

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***Abstract*. The propagation of a narrow-band signal (wave packet) in a medium with population inversion is considered. It is shown that in an optically thick medium layer the signal decomposes into the initial and amplified signals propagating at different velocities and having different durations. The initial signal propagates without distortions at the velocity c/n_0 (n_0 is the refractive index of the medium away from the resonance frequency) and plays the role of a precursor of the amplified signal. The amplified signal moves at a lower velocity. It lags behind the initial signal during propagation and acquires a universal Gaussian shape (irrespective of the shape of the initial signal and the spectral line profile). The appearance of the amplified signal, which substantially differs from the initial signal in all the parameters, as well as the interference between the amplified and initial signals looks like a 'extra-distortion' of the initial signal.**

Keywords: *amplifying medium, wave packet, dispersion, signal propagation velocity.*

1. This paper is devoted to the study of propagation of a signal (wave packet) whose carrier frequency ω_0 is close to the frequency of one of the spectral gain lines of an active (amplifying) medium. The propagation of a wave packet in a dispersion medium is usually considered in the first- or second-order approximation of the classical theory of dispersion of a wave packet in a nonabsorbing medium [1, 2]. This approach is well justified in the case of a conservative or weakly absorbing medium. If, however, there are one or several spectral gain lines in a medium and the width of the signal spectrum is not small[1], the situation changes qualitatively. The matter is that the exponential increase in the signal power occurs first of all at frequencies close to the central frequency of the spectral gain line. As a result, when the optical path is sufficiently long, the signal spectrum becomes almost completely coincident with the observed form factor of the spectral gain line and is virtually independent of the initial parameters of the signal (except a linear dependence on the initial density of the signal spectrum at the spectral line centre). Moreover, because the observed gain-line profile experiences the so-called normalisation with increasing medium layer thickness, the signal shape becomes universal (Gaussian) for any shape of the gain line [3, 4].

This paper is urgent due to extensive recent studies of the propagation of light pulses at superluminal (or negative) group velocities in amplifying media with a strong dispersion [5 – 7]. In these studies, an increase in the signal power (resulting in nonlinear effects like the gain saturation) is an impeding factor. It is eliminated by the frequency-nonselective (or weakly selective) signal attenuation, which allows one to use a linear approximation even for a the large optical thickness of a medium layer. As a result, the problem of propagation of a light pulse through an optically thick layer appears, which is considered in this paper.

2. Consider the propagation of the signal $E(z,t)$ with the carrier frequency ω_0 and the complex envelope $A(z,t)$ along the z axis. Assuming that the signal is narrow-band (the width of the signal spectrum is small compared to the carrier frequency ω_0 but is not necessarily small compared to the spectral line width), we obtain the following obvious relations for the time and spectral characteristics of the signal:

$$E(z,t)=\int_{-\infty}^{+\infty}E(z,\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega,$$

$$E(z,\omega)=(2\pi)^{-1}\int_{-\infty}^{+\infty}E(z,t)\exp(\mathrm{i}\omega t)\mathrm{d}t,$$

$$E(z,t)=A(z,t)\exp(-\mathrm{i}\omega_0 t)+A^*(z,t)\exp(\mathrm{i}\omega_0 t),$$

$$A(z,t)=\int_{-\infty}^{+\infty}A(z,\Omega)\exp(-\mathrm{i}\Omega t)\mathrm{d}\Omega, \qquad (1)$$

$$A(z,\Omega)=(2\pi)^{-1}\int_{-\infty}^{+\infty}A(z,t)\exp(\mathrm{i}\Omega t)\mathrm{d}t,$$

$$A(z,\Omega)=E(z,\omega),\quad \omega=\omega_0+\Omega.$$

Here, $E(z,t)$ and $E(z,\omega)$ are the high-frequency signal and its high-frequency spectrum (located near the carrier frequency ω_0), respectively; $A(z,t)$ and $A(z,\Omega)$ are the low-frequency complex envelope and its low-frequency

[1]This situation inevitably appears finally with increasing optical thickness of a medium layer (due to the well-known spectral-line narrowing) for any spectral line width and any initial signal duration.

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Received 23 June 2003; revision received 4 November 2003
Kvantovaya Elektronika **34** (4) 299 – 306 (2004)
Translated by M.N. Sapozhnikov

spectrum, respectively. All these functions depend on z due to the distortion of the signal propagating in the medium.

We assume that the signal propagates in the medium with the refractive index

$$n(\omega) = n_0 + \Delta n(\omega), \quad (2)$$

where n_0 is the nonresonance part of the refractive index[2], which weakly depends on the signal frequency near the resonance frequency ω_0 and $\Delta n(\omega)$ is the resonance part of the refractive index of the medium, which strongly depends on the signal frequency near ω_0. Then, we obtain for the complex transfer function of a medium layer of thickness z the expression

$$F(z,\omega) = \exp[\mathrm{i}kn(\omega)z] = \exp(\mathrm{i}kn_0z)\exp[\mathrm{i}k\Delta n(\omega)z], \quad (3)$$

where $k = \omega/c$. By introducing the amplitude gain of light at the spectral-line centre

$$\alpha_0 = \mathrm{i}k_0\Delta n(\omega_0), \quad k_0 = \omega_0/c \quad (4)$$

and the complex form factor of the line normalised to unity at the spectral-line centre ω_0

$$g(\Omega) = \mathrm{i}k\alpha_0^{-1}\Delta n(\omega_0 + \Omega), \quad (5)$$

we can represent the transfer function of the medium layer in the form

$$F(z,\omega) = \exp(\mathrm{i}kn_0z)\exp[\xi g(\Omega)], \quad (6)$$

where $\xi = \alpha_0 z$ is the optical thickness of the medium layer. This function can be also represented in the form

$$F(z,\omega) = \exp(\mathrm{i}kn_0z)\left[1 + F^{\mathrm{f}}(\xi,\Omega)\right], \quad (7)$$

where $\exp(\mathrm{i}kn_0z)$ is the transfer function of the medium layer away from the spectral line and

$$F^{\mathrm{f}}(\xi,\Omega) = (\exp\xi - 1)\gamma(\xi,\Omega) \quad (8)$$

is the additional (related to the spectral line) transfer function of the medium layer. It is natural to call the function

$$\gamma(\xi,\Omega) = \frac{\exp[\xi g(\Omega)] - 1}{\exp\xi - 1} \quad (9)$$

introduced in (8) the observed complex form factor of the spectral gain line. This function is a complex analogue of the spectral line profile observed for the nonzero optical thickness of the medium layer [3]. It has a 'proper' asymptotic $[\gamma(\xi,0) = 1,\ \gamma(\xi,\pm\infty) = 0]$ and transfers to the usual complex form factor of the spectral line $[\gamma(0,\Omega) = g(\Omega)]$.

