Quantum Electronics **34** (6) 487 – 505 (2004)

REVIEW

PACS numbers: 42.55.Rz; 42.60.Da; 42.60.Mi; 42.65.Sf
DOI: 10.1070/QE2004v034n06ABEH002712

Lasing regimes in solid-state ring lasers with modulated parameters

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***Abstract*.** **The nonlinear dynamics of solid-state ring lasers with the homogeneously broadened luminescence line is studied upon periodic modulation of their parameters. The main temporal, spectral, and phase characteristics of counterpropagating waves are considered and non-stationary lasing regimes are classified.**

Keywords: *ring solid-state laser, nonlinear radiation dynamics, lasing regimes, self-oscillations, dynamic chaos.*

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Received 20 January 2004
Kvantovaya Elektronika **34** (6) 487 – 505 (2004)
Translated by M.N. Sapozhnikov

1. Introduction

One of the most important achievements in the field of nonlinear dynamics in recent decades has been the discovery of dynamic (deterministic) chaos. Numerous theoretical and experimental studies have shown that in many rather simple dynamic systems with a small number of degrees of freedom, along with stationary periodic and quasi-periodic dynamic regimes, the non-stationary regimes can appear, which are characterised by irregular, chaotic variations of dynamic variables in time. Such regimes are paradoxical because irregular motions appear in the absence of any external random factors and their characteristics are completely determined by the initial conditions. At present the problem of chaos attracts great attention in studies of dynamic systems of a different nature, including lasers of different types (gas, solid-state, and semiconductor lasers). Numerous non-stationary lasing regimes, including chaotic ones, are also observed in solid-state ring lasers (SRLs). This is explained, first, by the presence of several relaxation frequencies in such lasers and, second, by the specific interaction of counterpropagating waves in them. In addition, SRLs of a new generation – diode-pumped monolithic ring lasers (ring chip lasers) – play a great role in modern fundamental physics and laser technology.

Being complex nonlinear systems, SRLs are very convenient for studying the general laws of the nonlinear dynamics of a variety of nonlinear systems, as well as the details of physical mechanisms of the nonlinear interaction between counterpropagating waves in an active medium. One of the urgent problems is the investigation of the conditions and mechanisms of the appearance of dynamic chaos in SRLs. The development of the mathematical model of SRLs is based on detailed theoretical studies of the nonlinear dynamics and a comparison of them with experiments. It is a carefully performed physical experiment that allows one to determine whether or not the mathematical model is adequate to a real nonlinear system and to find experimentally the field of its applicability.

The principal possibility of the appearance of nonperiodic modulation of the radiation intensity in a laser under certain conditions (for certain values of parameters) was discovered by Grasyuk, Zubarev, and Oraevsky as early as 1962 [1, 2].

Attempts to find deterministic chaotic lasing regimes in autonomous single-mode SRLs have failed for a long time. Dynamic chaos in a unidirectional SRL was first observed experimentally in Ref. [3], where it was shown that in the presence of the frequency nonreciprocity of the resonator, there exists the region of the SRL parameters in which a resonance appears between the two branches of relaxation oscillations, resulting in their instability and appearance of deterministic chaos. In Refs [4, 5], another mechanism of dynamic chaos in a bidirectional SRL was found, which was related to the parametric resonance between self-modulation (ω_m) and relaxation (ω_r) oscillations.

The regions of laser parameters in which quasi-periodic and chaotic lasing regimes appear are much broader in SRLs with modulated parameters, and in this case it is much simpler to realise the conditions for the appearance of dynamic chaos by exciting relaxation oscillations.

Although quasi-periodic and chaotic lasing regimes in solid-state lasers and systems of coupled lasers have been studied in many papers [6–15], there are still many blind spots in this field. This is explained to a great extent by the fact that it is extremely difficult to study dynamic chaos analytically, whereas numerical methods can be only used to investigate particular cases, which restricts the possibilities of generalisations and prediction of the nonlinear dynamics of lasers in a broad range of parameters. In addition, it was difficult to perform detailed experiments for a long time because of a high level of technical fluctuations of the parameters of flashlamp-pumped solid-state lasers.

The radiation dynamics of bidirectional SRLs has much in common with the dynamics of two coupled linear lasers investigated in many papers. However, along with common properties, ring lasers fundamentally differ in respect of the nature of coupling between counterpropagating waves (nonlinear coupling on the inverse population gratings induced in an active medium). Note also that the use of ring lasers opens up much greater possibilities for controlling lasing regimes due to the phase and amplitude nonreciprocities of the ring resonator.

Recall that the frequency (phase) nonreciprocity is manifested in the inequality of the eigenfrequencies ($\omega_1 \neq \omega_2$) of the ring resonator for counterpropagating waves. This takes the place despite the fact that the longitudinal and transverse mode indices are identical. In the case of the amplitude nonreciprocity, the eigenfrequencies of the resonator for counterpropagating waves are identical ($\omega_1 = \omega_2$), whereas the Q factors of the resonator (i.e., intracavity losses) for counterpropagating waves are different ($Q_1 \neq Q_2$).

In this paper, we considered the main characteristics, conditions for the appearance, and evolution of non-stationary lasing regimes in SRLs with modulated parameters and made an attempt to classify these regimes. It should be emphasized that we analysed here the nonlinear radiation dynamics only for single-frequency SRLs with the homogeneously broadened gain line (i.e., for ring lasers in which one mode with the same longitudinal and transverse indices is excited in each direction).

2. Solid-state ring laser and its mathematical model

Modern diode-pumped monolithic SRLs substantially differ from their predecessors – conventional flashlamp-pumped solid-state lasers consisting of separate elements. Monolithic ring lasers (ring chip lasers) feature a high stability of all parameters providing an extremely low level of technical fluctuations and a high stability of the laser frequency. It is this circumstance that allows us to study in detail their nonlinear dynamics under strictly controlled conditions.

2.1 Design of highly stable SRLs

The most stable characteristics of radiation can be achieved in a longitudinally diode-pumped monoblock ring laser. In this case, technical perturbations of the laser are greatly reduced, the laser design is rigid and stable, and good thermal stabilisation is provided (due to a small size of the laser and weak thermal release).

High-quality Nd^{3+} : YAG single crystals are used as an active medium in most papers devoted to the study of the radiation dynamics of SRLs with the homogeneously broadened luminescence line. This is explained by their good performance parameters: low lasing threshold, high heat conduction and optical homogeneity, and high rigidity.

Note that almost all lasing regimes can be excited in the same monolithic laser by varying coupling coefficients between counterpropagating waves, the pump power excess over the lasing threshold, the value of the frequency and amplitude nonreciprocity or by modulating laser parameters. This allows the use of a few ring chip Nd^{3+} : YAG lasers in experiments, which differ only in the resonator flatness, the reflectivity of the output mirror (determining the relaxation oscillation frequency), and in the effective coupling coefficients between counterpropagating waves, which determine the self-modulation oscillation frequency.

We used in our experiments (except the study of unidirectional ring lasers) a ring chip laser fabricated of a high-quality Nd^{3+} : YAG single crystal in the form of a complex polyhedral prism, with one of the faces having a spherical surface (Fig. 1a). The prism configuration pro-

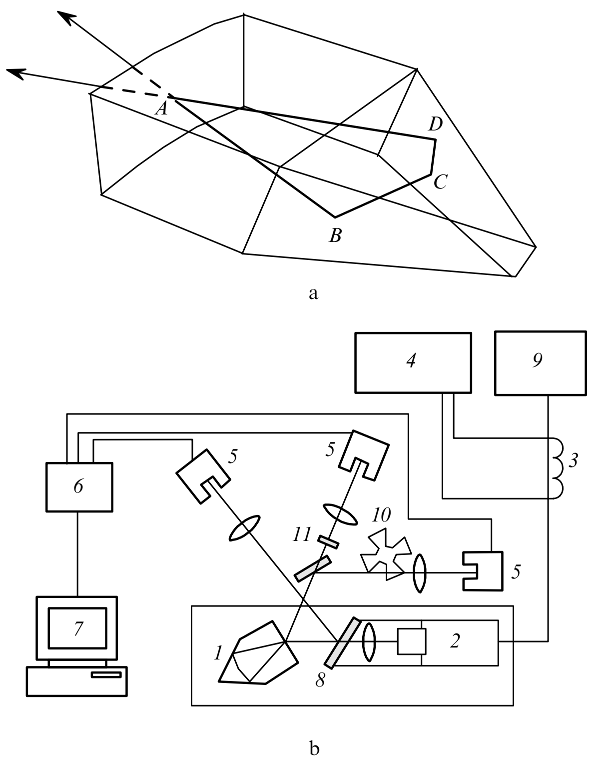

Figure 1. Design of the active element of a monolithic ring chip laser (*ABCD* is the laser-beam axis) (a) and the principal scheme of the laser setup (b): (*1*) monolithic ring chip laser; (*2*) pump laser diode; (*3*) transformer for pump-current modulation; (*4*) alternating-current generator; (*5*) LFD-2 photodetector; (*6*) ASK-3151 digital oscilloscope; (*7*) PC; (*8*) selective mirror; (*9*) laser-diode power supply; (*10*) obturator; (*11*) IR filter.

vided, due to total internal reflections from plane faces and a dichroic mirror applied to the spherical face, the existence of a closed loop inside the prism, serving as the axis of a ring resonator [16–18]. The ring laser was excited by radiation from a laser diode, which was focused and directed to the active element through a dichroic mirror. The high stability of the chip ring laser was provided by thermal stabilisation of the active element with an accuracy of 0.05 °C and by stabilisation of the laser-diode temperature with an accuracy of 0.1 °C. The design of highly stable solid-state chip lasers is described in more detail in papers [18–20]. The scheme of the experimental setup is shown in Fig. 1b.