The evolution of the spectrum of the signal envelope with increasing distance z is described by the equation

$$A(z,\Omega) = A(0,\Omega)F(z,\omega_0 + \Omega) = A^{\mathrm{p}}(z,\Omega) + A^{\mathrm{a}}(z,\Omega), \quad (10)$$

where

$$A^{\mathrm{p}}(z,\Omega) = A(0,\Omega)\exp(\mathrm{i}kn_0z),$$
$$A^{\mathrm{a}}(z,\Omega) = A(0,\Omega)\exp(\mathrm{i}kn_0z)F^{\mathrm{f}}(z,\Omega) \quad (11)$$

are the spectra of two components of the signal, which substantially differ in their parameters. The component A^{p} simply corresponds to the delayed initial signal, and we will call it below the initial signal. The component A^{a} corresponds to the part of the signal whose properties are determined first of all by the exponential increase in the amplitude and by the narrowing and normalisation of the spectrum with increasing optical thickness of the layer. We will call this part of the signal the amplified signal.

3. To find the time dependence of $A(z,t)$ at any point z from the dependence $A(0,t)$, is sufficient to perform the direct [to determine $A(0,\Omega)$] and inverse (to determine $A(z,t)$] Fourier transforms. In the general case, this procedure can be performed only numerically. Therefore, to obtain numerical estimates, we will use the method of cumulants [8, 9], i.e., represent the signal spectrum in the form of the expansion

$$A(z,\Omega) = \frac{M_0(z)}{2\pi}\exp\left[\sum_{n=1}^{\infty}\frac{(\mathrm{i}\Omega)^n}{n!}k_n(z)\right], \quad (12)$$

where $M_0(z)$ is the density of the signal spectrum at the point z at the frequency $\Omega = 0$ and functions $k_n(z)$ are the cumulants of the signal at the point z. To find the cumulants of the envelope, it is not necessary to know its Fourier transform because cumulants $k_n(z)$ of the envelope are related to its moments $m_n(z)$ by the known expressions $k_1 = m_1$, $k_2 = m_2 - m_1^2$, $k_3 = m_3 - 3m_1m_2 + 2m_1^3$, etc. [8, 9], and the moments can be directly found from the envelope:

$$m_n(z) = M_0^{-1}(z)\int_{-\infty}^{+\infty}t^nA(z,t)\mathrm{d}t,\ M_0(z) = \int_{-\infty}^{+\infty}A(z,t)\mathrm{d}t. \quad (13)$$

The cumulants of the signal envelope have a simple physical sense (this is the first advantage of the method of cumulants[3]). The first cumulant $k_1(z)$ corresponds to the middle point of the signal, and therefore can be reasonably identified [5, 10] with the instant $\tau(z)$ of propagation of the 'centre of gravity' of the signal through the observation

[2]Strictly speaking, n_0 is not a constant but the part of the refractive index that smoothly depends on frequency (near ω_0), which is caused by contributions from all other (except ω_0) resonance frequencies of the medium. We neglect this smooth frequency dependence assuming that the signal spectrum is sufficiently narrow – of course, not compared to the spectral line width but compared to the characteristic interval of a noticeable change in the nonresonance part of the refractive index.

[3]The method of moments (which coincide with the corresponding cumulants for $n \leqslant 2$) has been long successfully used [10] to study the evolution of the averaged parameters of a wave packet in a dispersion medium. The peculiarity of the problem under study is that the use of higher-order ($n > 2$) cumulants (which are no longer coincident with the corresponding moments) allows us to prove that the signal envelope is normalised with the distance propagated by the signal. As a result, the first two cumulants (i.e., the first initial and second central moments) of the signal characterise not only the 'centre of gravity' and 'the average width' of the signal, but also completely describe the signal shape (Gaussian) (when a medium layer is sufficiently thick).

point z. The second cumulant $k_2(z)$ can be treated as the square of the root-mean-square signal duration $T(z)$. Let us introduce the notation

$$\tau(z) \equiv k_1(z), \quad T(z) \equiv [k_2(z)]^{1/2}. \tag{14}$$

The third cumulant (which is called the asymmetry coefficient in the probability theory) determines the signal asymmetry in time with respect to its middle point $\tau(z)$. The normalised asymmetry coefficient is $\varkappa_3(z) = k_3(z)/k_2^{3/2}(z)$. For $\varkappa_3(z) > 0$, the leading edge of the signal is steeper than its trailing edge and the local minimum of the envelope is shifted from the middle point of the signal $\tau(z)$ to its leading part. For $\varkappa_3(z) < 0$, the situation is opposite.

Let us assume that the expansion of the observed complex form factor of the spectral line as a power series in the detuning Ω has the form

$$\gamma(\xi, \Omega) = \exp\left[\sum_{n=1}^{\infty} \frac{(\mathrm{i}x)^n}{n!} \mu_n(\xi)\right], \tag{15}$$

where $x = 2\Omega/\Delta\Omega_{1/2}$ is the normalised detuning and $\Delta\Omega_{1/2}$ is the half-width of the spectral line (the characteristic coherence time of radiation related to $\Delta\Omega_{1/2}$ by the expression $\tau_{\mathrm{coh}} = 2/\Delta\Omega_{1/2}$). The functions $\mu_n(\xi)$ in (15) are determined by expressions

$$\mu_1(\xi) = p(\xi)\xi\beta_1, \quad p(\xi) = \frac{1}{1 - \exp(-\xi)},$$

$$\mu_2(\xi) = p(\xi)\xi[\beta_2 + \beta_1^2(1 - p)\xi], \tag{16}$$

$$\mu_3(\xi) = p(\xi)\xi\left[\beta_3 + 3\beta_1\beta_2(1 - p)\xi + \beta_1^3(p - 1)(2p - 1)\xi^2\right],$$

where an infinite set of coefficients

$$\beta_n = (-\mathrm{i})^n \left.\frac{\mathrm{d}^n g}{\mathrm{d}x^n}\right|_{x=0} \tag{17}$$

can be considered as one of the methods for specifying the form factor of the spectral line $g(x)$. In the cases of the Lorentzian (L) (collision or radiative line broadening), Gaussian (G) (Doppler broadening), and time-of-flight (T) spectral line shapes, we have [11, 12]

$$\beta_1^{\mathrm{L}} = 1, \ \ \beta_2^{\mathrm{L}} = 2, \ \ \beta_3^{\mathrm{L}} = 6, \ \ \beta_1^{\mathrm{G}} = 2\left(\frac{\ln 2}{\pi}\right)^{1/2} \approx 0.939,$$

$$\beta_2^{\mathrm{G}} = 2\ln 2 \approx 1.386, \quad \beta_3^{\mathrm{G}} = 8\ln 2\left(\frac{\ln 2}{\pi}\right)^{1/2} \approx 2.605,$$

$$\beta_1^{\mathrm{T}} = \frac{2x_0}{3} \approx 0.927, \ \ \beta_2^{\mathrm{T}} = \frac{2x_0^2}{3} \approx 1.288, \ \ \beta_3^{\mathrm{T}} = \frac{4x_0^3}{5} \approx 2.148,$$

respectively, where $x_0 \approx 1.39$ is the root of the equation $\mathrm{sinc}^2 x = 1/2$.

For the zero thickness of an optical layer ($\xi = 0$), we have

$$\mu_1 = \beta_1, \ \ \mu_2 = \beta_2 - \beta_1^2, \ \ \mu_3 = \beta_3 - 3\beta_1\beta_2 + 2\beta_1^3, \tag{18}$$

whereas for an optically thick layer ($\xi \gg 1$),

$$\mu_n(\xi) \to \beta_n\xi. \tag{19}$$

Now the cumulant representation of the transfer function of the amplified signal has the form

$$F^{\mathrm{f}} = M_0^{\mathrm{f}} \exp\left[\sum_{n=1}^{\infty} \frac{(\mathrm{i}\Omega)^n}{n!} k_n^{\mathrm{f}}\right], \tag{20}$$

where

$$M_0^{\mathrm{f}} = \exp\xi - 1; \quad k_n^{\mathrm{f}} = \mu_n(\xi)\tau_{\mathrm{coh}}^n. \tag{21}$$

The truncation of the infinite series in the exponent in (20) does not violate a proper asymptotic behaviour of the function $\gamma(\xi, \Omega)$ and the transfer function $F(z, \omega_0 + \Omega)$ of the medium layer at $\Omega \to \pm\infty$. This is the second main advantage of the method of cumulants. The truncation of this series in the probability theory is used to define a random quantity by a finite number of cumulants [most often the normal (two terms) or excess (four terms) approximations are used].

The filter F^{f} not only narrows down with increasing optical thickness ξ of the layer [11, 12] as $\sim\xi^{-1/2}$ but it is also normalised (i.e., becomes Gaussian in the limit $\xi \to \infty$ [3]). This can be easily verified by considering the behaviour of normalised cumulants with indices larger than 2 for $\xi \to \infty$ taking into account (19): $\varkappa_n^{\mathrm{f}}(\xi) = k_n^{\mathrm{f}}(\xi)[k_2^{\mathrm{f}}(\xi)]^{-n/2} \to \beta_n\beta_2^{-n/2}\xi^{1-n/2} \to 0$.

Therefore, as the optical thickness ξ of a layer increases, the function F^{f} becomes a Gaussian and is completely characterised by the first two cumulants $k_1^{\mathrm{f}}(\xi)$ and $k_2^{\mathrm{f}}(\xi)$ and the normalisation factor $M_0^{\mathrm{f}}(\xi)$.

4. Let us assume that the time dependence of a signal at $z = 0$ is determined by the function $A^{(0)}(t) \equiv A(0, t)$. Then, the 'start' parameters of the signal are described by the expressions

$$M_0^{(0)} = \int_{-\infty}^{+\infty} A^{(0)}(t)\mathrm{d}t,$$

$$\tau_0 = k_1^{(0)} = \int_{-\infty}^{+\infty} tA^{(0)}(t)\mathrm{d}t \Big/ \int_{-\infty}^{+\infty} A^{(0)}(t)\mathrm{d}t, \tag{22}$$

$$T_0 = \left(k_2^{(0)}\right)^{1/2} = \left[\int_{-\infty}^{+\infty} (t - \tau_0)^2 A^{(0)}(t)\mathrm{d}t \Big/ \int_{-\infty}^{+\infty} A^{(0)}(t)\mathrm{d}t\right]^{1/2}.$$

It is clear [see (11)] that the evolution of the initial signal consists in its shift in time (by the time delay of the signal in the nonresonance medium) and the corresponding phase shift,

$$A^{\mathrm{p}}(z, t) = A^{(0)}(t - \tau_{\mathrm{v}}(z))\exp(\mathrm{i}k_0 n_0 z), \quad \tau_{\mathrm{v}} = zn_0/c. \tag{23}$$

Because the multiplication of the Fourier transforms of signals is equivalent to the summation of their cumulants (this is the third advantage of the method of cumulants), the parameters of the amplified signal have the form

$$\begin{aligned} M_0^{\mathrm{a}}(z) &= M_0^{(0)} M_0^{\mathrm{f}}(\xi)\exp(\mathrm{i}k_0 n_0 z), \\ k_n^{\mathrm{a}}(z) &= k_n^{(0)} + \delta_{n1}\tau_{\mathrm{v}}(z) + \tau_{\mathrm{coh}}^n\mu_n(\xi). \end{aligned} \tag{24}$$

In particular, the instant of propagation of the middle point of the amplified signal through the point z [$\tau_{\mathrm{a}}(z)$] and the root-mean-square duration $T_a(\xi)$ of the amplified signal are described by the expressions

$$\tau_a(z) = \tau_0 + \tau_v(z) + \tau_{coh}\mu_1(\xi), \quad T_a^2(\xi) = T_0^2 + \tau_{coh}^2\mu_2(\xi). \quad (25)$$