2.2 Mathematical model of a SRL

A bidirectional ring laser is a complex nonlinear oscillatory system with output characteristics depending on many parameters (the excess over the pump power threshold, the amplitude and frequency nonreciprocity of the resonator, its polarisation properties, the detuning of the laser frequency from the gain line centre, the amplitude and phase of the effective coupling coefficient of counterpropagating waves, and the Q factors of the resonator for counterpropagating waves).

The radiation dynamics of the SRL is theoretically analysed, as a rule, using the following assumptions:

(i) only one (fundamental) longitudinal mode is generated in each direction (in the presence of the optical nonreciprocity in the resonator, the frequencies of counterpropagating waves can be different);

(ii) polarisation of the waves is linear and identical for counterpropagating waves;

(iii) spatial inhomogeneity in the transverse direction is absent both for the population inversion and pump radiation;

(iv) population inversion gratings produced during gain saturation in the field of counterpropagating waves are sinusoidal.

The radiation field in the ring laser is written as the sum of two counterpropagating waves travelling along the resonator axis z:

$$E(z,t) = \mathrm{Re}\Big\{ \sum_{1,2} \boldsymbol{e}_{1,2}\tilde{E}_{1,2}\exp[\mathrm{i}(\omega t \pm kz)] \Big\}, \tag{1}$$

$$\tilde{E}_{1,2} = E_{1,2}\exp \mathrm{i}\varphi_{1,2},$$

where $\tilde{E}_{1,2}$, $E_{1,2}$, and $\varphi_{1,2}$ are the complex amplitudes, moduli, and phases of the fields of counterpropagating waves, respectively; $\boldsymbol{e}_{1,2}$ are the unit vectors; ω is the optical frequency; and k is the wave number.

The lasing dynamics of a SRL in the absence of external perturbations is well described by the system of differential equations [21–23]

$$\frac{\mathrm{d}\tilde{E}_{1,2}}{\mathrm{d}t} = -\frac{\omega}{2Q_{1,2}}\tilde{E}_{1,2} \pm \mathrm{i}\frac{\Omega}{2}\tilde{E}_{1,2} + \frac{\mathrm{i}}{2}\tilde{m}_{1,2}\tilde{E}_{2,1}$$

$$+ \frac{\sigma l}{2T}(1 - \mathrm{i}\delta)(N_0\tilde{E}_{1,2} + N_{\pm 2}\tilde{E}_{2,1}),$$

$$T_1\frac{\mathrm{d}N_0}{\mathrm{d}t} = N_{\mathrm{th}}(1+\eta) - N_0[1 + a(|E_1|^2 + |E_2|^2)] \tag{2}$$

$$- N_{+2}aE_1E_2^* - N_{-2}aE_2E_1^*,$$

$$T_1\frac{\mathrm{d}N_{+2}}{\mathrm{d}t} = -N_{+2}[1 + a(|E_1|^2 + |E_2|^2)] - N_0aE_2E_1^*.$$

Here N_{th} is the threshold inversion population; $\omega/Q_{1,2}$ are widths of the resonator bands for counterpropagating waves; L is perimeter length of the ring resonator; $T = L/c$ is the round-trip transit time for light in the resonator; T_1 is the longitudinal relaxation time; l is the active element length; $a = T_1c\sigma/(8\hbar\omega\pi)$ is the saturation parameter; $\sigma = \sigma_0/(1+\delta^2)$ is the cross section for the laser transition; $\delta = (\omega - \omega_0)/\Delta\omega_{\mathrm{g}}$ is the relative detuning of the laser frequency from the gain line centre; $\Delta\omega_{\mathrm{g}}$ is the gain linewidth; $\tilde{m}_{1,2} = m_{1,2}\exp(\pm\mathrm{i}\theta_{1,2})$ are the complex coefficients of coupling between counterpropagating waves through backscattering; $m_{1,2}$ and $\theta_{1,2}$ are the moduli and phases of the coupling coefficients; $\varDelta = \frac{1}{2}(\omega/Q_2 - \omega/Q_1)$ is the amplitude nonreciprocity of the resonator. The rest of the terms determine changes in the spatial harmonics N_0 and $N_{\pm 2}$ during saturation of the population of operating levels by the field of interfering counterpropagating waves.

These equations were derived assuming that the optical frequency ω is equal to the half-sum of the resonator eigenfrequencies: $\omega = (\omega_1 + \omega_2)/2$.

The first term in the right-hand side in equations for $\tilde{E}_{1,2}$ determines the decay rate of the fields of counterpropagating waves due to intracavity losses. The second term takes into account changes in the complex amplitudes of counterpropagating waves at frequencies $\omega_1 - \omega = \Omega/2$ and $\omega_2 - \omega = -\Omega/2$, where $\Omega = \omega_1 - \omega_2$ is the frequency nonreciprocity of the resonator. The third term in equations for $\tilde{E}_{1,2}$ determines changes in the fields of counterpropagating waves due to a linear feedback between them, which appears because of backscattering of the waves in the resonator. The last term takes into account the amplification and nonlinear coupling between counterpropagating waves, which are determined by polarisation of the medium.

During the interaction of counterpropagating waves with the active medium, inversion population gratings are induced in the latter, which can be described by the spatial Fourier components

$$N = \sum_{-\infty}^{+\infty} N_s \exp \mathrm{i}k_s z.$$

With a sufficiently good accuracy, we can take into account only the three first Fourier components

$$N_0 = \frac{1}{l}\int_0^l N\mathrm{d}z, \quad N_{\pm 2} = \frac{1}{l}\int_0^l N\mathrm{e}^{\pm \mathrm{i}2kz}\mathrm{d}z, \quad N_{+2} = N_{-2}^*,$$

which are the complex amplitudes of the spatial harmonics of the inverse population N.

The pump rate in equations for these components is written in the form $N_{\mathrm{th}}(1+\eta)/T_1$, where η is the excess of the pump power over the lasing threshold.

It was shown in many papers (see Refs [24–30] and other papers) that this model well describes a real situation, although in some cases (for example, for the description of polarisation characteristics of radiation) it should be refined. The mathematical model of the SRL allows us not only to find possible lasing regimes but also to determine the regions of their existence, and to solve the problem of stability of these regimes and their bifurcations.

This model can be applied to both bidirectional ring lasers and travelling-wave lasers (unidirectional lasing). In the latter case, the Q factors of the resonator for counterpropagating waves should be assumed different (for example, $Q_1 \gg Q_2$).

3. Non-stationary lasing regimes of solid-state ring lasers

All nonlinear systems (including ring lasers) can be divided into two groups: autonomous and non-autonomous. The operating regime of autonomous systems is completely determined only by their internal properties, whereas the properties of nonautonomous systems depend on the external control. In particular, ring lasers with modulated parameters (for example, ring lasers with the modulated pump power) belong to nonautonomous systems. Note that the excitation of non-stationary chaotic lasing regimes in a nonautonomous laser is considerably alleviated.

3.1 Methods of the dynamic chaos identification

The study of chaotic processes appearing in deterministic nonlinear systems (in our case, in SRLs) is one of the fundamental problems of physics. It has been shown in many experiments that chaotic deterministic lasing regimes observed in lasers appear not due to external and/or internal fluctuations (technical or quantum) but due to the realisation of the conditions provided by a certain combination of laser parameters, under which the phase trajectories come apart exponentially. Note that radiation parameters in the case of deterministic chaos fundamentally differ from the characteristics of chaotic lasing stimulated by external or internal fluctuations.

To describe all the features of deterministic quasi-periodic and chaotic oscillations and to analyse the mechanisms of transitions between different lasing regimes caused by a change in the control parameter, the study of only one of the oscillation characteristics is insufficient. It is necessary, as a rule, to use a combination of different characteristics (temporal realisations of the process, spectral power density, spectrum of the Lyapunov exponents (coefficients), Poincare mapping, bifurcation diagrams, etc.).

Consider some criteria that permit the identification of dynamic chaos in SRLs. Among radiation characteristics allowing the classification of non-stationary lasing regimes in ring lasers, the time dependences of the intensities $I_{1,2}(t)$ of counterpropagating waves, their power spectra $J_{1,2}(\omega)$, and phase portraits play an important role. Speaking about the time dependences, we should bear in mind that lasing regimes are possible in which the repetition periods of chaotic radiation pulses can coincide with the modulation period or will be a multiple of it. In addition the intervals between the pulses can be irregular and not related to the modulation period.