The higher normalised cumulants of the amplified signal have the form $\varkappa_n^a(\xi) = [\varkappa_n^{(0)}(T_0/\tau_{coh})^n + \mu_n(\xi)][(T_0/\tau_{coh})^2 + \mu_2(\xi)]^{-n/2}$. As expected, for $\xi \to \infty$ (more exactly, for $\xi \gg \max\{1, (T_0/\tau_{coh})^2\}$) and $n > 2$, these cumulants are small and the amplified signal proves to be a Gaussian wave packet. Assuming that all the cumulants of the signal with $n > 2$ are zero, we can easily obtain the expression for the complex envelope of the signal in the Gaussian approximation:

$$A(z,t) = A^p(z,t) + A^a(z,t),$$

$$A^p(z,t) = A^{(0)}(t - \tau_v(z))\exp(ik_0n_0z), \quad (26)$$

$$A^a(\xi,t) = \frac{M_0^{(0)}(\exp\xi - 1)}{\sqrt{2\pi}T_a(\xi)}\exp\left\{-\frac{[t-\tau_a(\xi)]^2}{2T_a^2(\xi)} + ik_0n_0z\right\}.$$

To perform calculations by formulas (25) and (26), it is sufficient to know the shape of the signal at $z = 0$, the gain α_0 at the line centre, the coherence time of radiation τ_{coh}, and the coefficients β_1 and β_2 of the complex form factor of the line. These formulas are simplified for optically thick layers ($\xi \gg 1$), when the term T_0^2 in (25) can be neglected compared to $\tau_{coh}^2\mu_2(\xi)$ and the function $\mu_n(\xi)$ can be described by expression (19):

$$A^a(\xi,t) = A_{max}^a(\xi)\exp\left[-\frac{(t - \tau_0 - zn_0/c - \tau_{coh}\beta_1\xi)^2}{2\tau_{coh}^2\beta_2\xi}\right],$$

$$A_{max}^a(\xi) = \frac{M_0^{(0)}\exp(ik_0z + \xi)}{\tau_{coh}(2\pi\beta_2\xi)^{1/2}}. \quad (27)$$

One can see that for $\xi \gg 1$, the parameters of the amplified signal depend neither on the shape nor on duration of the initial signal and are completely determined by the optical thickness ξ of the medium layer and the density $M_0^{(0)}$ of the spectrum of the initial signal at the spectral line frequency. The energy of the amplified signal propagating in the amplifying medium increases with ξ as $\xi^{-1/2}\exp 2\xi$, the maximum (in time) amplitude increases as $\xi^{-1/2}\exp\xi$, and the maximum (in time) intensity increases as $\xi^{-1}\exp 2\xi$. The duration of the amplified signal increases as $\xi^{1/2}$, i.e., noticeable slower than the duration of a usual signal in a dispersion medium ($\sim z$, see Refs [1, 2]).

By measuring the signal velocity v from the relation $v = [d\tau(z)/dz]^{-1}$, we can easily see that, for $\xi \gg 1$, the velocity of the amplified signal is independent of the distance propagated by the signal and is exactly equal to the usual group velocity of the wave packet $v_{gr} = (\partial k/\partial\omega)^{-1}$ with the carrier frequency ω_0 propagating in a medium with refractive index (2). The propagation velocity c/n_0 of the initial signal coincides with the phase (or group[4]) velocity of the signal away from the spectral line. Therefore, the velocity of the amplified signal can substantially differ from that of the initial signal, which propagates at the 'nonresonance' velocity and plays the role of a precursor in this case.

The appearance of the Brillouin – Sommerfeld precursor is usually associated with the presence of frequencies in the spectrum that are higher than the intrinsic frequencies of the medium (in our case, $\omega \gg \omega_0$, $n(\omega) \approx 1$), i.e., with the discontinuous or almost discontinuous nature of the signal. In this case, the precursor is usually a broadband (nonquasi-monochromatic) signal. Such a situation is not the only possible. For example, a signal can be continuous, but its duration can be small compared to the period of any vibrations in the medium, i.e., the width of the signal spectrum $\Delta\omega \gg \omega_0$. In this case, almost total signal falls in the region $n(\omega) \approx 1$, and the precursor virtually coincides with the initial signal.

The situation considered in this paper is also possible, when the duration of the quasi-monochromatic signal is long compared to the period of natural vibrations in the medium but is short compared to their relaxation time ($\Delta\Omega_{1/2} \ll \Delta\omega \ll \omega_0$). In this case, the signal is 'broadband for the given medium' ($\Delta\omega \gg \Delta\Omega_{1/2}$) but is 'narrow-band in itself' ($\Delta\omega \ll \omega_0$), i.e., quasi-monochromatic. Because in this case (as in the previous one) almost total initial signal falls into the 'dispersionless region' [$n(\omega) \approx n_0$, $\omega \notin (\omega_0 - \Delta\Omega_{1/2}, \omega_0 + \Delta\Omega_{1/2})$] and its precursor virtually coincides with the initial signal itself, the precursor proves to be a quasi-monochromatic (narrow-band) signal propagating without distortions, however, not at the vacuum but at the 'nonresonance' speed of light c/n_0 in the given medium.

For example, in the active medium of a He – Ne laser (when a signal is amplified at a wavelength of 3.39 μm), we have [12] $\alpha_0 = 2.3$ m^{-1} (the intensity gain is 20 dB m^{-1}), $\tau_{coh} = 1.1$ ns ($\Delta\nu_D = 300$ MHz), and $\beta_1^G = 0.94$ (Doppler broadening). For these parameters of the medium, the velocity of the amplified signal is $v_a = 0.6c$. Similarly for the active medium of a low-pressure CO_2 laser [12] [for the signal gain equal to $\alpha_0 = 0.46$ m^{-1} at 10.6 μm (the intensity gain is 4 dB m^{-1}), $\tau_{coh} = 6.4$ ns ($\Delta\nu_D = 50$ MHz), $\beta_1^G = 0.94$ (Doppler broadening)], we have $v_a = 0.55c$. In both cases, the velocity of the amplified signal is almost half the 'nonresonance' speed of light in these gases (which is virtually equal to the speed of light in vacuum).