Depending on the relative temporal positions of counterpropagating radiation pulses and their amplitudes, radiation pulsations can be either in-phase or out-of-phase. In the case of in-phase oscillations, pulsations in the counter directions can be in-phase (when a greater peak in one direction corresponds to a greater peak in the opposite direction) or out-of-phase (when a greater peak in one direction corresponds to a smaller peak in the opposite direction). Finally, the so-called generalised chaos is possible. In this case, the chaotic intensities of counter-propagating waves are related by the functional dependence $I_2(t) = f(I_1(t))$.

An important characteristic used to classify the lasing regimes in ring lasers is the radiation power spectrum in counter directions. While the power spectrum of the periodic or quasi-periodic regime is characterised by a limited set of discrete spectral components, this spectrum in the dynamic chaos regime is characterised by a relatively broad band. The power spectrum of chaotic oscillations can exhibit against a broad noise background the intense discrete components at frequencies coinciding with or multiples of the frequencies of self-modulation and relaxation oscillations, as well as of the modulation frequency of the control parameter. In some cases, the power spectrum can be of the 'noisy' type. One should bear in mind in this case that the radiation power spectra in counter directions can be either identical or different (i.e., the spectral nonreciprocity is possible). If the spectrum contains discrete components, then components with the maximum intensity can have different frequencies in counter directions.

Lasing regimes can be also classified with the help of phase portraits and Poincare cross sections. The criterion showing the presence of chaotic oscillations is also the form of the correlation function $K = \langle I_1(t+\tau)I_1(t)\rangle$, which exponentially decays with increasing τ in the case of dynamic chaos.

The criterion allowing one to determine unambiguously the possibility of the dynamic chaos appearing is the presence of the positive Lyapunov coefficients (exponents), whose spectra give quantitative information on the average stability of a phase trajectory. The type of lasing can be determined directly by calculating the Lyapunov coefficients from experimental data (which is, however, rather difficult to do). It is also possible to reproduce first the experimental realisation of the nonlinear process using numerical simulations and then to calculate the Lyapunov coefficients for this model.

Note that the difference between various quasi-periodic and chaotic lasing regimes in nonautonomous ring lasers can be caused by the difference in the phase dynamics of counterpropagating waves [31 – 34].

3.2 Conditions for the appearance of dynamic chaos in ring lasers

The possibility (or impossibility) of existence of chaotic regimes in a nonlinear system is determined from the theoretical point of view by the dimensionality of a mathematical model describing the nonlinear system or by the dimensionality of the corresponding phase space [35 – 38].

For example, the mathematical model of an autonomous unidirectional ring laser oscillating at the centre of the homogeneously broadened gain line represents the second-order differential equation. In this case, the phase space is a plane, and only periodic self-oscillations can appear. The geometrical image of the established self-oscillations in the phase space is an attractor – a trajectory (or a set of trajectories) located within a limited region of the phase space and attracting all the near trajectories. Because the trajectories cannot intersect in the phase plane, a simplest nontrivial attractor appears in the system – a closed trajectory, which is called the limit cycle. The limit cycle determines the amplitude, period (the time of motion of an imaging point over the limit cycle), and shape of oscillations.

In the case of single-mode bidirectional SRLs, which are described in the general case by the seven-order system of differential equations, the situation is more complicated [21 – 23]. This model already admits the existence of not only periodic and quasi-periodic but also chaotic oscillations.

In the case of excitation of dynamic chaos, an attractor is also present: all the trajectories in the phase space are located within a limited region, which, however, contains neither stable states nor limit cycles. Such an attractor is called a strange attractor [35 – 38]. It represents a set of attracting trajectories, each of them being unstable. The strange attractor has two substantial features: its trajectories are non-periodic and are not closed.

Note that, although the nonlinear system of equations describing bidirectional lasing in an autonomous SRL admits the existence of quasi-periodic and chaotic oscillations in the laser, nevertheless dynamic chaos can be realised in the autonomous ring laser only within rather narrow regions of variation of laser parameters [3 – 5, 22].

The modulation of parameters of a ring laser can be efficiently used to control the nonlinear dynamics and excite quasi-periodic and chaotic oscillations. A specific feature of bidirectional ring lasers is a large number of control parameters (compared to a unidirectional ring laser), whose modulation can change the type of lasing, resulting, in particular, in the appearance of quasi-periodic and chaotic regimes. Such parameters include the excess over the pump threshold, intracavity losses, frequency and amplitude nonreciprocities of the resonator, coupling coefficients for counterpropagating waves, and some other parameters.

The theoretical and experimental studies of SRLs have shown that the modulation of laser parameters can produce a number of instabilities appearing before the realisation of dynamic chaos. Among the most probable scenarios of transition to chaos with increasing the control parameter, we can point out the transition through bifurcations of the period doubling. The second probable scenario is a quasi-periodic transition to chaos: the chaotic regime appears due to instability of quasi-periodic lasing. The transition to chaos can also occur through intermittencies characterised by alternating periods of quasi-periodic and chaotic pulsations of radiation [35 – 37].

Note that, when a new regime appears, the previous regime can lose its stability or even disappear, while the new regime can either acquire some properties of the previous one or can be fundamentally different (bifurcation).

The asymmetry of the system plays an important role in nonlinear systems with strange attractors [39]. In particular, the degree of asymmetry (for example, the amplitude nonreciprocity of the resonator) determines to a great extent the dynamic properties of ring resonators in the case of parametric resonance [4].

3.3 Classification of non-stationary lasing regimes in ring lasers

Many studies [40 – 72] have shown that a variety of non-stationary regimes can exist (along with periodic self-modulation lasing regimes) under certain conditions in SRLs. In this paper, we focus our attention on quasi-periodic and chaotic lasing regimes in autonomous lasers and also nonautonomous lasers with harmonically modulated parameters. The difference between lasing regimes is manifested in a number of their qualitative features: the type of time dependences of the intensities of counterpropagating waves, their phase relations, the difference in the spectra of counterpropagating waves, phase portraits, Lyapunov coefficients, etc. Note that lasing regimes cannot be classified unambiguously with the help of only one of the radiation parameters.

At present there exists a distinct classification of possible non-stationary lasing regimes in autonomous ring lasers. The main non-stationary regimes are self-modulation of the first and second kinds, beats of counterpropagating waves, and dynamic chaos regimes. In lasers with periodically modulated parameters, much more non-stationary regimes can exist, which have not been classified so far.

By analysing the results of numerous experimental and theoretical studies, we can distinguish the following non-stationary lasing regimes appearing in nonautonomous SRLs, classifying them mainly by the temporal and spectral radiation parameters of ring lasers:

(i) Quasi-sinusoidal (QS) lasing regimes in which the time dependence of the radiation intensity is described by sinusoidal oscillations with a periodic low-frequency envelope. Such regimes appear when the modulation amplitude of the control parameter is small; and the radiation intensity is well described by the expression $I_{1,2} = I_0 \cos(\omega_p t) \times \cos(\omega_m t \pm \varphi_{1,2})$. The QS regimes can be of two types: the QS-I regime, when a component at the frequency ω_m of self-modulation oscillations of an autonomous laser dominates in the emission spectrum, and the QS-II regime, when a component at the modulation frequency ω_p of the control parameter dominates. In a certain region of parameters, both types of the QS regime can be stable (bistability). The QS-I regime is characterised by the antiphase dynamics of the appearance of quasi-periodic radiation pulses of counterpropagating waves. In this case, the total intensity is $I = I_1 + I_2$ and oscillations at the self-modulation frequency ω_m are suppressed or can be even absent under certain conditions. In the QS lasing regimes, the spectrum of the Lyapunov coefficients does not contain positive coefficients.

(ii) Regimes of forced synchronisation of self-modulation oscillations (FS regimes) representing harmonic oscillations of the intensities of counterpropagating waves at the frequency ω_p.

(iii) Periodic and quasi-periodic impulse (QPI) regimes in which the counterpropagating waves follow in the form of rather short pulses at the pulse repetition period equal to or a multiple of the modulation period of the control parameter. The peak intensity of radiation pulses can be either constant or can have a low-frequency envelope. In the latter case, regimes are possible in which the envelopes for counterpropagating waves are in phase (the QPI-1A regime) or out of phase (the QPI-1B regime).

Hereafter, the numbers 1 or 2 in the regime notation mean that synchronous or nonsynchronous pulsations of the intensities of counterpropagating waves occur. The letters characterise the behaviour of the low-frequency envelopes of counterpropagating waves: A or B mean that oscillations occur in phase or out of phase, respectively; while C means that there exists a complicated functional dependence between the envelopes of the intensities of counterpropagating waves. If the envelope period is a multiple of the pulse repetition period, such a regime is periodic, otherwise it is quasi-periodic. Note that there exist periodic pulsed regimes, which differ in the number n of pulses per period (nT regimes). The emission spectrum in QPI regimes is discrete. The spectrum of the Lyapunov coefficients, as in the case of the QS and FS regimes, does not contain positive coefficients.

(iv) Strange-nonchaotic-attractor (SNA) regimes. Unlike periodic and quasi-periodic regimes, SNA regimes are characterised by a complex geometrical structure of the attractor, the irregular temporal structure, and a complicated emission spectrum. The time dependence of the leading Lyapunov exponent λ_0 is irregular and its mean value is zero. However, the local values of λ_0 are positive during some time intervals. The distribution of local values of λ_0 is characterised by a function with a maximum at $\lambda_0 = 0$.

In the case of chaotic regimes, two phase trajectories located within a small region of the phase space come apart with time (mixed) and occupy the entire region of the attractor (the so-called trajectory mixing; see details, for example, in Ref. [37]). In the case of the strange nonchaotic attractor, no such mixing occurs.

(v) Discrete-spectrum chaotic (DSC) regimes. The oscillation spectra in these regimes exhibit intense discrete components against a relatively weak 'noise' background (for example, at the frequencies equal to or multiple of the external modulation frequency ω_p, self-modulation (ω_m) or relaxation (ω_r) frequencies or their combinations). As all other regimes of dynamic chaos, DSC regimes are characterised by the exponential instability of phase trajectories, which is determined by the presence of the positive Lyapunov coefficients. The narrowband chaos varieties can also exist, when the component at the frequency ω_m or ω_p can be the most intense component in the spectrum. As a rule, DSC regimes appear in bidirectional ring lasers when the modulation frequency of the control parameter is lower than the main relaxation frequency ω_r.

(vi) Continuous-spectrum chaotic (CSC) regimes. These regimes are characterised by a broad continuous spectrum, which can exhibit weak spectral components at the frequencies ω_p, ω_m, and ω_r. The regimes are also characterised by the presence of several positive Lyapunov coefficients.

(vii) Intermittent chaotic (IC) regimes in which the regions of chaotic oscillations with different types of radiation are alternated through irregular time intervals.

The above classification is not exhaustive. In particular, it contains no information on the presence or absence of synchronisation between counterpropagating waves, which could be used to divide the above-considered regimes into several groups.

Note that synchronisation processes have a number of specific features in the case of chaotic lasing regimes. It may appear that the synchronisation of chaotic oscillations is impossible because the phase trajectories in the interacting systems are unstable. However, it has been shown in many studies that in this case the relative motion of phase trajectories of the interacting subsystems often proves to be stable, resulting in the synchronisation of chaotic oscillations.

Studies have shown that several different synchronisation regimes can exist in chaotic systems. For example, the identical-synchronous-chaos regime can exist in which the temporal and spectral characteristics of coupled chaotic systems coincide. In a more general case, the synchronisation of chaotic oscillations is manifested in the establishment of a certain functional dependence $x_1(t) = F(x_2(t))$ between chaotic oscillations $x_1(t)$ and $x_2(t)$ of the subsystems. This regime is called the generalised synchronisation regime [73].

In the synchronous chaos regimes, the dynamics of the intensity and/or optical phases of counterpropagating waves are correlated, unlike the case of nonsynchronous chaos. The transition from synchronous to nonsynchronous chaos can occur in certain regions of the SRL parameters.

Therefore, depending on the presence or absence of synchronisation in counterpropagating waves, the DSC and CSC regimes can be divided into two subgroups:

(1) The DSC-1 and CSC-1 regimes, which are characterised by the synchronisation of chaotic pulsations in counterpropagating waves; and

(2) the DSC-2 and CSC-2 regimes, in which synchronisation of counterpropagating waves is absent.

In turn, there exist among synchronous chaos regimes (the first-group regimes) at least three different regimes: identical synchronous chaos (DSC-1A and CSC-1A), synchronous chaos with the antiphase dynamics of counterpropagating waves (DSC-1B and CSC-1B), and generalised synchronous chaos (DSC-1C and CSC-1C).

In the case of chaotic oscillations, there also exist regimes of phase synchronisation, when the correlation between the amplitudes of chaotic oscillations can be absent, the instant frequencies of the interacting systems fluctuate and can differ from each other, but their mean values prove to be identical. In this case, the phase difference of oscillations in coupled subsystems varies within a limited interval.

Synchronisation processes in ring lasers can be considered both applied to oscillations at the optical frequencies of counterpropagating waves and to the oscillations of their envelopes. Note that it is the dynamics of envelopes (intensities) of counterpropagating waves that has been analysed in most experimental studies, while information on the optical phases of the fields is often absent.

In the case of a bidirectional ring laser, the interacting subsystems are counterpropagating waves. As was mentioned above, the coupling between them is determined by backscattering and self-diffraction of radiation from inverse population gratings induced by the interfering fields in the active medium. The coupling between counterpropagating waves on inverse population gratings is nonlinear, which introduces certain features to the phase dynamic of radiation in the regimes of synchronisation of counterpropagating waves. Upon modulation of the parameters of a ring laser, the laser itself and an external oscillator providing modulation can be treated as the interacting subsystems. In this case, one of the subsystems proves to be completely independent.

4. Parametric processes and dynamic chaos in autonomous ring lasers

In autonomous ring lasers, a number of stationary lasing regimes are possible, whose characteristics are well studied, and the regions of existence of these regimes are known. It has been found that single-mode cw SRLs in the absence of external perturbations (at different values of their parameters) can operate in the travelling-wave regime (unidirectional single-frequency modulation), self-modulation regimes of the first and second kinds, the beat regime, parametric-resonance regime and strong-coupling regimes, and some other regimes [42–50].

4.1 Self-modulation regime of the first kind

The most interesting and widespread lasing regime in SRLs is the self-modulation regime of the first kind, which is characterised by the sinusoidal antiphase modulation of the intensities of counterpropagating waves. The frequency of self-modulation oscillations depends on the coupling strength of counterpropagating waves, the frequency and amplitude nonreciprocity of the ring resonator and a number of other parameters. In the absence of nonreciprocities, the self-modulation frequency is described in the general case by the expression

$$\omega_{\rm m0}^2 = m_1 m_2 \cos(\theta_1-\theta_2) - \delta m_1 m_2 \sin(\theta_1-\theta_2) + \frac{(1+\delta^2) m_1^2 m_2^2 \sin^2(\theta_1-\theta_2)}{m_1^2+m_2^2+2m_1m_2\cos(\theta_1\theta_2)]}. \quad (3)$$

This expression is significantly simplified if the lasing frequency coincides with the gain line centre and the coupling between counterpropagating waves is symmetrical: $\omega_{\rm m0} = m\cos[(\theta_1-\theta_2)/2]$. If the coupling is caused, for example, by the inhomogeneities of the dielectric constant of the medium, then we have $|\theta_1-\theta_2| \ll 1$. In the presence of the frequency nonreciprocity Ω, the frequency of self-modulation oscillations is determined by the expression $\omega_{\rm m} = (\omega_{\rm m0}^2+\Omega^2)^{1/2}$. The conditions of the appearance and stability of this regime (in the absence of the resonator nonreciprocity) are determined by the inequalities

$$\left(\frac{\eta\omega}{QT_1}\right)^{1/2}\left|\cos\frac{\theta_1-\theta_2}{2}\right|^{-1} < m < \frac{\eta\omega}{3Q}\left|\sin\frac{\theta_1-\theta_2}{2}\right|^{-1}, \quad (4)$$

which are satisfied in a broad region of the SRL parameters. Here, we assume for simplicity that $m_1 = m_2 = m$.

Studies [21–25] have shown that, if in the absence of the frequency nonreciprocity (i.e., for $\Omega = 0$) the laser operates in the self-modulation regime of the first kind, this regime is preserved with increasing $|\Omega|$ in the region $|\Omega| \leqslant \Omega_1$. The boundary value of Ω_1 can be found from the expression

$$m\left|\sin\frac{\theta_1-\theta_2}{2}\right| = \frac{\omega}{Q}\eta\frac{m(m^2+\Omega_1^2)^{1/2}}{3m^2+2\Omega_1^2}.$$

For $|\Omega| > \Omega_1$, the self-modulation regime passes to stationary lasing with counterpropagating waves having unequal intensities. In the self-modulation regime of the first kind, the frequency of self-modulation oscillations increases with increasing $|\Omega|$, while the mean values of the intensities of counterpropagating waves become unequal. The frequency of self-modulation oscillations of the SRL in the absence of the resonator nonreciprocity is mainly determined by the strength of coupling between counterpropagating waves through backscattering and can vary from hundredths to a few megahertz.

Because the inverse population relaxes slowly ($\omega/Q_{1,2} \gg 1/T_1$), non-stationary processes of the establishment of the stationary state of self-modulation oscillations are of the oscillator type and are described by relaxation frequencies. The number of relaxation frequencies depends on the dimensionality (the number of independent variables) of the dynamic system.