The analogy between the Sommerfeld – Brillouin precursor and the narrow-band initial signal pointed out above should not be understood too literally – simply both these phenomena have a common reason and similar manifestations. The reason is that, strictly speaking, neither medium can simply attenuate or change an incident (initial) signal. It can only add its own induced signal (called the amplified signal in this paper) whose interference with the incident signal can modify the latter. If, for example, the sum of the initial and induced signals is similar to the delayed or attenuated initial signal, the signal is said to be delayed or attenuated by the medium. In any medium, the induced signal can appear only near some (resonance for the give medium) frequencies. For this reason, the part of the initial signal whose spectrum does not fall into the vicinity of these resonance frequencies (precursor) 'does not feel' them and propagates as in a nonresonance medium.

For example, the Sommerfeld – Brillouin precursor appears due to the presence of frequencies in the signal spectrum that are higher than any natural frequencies of the medium ($|\omega| \gg \omega_0$). It propagates at the speed of light in vacuum because the medium cannot produce a signal at

[4]Because we have already neglected the frequency dependence of the nonresonance part of the refractive index of the medium, the 'nonresonance phase' and 'nonresonance group' velocities coincide.

these frequencies. As a result, the corresponding part of the initial signal (precursor) cannot interfere with stimulated radiation of the medium (simply because the latter is absent) and therefore 'does not feel' the medium and propagates as in vacuum.

The initial signal considered in this paper can be treated as the precursor of the envelope of a narrow-band signal, which appears due to the presence of frequencies in the envelope spectrum that exceed the spectral line width ($|\Omega| \gg \Delta\Omega_{1/2}$). This signal propagates at the 'nonresonance' speed of light in the given medium because the given spectral line (more exactly, the corresponding resonance of the medium) cannot produce a signal at these frequencies. As a result, the corresponding part of the initial signal cannot interfere (more exactly, only weakly interferes, see below) with stimulated radiation of the medium related to the given resonance and, therefore, 'does not feel' the given spectral line and propagates as in a nonresonance medium.

5. Let us apply the general relations obtained above to analyse the propagation of a Gaussian pulse of duration T_1 in an active medium. Let the time dependence $E(t)$ of the signal at $z = 0$ has the form

$$E(t) = A_1^{(0)}(t)\exp(-\mathrm{i}\omega_1 t) + A_1^{(0)*}(t)\exp(\mathrm{i}\omega_1 t),$$
$$A_1^{(0)}(t) = \exp(-t^2/T_1^2), \qquad (28)$$

where $|\omega_1 - \omega_0| \ll \omega_0$. Because the carrier frequency ω_1 does not coincide in the general case with the gain line centre ω_0, it is necessary to represent (28) in form (1) in order to use general relations (22)–(26). For this purpose, it is sufficient to introduce the function $A^{(0)}(t)$ as

$$A^{(0)}(t) = \exp\left(-t^2/T_1^2 - \mathrm{i}\Delta\omega t\right),$$

where $\Delta\omega = \omega_1 - \omega_0$ is the shift of the wave-packet carrier frequency with respect to the spectral line centre. Thereafter we obtain for cumulants of signal (28)

$$M_0^{(0)} = T_1\sqrt{\pi}\exp(-x_1^2), \quad \tau_0 = -\mathrm{i}x_1 T_1, \quad T_0^2 = T_1^2/2, \qquad (29)$$

where $x_1 = \Delta\omega T_1/2$ is the pulse-duration-normalised shift of the pulse carrier frequency with respect to the line centre.

For $x_1 \neq 0$, the signal envelope (at the frequency ω_0) is not a real function, and therefore the signal cumulants lose their clearness (see section 3). Nevertheless, their use leads (as shown below) to correct results. Thus, it is these cumulants (which can be positive, negative, and even complex) that still correctly describe the signal spectrum in the vicinity of point $\Omega = 0$ ($\omega = \omega_0$) and it is this part of the signal spectrum that determines first of all the parameters of the amplified signal.

One can see from (29) that for $x_1 \ll 1$, the shift of the signal carrier frequency with respect to the spectral line centre can be neglected. In this case, $M_0^{(0)} = T_1\sqrt{\pi}$, $\tau_0 = 0$ and $T_0^2 = T_1^2/2$. If the relation $T_1 \ll \tau_{\mathrm{coh}}$ is also satisfied, i.e., the signal duration is small compared to the coherence time of the line, then we can set $\tau_0 = T_0 = 0$ in (25) for any optical thickness of matter. As a result, we arrive at the problem of propagation of the delta pulse in an active medium. In this case, with an accuracy of normalisation, the envelope of the amplified signal proves to the complex envelope of the response of a medium layer. It is clear that results obtained by solving this problem are applicable to any sufficiently short signal if its duration is small compared to the coherence time of the spectral line and the shift of the carrier frequency with respect to the spectral line centre is small compared to the width of the signal spectrum. For the parameters of a medium presented above, which are typical for He–Ne and CO_2 lasers, the pulse duration should be a few fractions of a nanosecond for a He–Ne laser and of the order of a nanosecond for CO_2 laser.

Figure 1 shows the results of numerical calculations (solid curves) and Gaussian approximations (26) (dotted curves) for the small ($\xi = 0.5$), intermediate ($\xi = 2$), and large ($\xi = 8$) optical thickness of the medium layer. On the abscissa the time $\vartheta = (t - zn_0/c)/\tau_{\mathrm{coh}}$ in the moving coordinate system is plotted (in the units of the coherence time of the spectral line). The ordinate is the dimensionless amplitude $\mathscr{A} = |A(z,t)|(\tau_{\mathrm{coh}}/M_0^{(0)})\exp(-\xi)$ [cf. (27)]. The initial signal (which is the delta function) is denoted by the vertical arrow. The choice of a scale over the ordinate takes explicitly into account (thereby excluding from a consideration) the exponential increase in the amplified signal

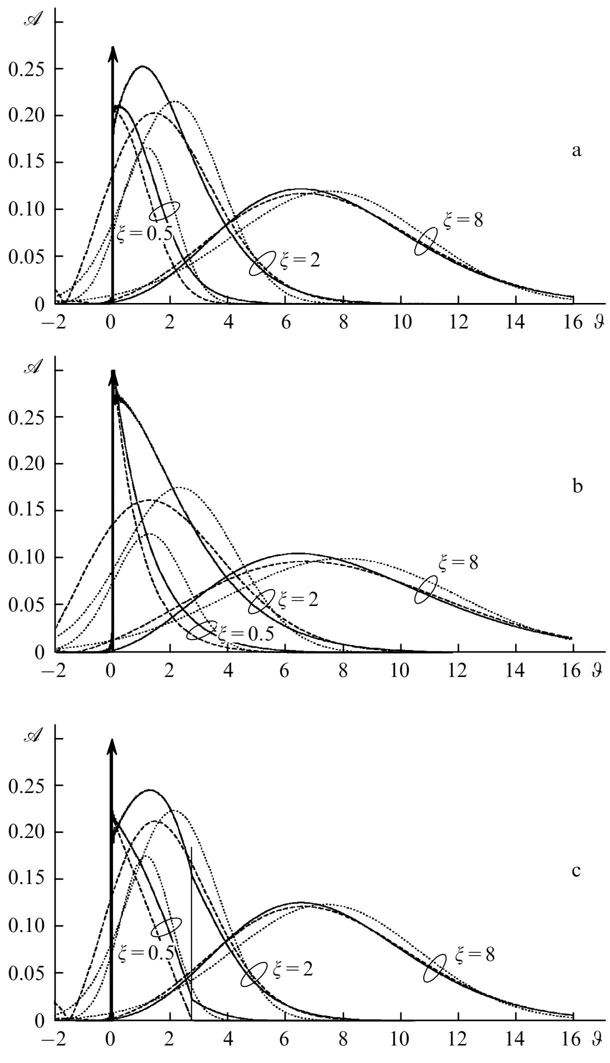

Figure 1. Response functions of gas layers of different thickness ξ calculated for the Doppler (a), collision (b), and time-of-flight (c) broadening mechanisms of a spectral line numerically (solid curves), in the Gaussian approximation (dotted curves), and using alternative approximations (dashed curves). The thin vertical straight line in Fig. 1c indicates the break in the modulus of the signal amplitude.

during its propagation in a medium layer. Note that the exponential increase in the signal is compensated for in many cases by introducing frequency-nonselective absorption.

The dashed curves in Fig. 1 correspond to calculations performed by using other (with respect to Gaussian) approximations. For $\xi = 2$ and 8, we used the three-cumulant approximation. In this approximation, the shape of the amplified signal was reconstructed not from two first cumulants (Gaussian approximation) but from three cumulants [i.e., taking the asymmetry coefficient $\varkappa_3(z)$ into account]. For $\xi = 0.5$, we used the polarisation response function of a medium as a reasonable alternative (see, for example, Ref. [1]), which is the Fourier transform of the complex form factor $g(\Omega)$ of the spectral line.

Indeed, for $\xi \ll 1$, it follows from (8) and (9) that $F^{\mathrm{f}}(\xi, \Omega) = \xi g(\Omega)$, and we can easily obtain for the complex amplitude A^{a} of the amplified signal instead of (26) the expression

$$A^{\mathrm{a}}(\xi, t) = \frac{T_1 \xi}{2\pi} g(t - \tau_{\mathrm{v}}(\xi)), \tag{30}$$

where

$$g(t) = \int_{-\infty}^{+\infty} g(\Omega) \exp(-\mathrm{i}\Omega t) \mathrm{d}t \tag{31}$$

is the Fourier transform of the complex form factor of the spectral gain line. As a result, we have for $\xi \ll 1$ instead of (26) the expressions

$$A_{\mathrm{G}}^{\mathrm{a}}(\xi, t) = \frac{T_1 \xi}{(\pi \ln 2)^{1/2} \tau_{\mathrm{coh}}} \exp\left\{ -\frac{[t - \tau_{\mathrm{v}}(z)]^2}{4 \ln 2 \tau_{\mathrm{coh}}^2} \right\} \eta(t - \tau_{\mathrm{v}}(z)),$$

$$A_{\mathrm{L}}^{\mathrm{a}}(\xi, t) = \frac{T_1 \xi}{\tau_{\mathrm{coh}}} \exp\left[-\frac{t - \tau_{\mathrm{v}}(z)}{\tau_{\mathrm{coh}}} \right] \eta(t - \tau_{\mathrm{v}}(z)), \tag{32}$$

$$A_{\mathrm{T}}^{\mathrm{a}}(\xi, t) = \frac{T_1 \xi}{x_0 \tau_{\mathrm{coh}}} \left[1 - \frac{t - \tau_{\mathrm{v}}(z)}{2 x_0 \tau_{\mathrm{coh}}} \right] \eta \left[1 - \frac{t - \tau_{\mathrm{v}}(z)}{2 x_0 \tau_{\mathrm{coh}}} \right] \times \eta(t - \tau_{\mathrm{v}}(z)),$$

for the Gaussian, Lorentzian, and time-of-flight profiles of the spectral line, respectively, where $\eta(x)$ is the Heaviside function and $x_0 \approx 1.39$.

By comparing (17) and (39), we can readily elucidate the physical sense of coefficients β_n. These coefficients are simply the moments of the function $g(t)$, which are expressed in the units of the coherence time τ_{coh} of the spectral line. The similar coefficients $\mu_n(0)$ (18) are the corresponding cumulants of the function $g(t)$. In particular, $\mu_1(0)\tau_{\mathrm{coh}} = \beta_1 \tau_{\mathrm{coh}}$ is the average delay time of the polarisation response of the medium to the delta-like perturbation (the pulse duration at the carrier frequency ω_0 should be small compared to τ_{coh} and large compared to $1/\omega_0$) and $\mu_2(0)\tau_{\mathrm{coh}}^2 = (\beta_2 - \beta_1^2)\tau_{\mathrm{coh}}^2$ is the square of the root-mean-square duration of the polarisation response of the medium to the delta-like perturbation.

Consider a general character of the propagation of a short (compared to the line coherence time) signal in an amplifying medium. One can see from Fig. 1 that parameters of the signal as a whole are not modified because we are dealing with the propagation of the initial signal (which behaves as in a nonresonance medium without dispersion) and the amplified signal, which is a ‘train’ of the initial signal. For low optical densities ξ of the layer, the ‘train’ amplitude increases linearly with the distance propagated by the signal and the time dependence of the ‘train’ coincides with that of the polarisation response of the medium to an external perturbation (and is different for different line-broadening mechanisms). At this amplification stage, the amplified signal is small compared to the initial signal and does not experience any self-action. This explains its invariable shape and its propagation behind the initial signal.