Relaxation oscillations in the SRL operating in the self-modulation regime of the first kind occur at two characteristic frequencies. One of these frequencies (the main relaxation frequency)

$$\omega_{\rm r} = \left(\frac{\omega}{Q}\frac{\eta}{T_1}\right)^{1/2} \quad (5)$$

is similar to the relaxation frequency of a linear solid-state laser, while another corresponds to low-frequency relaxation oscillations and is determined by the expression [21]

$$\omega_{\rm r1} = \left\{\frac{1}{2}[\omega_{\rm r}^2+\omega_{\rm m}^2-(\omega_{\rm m}^4+2\omega_{\rm r}^2\Omega^2)^{1/2}]\right\}^{1/2}. \quad (6)$$

4.2 Dynamic chaos regimes in a bidirectional autonomous laser

The study of the self-modulation regime of the first kind showed that this regime is stable within the entire region of values of m determined by condition (4), except regions of parametric resonances. In an autonomous laser operating in the self-modulation regime, the parametric interaction between self-modulation oscillations is the main mechanism leading to bifurcations and a passage to dynamic chaos. It seems that chaotic oscillations can be excited only when the system is asymmetric: the conditions $m_1 \neq m_2$ and/or $Q_1 \neq Q_2$ are necessary. In the regions of parametric resonance, the self-modulation regime is unstable, period-doubling bifurcations appear, and a passage to dynamic chaos is observed under certain conditions. As shown in Refs [4, 5], the main parametric resonances are observed at

$$\omega_{\rm m} \simeq 2\omega_{\rm r},\ \omega_{\rm m} \simeq 2\omega_{\rm r1}, \text{and } \omega_{\rm m} \simeq \omega_{\rm r} + \omega_{\rm r1}. \tag{7}$$

The modulation of the laser parameters required for the appearance of parametric phenomena is produced due to periodic energy transfer between counterpropagating waves, which is typical for this lasing regime.

The self-modulation regime becomes unstable not only when conditions (7) are exactly satisfied but also in some region near the exact resonance (the width of this region can be a few tens of kilohertz). The parametric build-up of oscillations, for example, at the frequency $\omega_{\rm m} = 2\omega_{\rm r}$ occurs if $\omega_{\rm m}$ lies in the region $\omega_{\rm m1} \leqslant \omega_{\rm m} \leqslant \omega_{\rm m2}$. The boundaries of this region ($\omega_{\rm m1}$ and $\omega_{\rm m2}$) at $\theta_1 = \theta_2$ are determined by the expressions

$$\omega_{\rm m1} = A - B, \quad \omega_{\rm m2} = A + B, \tag{8}$$

where

$$A = 2\omega_{\rm r} + \frac{\omega_{\rm r}^2 + \Omega_0^2/4}{16\omega_{\rm r}} - \frac{(7\Omega_0^2 + 12\omega_{\rm r}^2)\omega_{\rm r}}{4(8\Omega_0^2 + 28\omega_{\rm r}^2)};$$

$$B = \left[\left(\frac{5\omega_{\rm r}^2}{(4\Omega_0^2 + 14\omega_{\rm r}^2)}\right)^2 4h^2\omega_{\rm r}^2 - \frac{1}{T_1^2}\right]^{1/2}; \tag{9}$$

$$\Omega_0^2 = (2\omega_{\rm r})^2 - m_1 m_2 > 0;$$

$$h = \frac{(m_1 m_2)^{1/2}(m_1 - m_2)}{2\omega_{\rm m}(m_1 + m_2)}.$$

One can see that the width of the region of parametric instability increases with increasing the difference of moduli of coupling coefficients. In the case of symmetric coupling ($m_1 = m_2$), the width of the instability region vanishes.

Studies of the radiation dynamics in parametric-resonance regions show that the build-up of relaxation oscillations and their interaction with self-modulation oscillations lead to a number of period-doubling bifurcations and the appearance of chaotic oscillations.

Dynamic chaos regimes can be excited in an autonomous SRL by several methods. Consider this question by the example of parametric resonance in the region $2\omega_{\rm r} = \omega_{\rm m}$. The frequency of self-modulation oscillations in the presence of frequency nonreciprocity Ω in the resonator depends on the parameters of the ring laser as [21]

$$\omega_{\rm m} = (\omega_{\rm m0}^2 + \Omega^2)^{1/2} + \Delta\omega, \tag{10}$$

where $\omega_{\rm m0}$ is determined by expression (3); $\Delta\omega = \omega_{\rm r}^2(\Omega^2 + \omega_{\rm m0}^2)/(4\omega_{\rm m0}^3)$ is the correction in the first approximation over a small parameter $\omega_{\rm r}^2/\omega_{\rm m0}^2$; and $\omega_{\rm r}$ is the main relaxation frequency [see (5)] [4].

In the presence of the amplitude nonreciprocity $\varDelta$ of the resonator and for $\Omega = 0$, the expression for the self-modulation frequency takes the form

$$\omega_{\rm m} = \left\{ m^2 \cos\frac{\theta_1 - \theta_2}{2} - \frac{[\omega\eta/(Q(1+\eta))]^2 \varDelta^2}{[2m\cos[(\theta_1 - \theta_2)/2]]^2} - \varDelta^2 \right\}^{1/2}. \tag{11}$$

It follows from expressions (3), (10), and (11) that the control parameter can be the frequency (Ω) and amplitude ($\varDelta$) nonreciprocities of the resonator, the laser frequency detuning δ from the gain line centre, the pump excess over the lasing threshold, the Q factor of the resonator, and moduli ($m_{1,2}$) and phases ($\theta_{1,2}$) of the coupling coefficients for counterpropagating waves.

The condition $2\omega_{\rm r} = \omega_{\rm m}$ was achieved in studies of lasing regimes near a parametric resonance [40, 52, 53] with the help of an external permanent magnetic field applied to an active element, which allowed the variation in the self-modulation frequency $\omega_{\rm m}$ in a broad range. In the absence of a magnetic field, the self-modulation frequency was $\omega_{\rm m}/2\pi = 78$ kHz and the frequency of relaxation oscillations was $\omega_{\rm r}/2\pi = 63$ kHz. A magnetic field applied to the active element of a chip laser provided the condition of the parametric resonance $2\omega_{\rm r} \simeq \omega_{\rm m}$. The external magnetic field also caused an increase in the amplitude nonreciprocity $\varDelta$ of the resonator. In the parametric resonance region, depending on the magnetic field strength and orientation, the period doubling was observed for self-modulation oscillations and also quasi-periodic or chaotic lasing regimes appeared. Figure 2 shows the characteristic temporal and spectral characteristics of the CSC and DSC lasing regimes, which appear in the chip laser in two magnetic fields with different strengths and orientations with respect to the chip-laser.

Due to the presence of several regions of parametric resonances, the lasing characteristics do not change monotonically depending on the frequency or amplitude nonreciprocity of the resonator: dynamic chaos passes to quasi-periodic pulsations, then dynamic chaos again appears, etc. It was shown experimentally that to excite the CSC regime, the amplitude nonreciprocity $\varDelta$ is needed, the width of the parametric-resonance region monotonically increasing with $\varDelta$ [22, 40]. A similar result was obtained by numerical simulations.

4.3 Dynamic chaos in a travelling-wave autonomous ring laser

The necessary condition for excitation of quasi-periodic and chaotic oscillations in a unidirectional autonomous ring laser is the presence of resonance between two relaxation frequencies [3–5, 22, 40]. In the case of a unidirectional ring laser, there exist three relaxation frequencies, one of which ($\omega_{\rm r}$) coinciding with the relaxation frequency of a linear solid-state laser (5) and the other two being described by the expressions

$$\omega_{\rm r1,2} = \left(\frac{\omega_{\rm r}^2}{2} + \frac{\Omega^2}{4}\right)^{1/2} \pm \frac{\Omega}{2}. \tag{12}$$

In the absence of the frequency nonreciprocity of the resonator, the frequencies $\omega_{\rm r1}$ and $\omega_{\rm r2}$ are degenerated. One can see from (12) that, when the condition

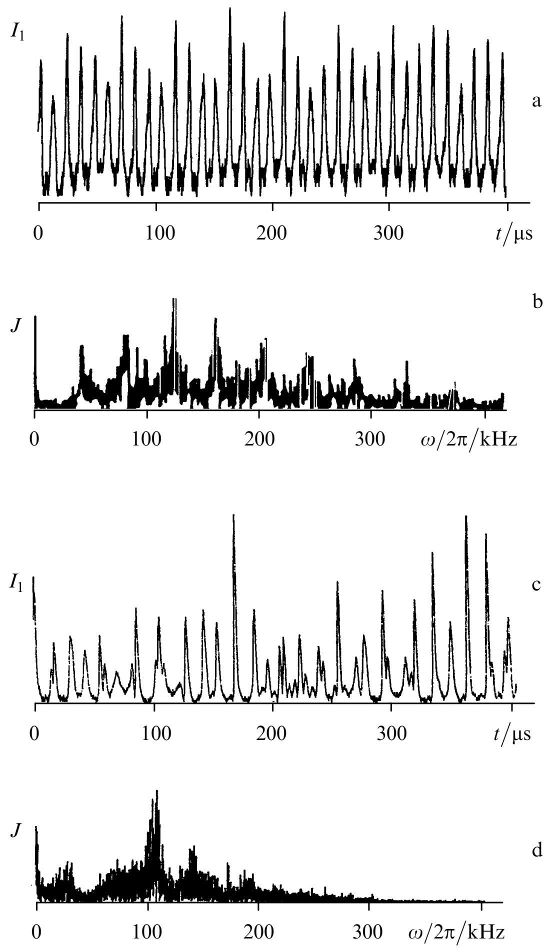

Figure 2. Radiation oscillograms for one of the counterpropagating waves (a, c) in the autonomous ring laser and its spectra (b, d) in the cases of the parametric resonance $2\omega_r \simeq \omega_m$ in the DSC (a, b) and CSC (c, d) regimes.

$$2\Omega = \omega_r \tag{13}$$

is fulfilled, the resonance appears between the two relaxation oscillations [in this case, $\omega_r = \omega_{r1}(\Omega)$]. The fulfilment of condition (13) violates the stability of the travelling-wave regime, and, as shown in Ref. [3], leads to the appearance of quasi-periodic and chaotic lasing regimes in the autonomous ring laser.

5. Lasing regimes of ring lasers with modulated parameters

Due to a high sensitivity of ring lasers to the modulation of their parameters, even relatively weak periodic modulation proves to be sufficient for the laser response to the external perturbation to become nonlinear, resulting in the appearance of a variety of quasi-periodic and chaotic lasing regimes classified above.

In a SRL with periodically modulated parameters, along with parametric resonances inherent in autonomous lasers, there also exist parametric resonances at the modulation frequencies equal to or multiple of the frequencies of relaxation and self-modulation oscillations. The SRL is very sensitive to the modulation of its parameters in parametric-resonance regions and passes to the dynamic chaos regime at a substantially lower degree of modulation of control parameters.

In nonlinear systems, along with the fundamental resonance, resonances at the overtones and undertones of the perturbing force are also possible, which substantially facilitates a change in lasing regimes [74]. A strong competition between counterpropagating waves in the active medium of the ring laser also favours the appearance of quasi-periodic and chaotic lasing regimes.

Consider now the basic properties of SRLs appearing during the external harmonic modulation of their parameters [65–69]. The nonlinear dynamics of bidirectional nonautonomous ring lasers is much more complicated than the lasing dynamics of unidirectional ring lasers. We consider first some features of the dynamics of travelling-wave lasers.

5.1 Bifurcations and the appearance of chaos in travelling-wave lasers

To obtain the travelling-wave regime in a ring laser, the amplitude nonreciprocity Δ is produced in the resonator (different Q factors of the resonator Q_1 and Q_2 for counterpropagating waves). In addition, the Q factors are periodically modulated:

$$\frac{\omega}{Q_{1,2}} = \pm\frac{\Delta}{2} + h\,\frac{\omega}{Q}\sin\omega_p t, \tag{14}$$

where ω/Q is the average width of the ring resonator band in the absence of modulation of losses; and h is the depth of modulation of losses. The lasing dynamics in the travelling-wave regime is well described by the system of equations (2) taking expression (14) into account.

Because a detailed analysis of all properties of the nonlinear dynamics of a travelling-wave ring laser in the different regions of parametric resonances is not the aim of this paper, we will consider here only the evolution of lasing regimes in a unidirectional ring Nd : YAG laser with the Q factor modulated at the frequencies lower than the relaxation frequency ω_r [3, 47]. In this region, a resonance can occur at the undertone of the fundamental relaxation frequency. Unidirectional lasing was provided in experiments by using an intracavity amplitude nonreciprocal element, and losses were modulated with the help of an intracavity amplitude modulator operating at the frequency $\omega_p/2\pi = 8 - 15$ kHz. The fundamental relaxation frequency $\omega_r/2\pi$ was 27 kHz.

Experimental studies and numerical simulations [47] showed that lasing regimes are changed with increasing the depth of modulation of losses in a laser in the following way. At first, the laser response to the modulation is harmonic, and a component at the frequency $\omega_p/2\pi$ appears in the emission spectrum. As the control parameter h is increased, the modulation depth of output radiation continuously increases and nonlinear distortions also increase. Then, at a critical value of $h = h_{cr}^{(1)}$ ($h_{cr}^{(1)} = 0.021$), lasing regime changes abruptly and the laser begins to emit pulses with the pulse repetition period $T = 2\pi/\omega_p$ (the QPI regime). As h is further increased, period-doubling bifurcations appear, and then, at $h_{cr}^{(2)} = 0.027$, dynamic chaos appears (the DSC regime). When $h_{cr}^{(3)} = 0.035$, the QPI lasing regime appears again; however, its period is already equal to $4T$.

5.2 Nonlinear dynamics in the region of modulation frequencies including a parametric resonance with self-modulation oscillations

Consider now quasi-periodic and chaotic lasing regimes and their basic characteristics in a bidirectional ring laser with parameters modulated periodically at the frequency ω_p [31–34, 48, 65–72]. The nonlinear dynamics can be studied experimentally by using various parameters of the ring laser as a control parameter. It is most convenient to modulate the pump power in monolithic chip lasers. The frequency and amplitude nonreciprocities of the resonator can be modulated quite easily by modulating a magnetic field applied to the active element. By producing periodic stresses in the active element, one can modulate the perimeter and Q factor of the resonator, its polarisation anisotropy, and the detuning of the laser frequency from the gain line centre. The additional feedback between counterpropagating waves provided by an external mirror makes it possible to change the moduli and phases of coupling coefficients. At present the nonlinear radiation dynamics of a bidirectional ring laser with modulated intracavity losses and pump power is studied in most detail.

The nonlinear dynamics of a nonautonomous laser is studied at fixed values of all laser parameters, except the control parameter, which is modulated at the frequency ω_p with the modulation depth h. In a ring laser operating in the self-modulation regime of the first kind, the regions of parametric resonances appear at the external-perturbation frequencies ω_p equal to or multiple of the relaxation frequencies ω_r and ω_{r1}, the self-modulation frequency ω_m, and the combination frequencies $n\omega_r + m\omega_m$, etc. In different regions of parametric resonances, a number of features appear in the nonlinear radiation dynamics of counterpropagating waves.

Consider first the nonlinear dynamics of a SRL in the region of modulation frequencies containing a parametric resonance with self-modulation oscillations, i.e., in the region of modulation frequencies close to the frequency ω_m. In this region, a number of specific effects appears, which are absent in other regions. Among these effects, the synchronisation of the frequency of self-modulation oscillations by an external modulating signal should be pointed out, which is possible only when the condition $\omega_p < \omega_m$ is fulfilled. Otherwise, synchronisation is absent [65]. In this region, multistability and hysteresis effects are observed [70, 71].

Theoretical and experimental studies of the radiation dynamics of bidirectional ring lasers with modulated pump or resonator losses showed that these modulation methods are almost identical. The only difference is that upon modulation of the Q factor, the modulation depth proves to be substantially greater. Below, we will consider mainly only a periodic modulation of the pump (because this modulation was used in most experiments). The theoretical analysis and numerical simulation in this case are based on a standard system of equations (2), in which the threshold-pump-power excess should be written in the form

$$\eta = \eta_0 + h\cos\omega_p t, \quad (15)$$

where η_0 is the threshold-pump-power excess in the absence of modulation.

The radiation dynamics was studied for monolithic chip lasers in which the self-modulation frequency was varied from 100 to 250 kHz, while the fundamental relaxation frequency was varied from 60 to 100 kHz. It was found (Fig. 3) that, when the modulation frequency ω_p of the control parameter approached the self-modulation frequency ω_m ($\omega_p < \omega_m$), the self-modulation frequency was locked to the external signal. In the locking region of width $\Delta\omega$, the self-modulation frequency proves to be equal to the frequency ω_p of the external force, and the emission spectrum consists of one component (the FS regime).

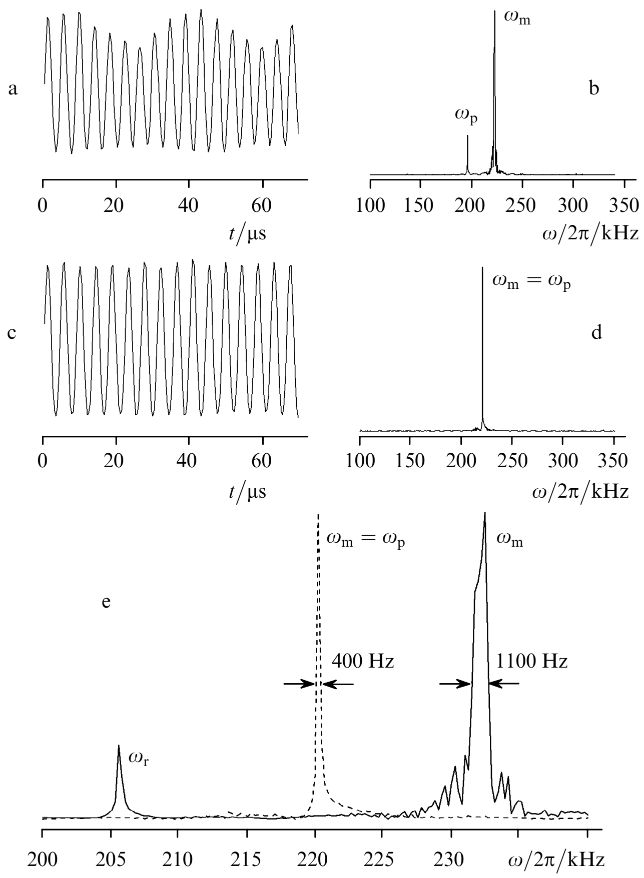

Figure 3. Oscillograms (a, c) and emission spectra (b, d) of the ring chip laser in the QS (a, b) and FS (c, d) regimes and the behaviour of the spectrum of self-modulation oscillations in the QS regime (solid curve) and FS regime (dashed curve) (e).