For $\xi \ll 1$, we are dealing with the perturbation of the medium polarisability in the fixed field approximation for the initial signal, which dominates over the amplified signal. Nevertheless, the amplitude of the amplified signal (‘train’) increases with increasing distance propagated by the signal, i.e., the perturbation of the medium polarisability is accumulated. As a result, for $\xi \sim 1$, the amplified signal amplitude increases so that the influence of the signal on the medium can no longer be neglected (the spectral density of the amplified signal at the centre of the gain line becomes comparable with that of the initial signal). For this reason, the self-amplification of the amplified signal begins and the signal ‘separates’ from the initial signal, i.e., the initial and amplified signals become separated in time, and the universal (Gaussian) shape of the amplified signal begins to form. For $\xi \gg 1$, the initial and amplified signals are finally separated, and the amplified signal has the universal (Gaussian) profile. The profile parameters are independent of the parameters of the initial signal but are determined by the distance propagated by the signal and the parameters α_0, τ_{coh}, β_1, and β_2 of the spectral line.

One can see from Fig. 1 that, when the optical thickness ξ of a medium layer is large (for example, $\xi = 8$), the analytic Gaussian approximation and numerical calculations are in quantitative agreement, whereas the agreement is only qualitative at moderate values of ξ (for example, $\xi = 2$). For small ξ (for example, $\xi = 0.5$), the agreement between analytic and numerical calculations leaves much to be desired. Nevertheless, the areas under the analytic and numerical curves, the positions of their middle points and characteristic durations of signals coincide for any ξ. This would be expected in the light of discussion of the meaning of averaged characteristics (cumulants) of a signal performed in section 3: the Gaussian approximation gives correct average characteristics of the signal for any ξ, whereas the time dependence of the signal can be calculated in this approximation only for large ξ.

The analytic calculation in the fixed field approximation (32) well agrees with numerical data for small ξ (for example, $\xi = 0.5$), however, it is completely inapplicable already for moderate values of ξ (for example, $\xi = 2$).

Consider now the difference between the results obtained for different mechanisms of spectral line broadening. A comparison of Figs 1a and 1c shows that dependences obtained for the Doppler and time-of-flight broadening are surprisingly close to each other. This is explained by the closeness of the coefficients β_n for the Gaussian and time-of-flight form factors of the line. In these two cases, the amplified signal distinctly separates from the initial signal already at $\xi = 2$, while for $\xi = 8$ the signal is almost completely normalised. In the case of a Lorentzian line (collision broadening), the separation of the amplified signal from the initial one only begins to show at $\xi = 2$ and the normalisation of the amplified signal at $\xi = 8$ is far from completion. These differences are explained by a rather large asymmetry coefficient of the Fourier transform $g(t)$ of the Lorentzian form factor $g(\Omega)$ of the line. Indeed, the normalised asymmetry coefficient for the response function $g(t)$ is $\varkappa_3(0) = (\beta_3 - 3\beta_1\beta_2 + 2\beta_1^3)(\beta_2 - \beta_1^2)^{-3/2}$, which gives $\varkappa_3(0) = 0.995$ for the Gaussian form factor, and 0.566 for the time-of-flight and 2 for the Lorentzian form factors, respectively.

We can also point out an 'unintended' illustration of the propagation of sharp (compared to the coherence time of the spectral line) fragments of the signal envelope at the 'nonresonance' velocity in Fig. 1c. The case in point is a break in the modulus of the signal amplitude for the time-of-flight broadening of the spectral line (it is indicated with a vertical straight line in Fig. 1c). This break appears at $\xi = 0$ as an inevitable consequence of the spectral line shape caused by the time-of-flight broadening [see (32)] and propagates at the velocity c/n_0 (as should be with any break of the signal envelope or its derivative), 'travelling' simultaneously over the time envelope of the signal from its tail to the leading part (because the envelope itself moves slower). Of course, no break was observed for the Gaussian approximation (which 'does not know' about the ultimate propagation velocity c/n_0 of perturbations in a nonresonance medium).

The next series of calculations was performed for a signal of duration $T_1 = 0.763\tau_{\text{coh}}$ and the Lorentzian lines shape, when the carrier frequency coincided with the spectral line frequency Fig. 2). In this case, the width of the signal spectrum is of the order of the spectral linewidth. The chosen values of parameters allow us to make comparison with similar numerical calculations [5]. Numerical calculations and analytic calculations using expressions (26) are presented in Fig. 2 by solid and dotted curves, respectively. The calculations were performed for the optical thickness of the layer $\xi = 0, 1, 2, 3$.

Figure 2 shows that, as ξ increases, the following situations are successively realised. When $\xi = 1$, the amplified signal is small compared to the initial signal and can be treated as its weak 'train'; when $\xi = 2$, this 'train' becomes equal to the initial signal; and when $\xi = 3$, the initial signal becomes small compared to the amplified signal and is virtually lost against its background. As the layer thickness further increases, the amplified signal becomes more and more normalised. In this case (when the carrier frequency coincides with the spectral line centre), neither spatial separation of the initial and amplified signals nor interference irregularities of the total signal profile were observed.

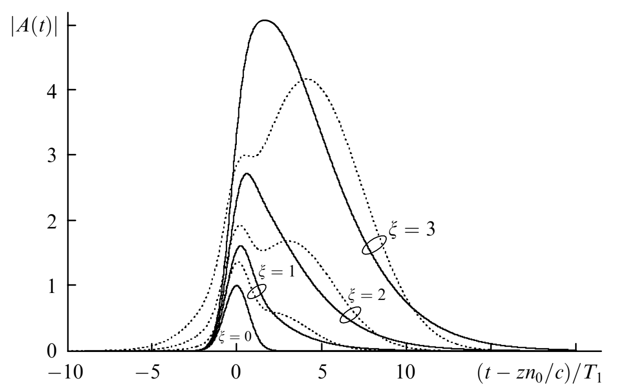

Figure 2. Deformation of a Gaussian wave packet in the case of collision broadening of a spectral line calculated for different optical thicknesses ξ of a layer numerically (solid curves) and in the Gaussian approximation (dotted curves). The carrier frequency of the wave packet coincides with the spectral line frequency.

We also performed calculations to study the effect of the shift of the signal carrier frequency with respect to the central frequency of the gain line (Fig. 3). The calculations were carried out for the same signal ($T_1 = 0.763\tau_{\text{coh}}$), whose carrier frequency was shifted with respect to the gain line centre ($\Delta\omega\tau_{\text{coh}} = 7$). In this case, the density of the spectrum of the initial signal near the central frequency and the intensity of the amplified signal drastically decrease [see (29)], the latter being observable only at $\xi \geqslant 7$. For this reason, we performed calculations for the optical thickness of the layer $\xi = 7, 8, 9$. Due to attenuation of the amplified signal and its late 'creation', it emerges being already well shaped (almost Gaussian) and delayed from the initial signal, so that these two signals prove to be sufficiently autonomous. Nevertheless, the exponential increase in the amplified signal remains (compared to the initial signal) and it soon becomes dominant.

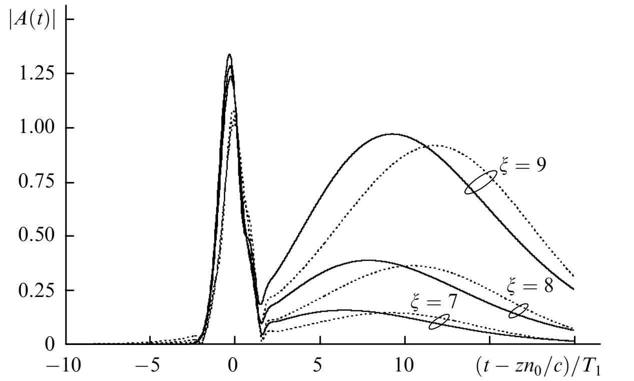

Figure 3. Deformation of a Gaussian wave packet in the case of collision broadening of a spectral line calculated for different optical thicknesses ξ of a layer numerically (solid curves) and in the Gaussian approximation (dotted curves). The carrier frequency of the wave packet does not coincide with the spectral line frequency.

It is clear that, if a medium has not one but several gain lines, there exist one initial signal and several (equal to the number of gain lines) amplified signals, each of them being asymptotically normalised. Nevertheless, the interference of amplified signals 'generated' by different spectral lines can produce a rather complicated pattern. Figure 4 illustrates the results of calculations performed for the same signal, whose carrier frequency is located in the middle between two identical gain lines ($\Delta\omega\tau_{\text{coh}} = \pm 7$). The calculations were carried out for $\xi = 8$. One can see that the interference between the two amplified signals in this case produces a strongly oscillating signal with the envelope of almost a Gaussian shape because interference occurs between two almost Gaussian signals with somewhat different carrier frequencies. A comparison of the results of numerical calculations performed here (Fig. 4) and in Ref. [5] (Fig. 5 in Ref. [5]) for the same values of parameters shows that they are close to each other and the accuracy of analytic expressions presented in our paper is good enough.

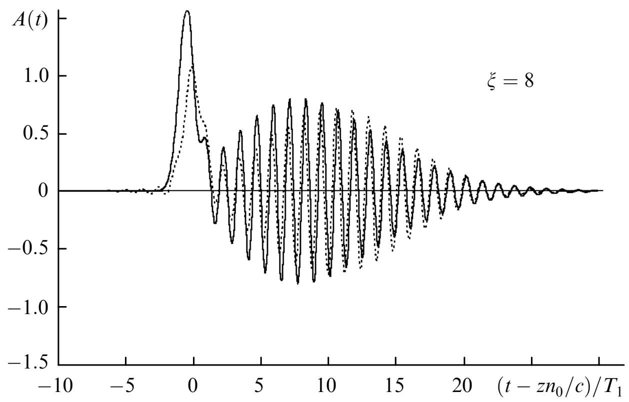

Figure 4. Deformation of a Gaussian wave packet in the case of collision broadening of a doublet of spectral lines calculated for the optical thickness of the layer $\xi = 8$ numerically (solid curves) and in the Gaussian approximation (dotted curves). The carrier frequency of the wave packet is located in the middle between identical spectral lines with different frequencies.

Thus, we can conclude that an extremely strong distortion of a signal in an amplifying medium considered in Ref. [5] can be interpreted simply as the appearance of an amplified signal, whose shape, duration, and propagation velocity substantially differ from these characteristics of the initial signal. Therefore, the estimates of the parameters of the amplified signal presented in this paper can be used to study the problem of the maximum possible path length propagated by a signal without substantial distortions. It is easy to verify, for example, that a numerically calculated drastic distortion of a Gaussian signal with the truncated leading front propagated over a sufficiently long path [7] occurs due to the appearance of an amplified signal (at the spectral line frequency) interfering with a signal at the initial carrier frequency.

6. Thus, a signal propagating in an amplified medium decomposes into the initial and amplified signals. The initial signal propagates without amplification at the 'nonresonance' speed of light in the given medium and plays the role of the precursor of the amplified signal. The amplified signal propagates at the group velocity corresponding to the central frequency of the spectral line. Its carrier frequency coincides with the spectral line frequency. During its propagation, this signal lags behind the initial signal and acquires the universal Gaussian shape (irrespective of the carrier frequency and the shape of the initial signal for any spectral line profile). As the optical thickness ξ of the layer increases, the amplitude of the amplified signal increases as $\sim\xi^{-1/2}\exp\xi$, its duration increases as $\sim\xi^{1/2}$, and energy as $\sim\xi^{-1/2}\exp 2\xi$.

When the optical thickness of a layer is small, the amplified signal is small compared to the initial one and either plays the role of its 'train' (when the initial pulse duration is small compared to the coherence time of the spectral line) or is lost against the initial signal background.

When the optical thickness of a layer is large, the initial signal is small compared to the amplified one. As a result (in the case of the only gain line), the total signal has the universal Gaussian shape and its parameters weakly depend on the features of the initial signal and the form factor of the spectral line. If the medium has several gain lines, several amplified signals appear and their interference may produce a rather complicated structure of the total amplified signal (i.e., its extremely strong distortion in the notation [5]).

When the optical thickness of a layer is moderate, the interference between the initial and amplified signals (which have comparable amplitudes) can be observed. If the initial signal duration is not small compared to the coherence time of the spectral line, this interference also can produce a rather complicated (extremely distorted) total signal.

Acknowledgements. The author thanks V.P. Makarov for stimulating criticism and useful advices.

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