Similar results were also obtained in the theoretical study of the radiation dynamics of a ring laser. System of equations (2) has a periodic solution describing the synchronisation of the self-modulation frequency by an external signal. Note that an exact analytic solution can be obtained only for small values of the parameter h. Analysis performed in paper [65] gave the relation

$$\frac{T_1\omega_r^4 h\omega_p}{4(1+\eta_0)\eta_0\omega_m} = (\omega_m^2 - \omega_p^2)^{1/2}(\omega_p^2 - \omega_r^2) \quad (16)$$

for determining the width of the locking region of the self-modulation frequency by the external signal.

The values of ω_p satisfying relation (16) determine the left boundary of the region of existence of the synchronisation regime. The right boundary proves to be equal to the self-modulation frequency ω_m. Inside this region, self-modulation oscillations are synchronised by the external signal (radiation represents a harmonic oscillation with a period equal to the modulation period). An interesting feature of the locking regime is a considerable narrowing (almost by a factor of three) of the spectrum of self-modulation oscillations locked to the external signal, which was observed in experiments. This is distinctly illustrated in Fig. 3 where the emission spectra of a ring chip laser pumped by modulated radiation are shown in the absence and presence of frequency locking [70, 71].

A specific feature of the FS regime is the spontaneous appearance of the amplitude nonreciprocity: the radiation intensities of counterpropagating waves become unequal. There exist two FS regimes which differ in the phase shift of the intensity oscillations for counterpropagating waves with respect to the modulating signal. In one of the regimes, the wave propagating clockwise is suppressed with increasing detuning ($\omega_m - \omega_p$), while in another regime – the counter-clockwise propagating wave is suppressed.

The specific features of the nonlinear radiation dynamics of a particular ring laser pumped by modulated radiation can be studied using numerical simulations. In particular, this method was used [70, 71] to study the radiation dynamics of a ring chip laser ($\omega_m/2\pi = 230$ kHz, $\eta_0 = 0.21$, $\omega_r/2\pi = 70$ kHz, and $\omega/Q = 7 \times 10^6$ Hz) pumped by modulated radiation. The positions of the right and left boundaries of the region of stable synchronisation and the width of the locking region were found, and the dependence of these parameters on the pump power modulation depth h was also studied. It was found that the right boundary of the synchronisation region is independent of h and coincides with ω_m. The stable FS regime was observed within a much narrower region than the region of its existence determined by expression (16). For $h > 0.2$, the width of the region of stable synchronisation can achieve 50 kHz.

Outside the locking region, for small modulation depths of the control parameter, the QS regime appears in the laser. In this regime, the periodic modulation of the amplitude of self-modulation oscillations appears at the frequency ω_p, resulting in the emergence of new spectral components at the combination frequencies $\omega_m \pm n\omega_p$ in the spectrum, whose number increases with increasing h.

Near the boundaries of the region of stable synchronisation, hysteresis effects are observed. There exists a bistability region where, depending on the initial conditions, either the FS or QS regime appears. The widths of bistability regions near the left and right boundaries of the synchronisation region of self-modulation oscillations are considerably different, being equal to ~ 25 and ~ 5 kHz, respectively.

The performed studies showed that a passage to dynamic chaos in this region of modulation of control parameters is hindered. This can be explained by the fact that, as the pump modulation depth is increased, the width of the synchronisation region increases and the regions of existence of the QS and QPI regimes narrow down.

5.3 Nonlinear dynamics in the region of modulation frequencies including parametric resonances with relaxation oscillations

Let us now discuss the characteristic features of nonlinear dynamics in the regions of modulation frequencies including parametric resonances with relaxation oscillations. Consider first the nonlinear dynamics of a ring laser in the case of low-frequency pump modulation, when the modulation frequency lies in the region of resonances with the fundamental relaxation frequency ω_r or the additional relaxation frequency ω_{r1}, and the amplitude and frequency nonreciprocities of the resonator are absent.

The sequence of appearance of different regimes with increasing the modulation depth h of the control parameter depends not only on the parameters of the chip laser itself but also to a great extent on the modulation frequency $\omega_p/2\pi$ of the control parameter. Consider, for example, the sequence of appearance of different lasing regimes in a ring chip laser ($\omega_m/2\pi = 170$ kHz, $\omega_r/2\pi = 66$ kHz, and $\eta = 0.21$) with increasing the modulation depth for the fixed modulation frequency $\omega_p/2\pi = 50$ kHz, close to the relaxation frequency. In this case, the FS regime does not appear. For $h = 0$, the laser operates in the self-modulation regime, which passes to the QS and then QPI regimes with increasing h, which, in turn, pass to the CSC and IC regimes. As h is further increased, the DSC regimes appear in the laser.

At small modulation depths, bistability can exists and two different QS regimes appear. In one of these regimes (QS-I), a component at the frequency ω_m dominates in the spectrum, while in another (QS-II) – a component at the frequency ω_p of the modulating signal dominates (Fig. 4). In the QS-I regime, the antiphase radiation dynamics takes place: the intensities of counterpropagating waves have the form of antiphase self-modulation oscillations with the low-frequency envelope whose frequency coincides with $\omega_p/2\pi$.

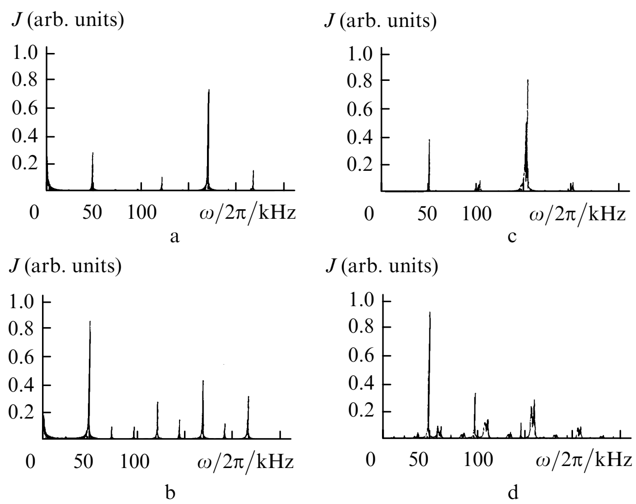

Figure 4. Calculated (a, b) and experimental (c, d) emission spectra of the ring laser in the QS-I (a, c) and QS-II (b, d) regimes for $h = 0.03$ and $\omega_p/2\pi = 50$ kHz.

As the modulation depth h is increased ($h > 0.1$), the QPI lasing regimes appear (Fig. 5) in which trains of short pulses of duration τ are emitted, the pulse duration being much shorter than the modulation period T_p coinciding with the pulse repetition period. In the presence of frequency nonreciprocity (see below), the QPI regimes can appear with the period equal to the doubled modulation period.

Dynamic chaos regimes emerge when a critical value h_{cr} of the modulation depth, which depends on the modulation frequency and laser parameters, is exceeded. In this case, a passage to chaos occurs according to a quasi-periodic

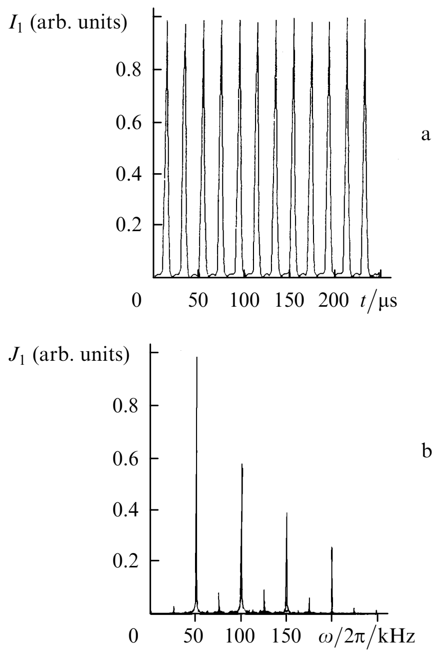

Figure 5. Oscillogram (a) and emission spectrum (b) of the ring laser in the QPI regime (experiment) for the pump modulation depth $h = 0.29$ and $\omega_p/2\pi = 50$ kHz.

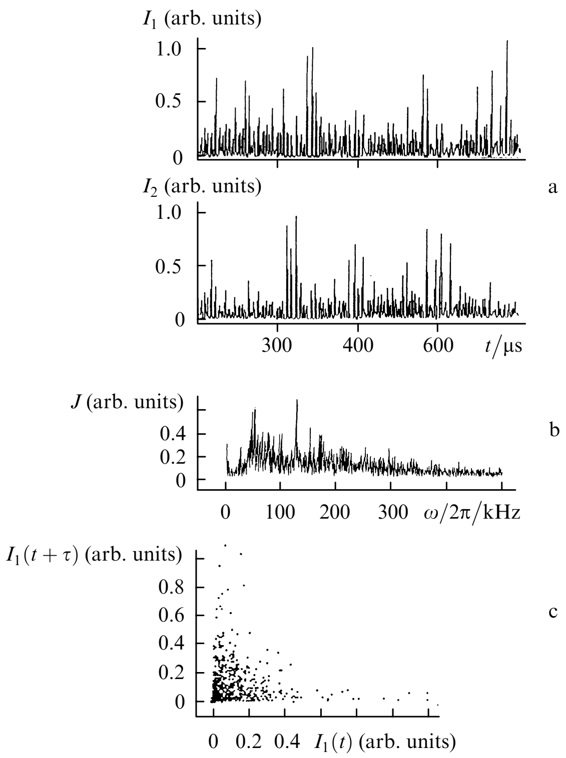

Figure 6. Emission characteristics in the CSC-2 regime (experiment): intensity oscillograms for counterpropagating waves (a), spectrum of one of the waves (b), and phase portrait (c) for $h = 0.4$ and $\omega_p/2\pi = 50$ kHz.

scenario: for $h < h_{cr}$, the attractor of the system represents a multidimensional torus, which becomes unstable at $h > h_{cr}$ and is destroyed, becoming a strange chaotic attractor (i.e., dynamic chaos appears).

In the presence of the amplitude nonreciprocity of the ring resonator, a regime with a complex time radiation dynamics, having a strange nonchaotic attractor, can appear before the dynamic chaos regime.

In the region of modulation frequencies of the control parameter, all the three types of chaotic regimes, DSC, CSC, and IC, can be observed. For $h > h_{cr}$, the CSC regime first appears, which passes to the DSC regimes with increasing the modulation depth h. The typical time dependence of the intensities of counterpropagating waves and the emission spectrum for one of the waves in the CSC-2 regime are shown in Fig. 6. In this regime, the intervals between adjacent pulses prove to be irregular and the emission spectrum is nearly continuous.

Figure 7 shows similar dependences for one of the DSC regimes. One can see that the pulses of counterpropagating waves with chaotic amplitudes propagate with regular time intervals equal to the modulation period, and the spectrum consists of intense discrete components against a weak noise background.

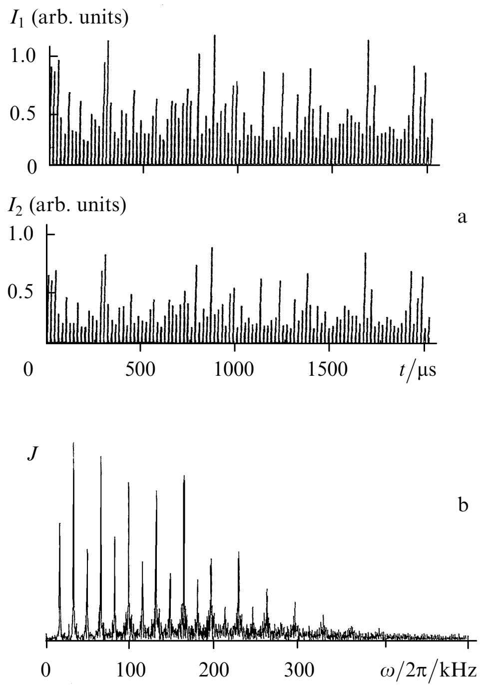

Figure 7. Emission characteristics in the DSC-1A regime (experiment): intensity oscillograms for counterpropagating waves (a) and spectrum of one of the waves (b) for $\omega_p/2\pi = 50$ kHz and $h = 0.46$.

There exists the region of laser parameters where the IC regime is observed. The radiation dynamics in this case is characterised by alternating regions with different lasing parameters. For example, regions can alternate in which the chaotic intensities of counterpropagating waves change synchronously with the regions of nonsynchronous chaos in counter directions. The IC regime was experimentally observed in a ring chip laser with the parameters $\omega_m/2\pi = 136$ kHz, $\omega_r/2\pi = 90$ kHz, $\eta = 0.5$, and $\omega_p/2\pi = 121$ kHz. Figures 8a, b show the intensity oscillograms for counterpropagating waves obtained in this regime at different time-base sweeps. The oscillograms demonstrate the alternating regions of synchronous chaos with regular intervals between pulses (equal to the pump modulation period) and the regions of nonsynchronous chaos in counter directions, when intervals between pulses are irregular.

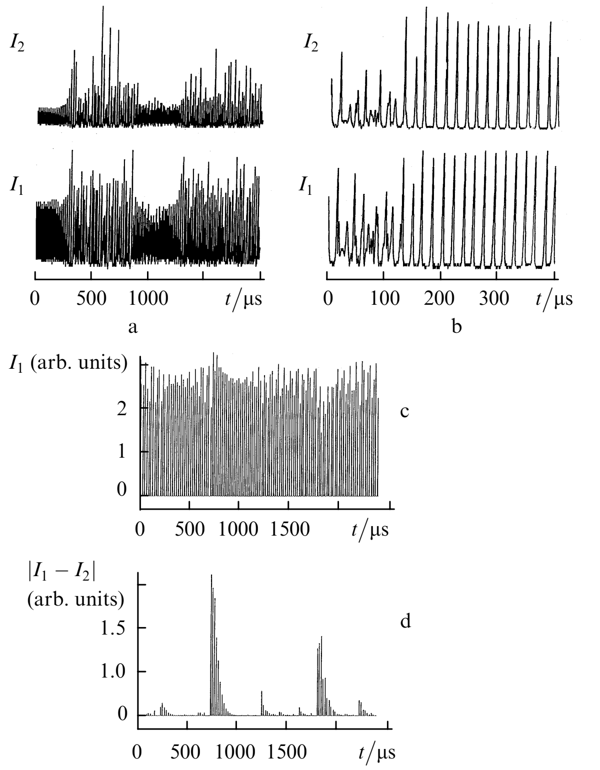

Figure 8. Intensity oscillograms for counterpropagating waves (on the right are fragments of the time-base sweep) in the IC regime (a, b) and the calculated time dependences of the intensity of one of the waves (c) and the modulus of the difference of intensities of counterpropagating waves (d) for $\omega_p/2\pi = 121$ kHz and $h = 0.27$.

It is convenient to study the IC regimes by detecting the time dependence of the difference between the intensities of counterpropagating waves. The oscillograms of the intensity difference allow one to detect the IC regime even when it is difficult to observe this regime by the wave intensity oscillograms. Thus, the time dependence of the radiation intensity of one of the waves shown in Fig. 8c does not give reliable information on the presence of intermittent regions. At the same time, the time dependence of the difference of intensities for counterpropagating waves for this case (Fig. 8d) demonstrates the presence of intermittent regions of synchronous and nonsynchronous chaos.

As mentioned above, in some regions of parameters of nonautonomous SRLs, bistability was observed and quasi-periodic bistable QS-I and QS-II regimes can appear. As the modulation depth is increased, these regimes pass to dynamic chaos at different values of h. As a result, the quasi-periodic regime and dynamic chaos can be stable simultaneously, and two different dynamic chaos regimes can be also bistable.

Note that the scenario of a passage to chaos somewhat changes in the case of the parametric resonance $\omega = 2\omega_r$. Figure 9 shows the results of numerical simulations obtained using parameters of a chip laser close to their experimental values ($\omega_m/2\pi = 170$ kHz, $\omega_r/2\pi = 66$ kHz, $\omega_p/2\pi = 130$ kHz and $\eta = 0.21$). The results demonstrate the dependences of the Poincare cross sections and the lasing spectra of the solid-state ring on the modulation depth. One can see that the passage to dynamic chaos is preceded by the torus-doubling bifurcation.

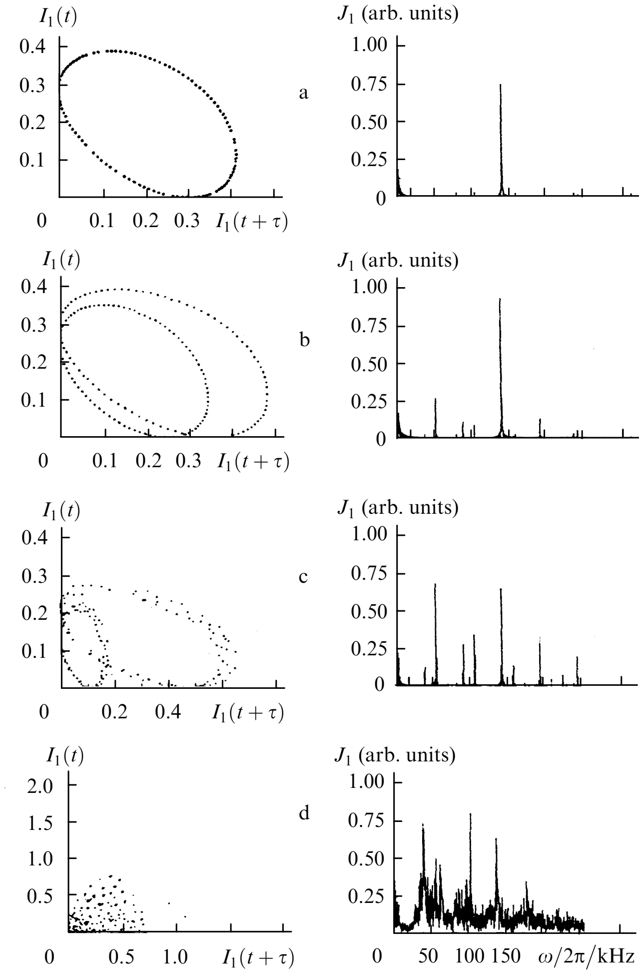

Figure 9. Scenario of a passage to chaos upon modulation of losses at the frequency $\omega_p/2\pi = 130$ kHz: changes in the Poincare cross sections (left) and lasing spectra (right) with increasing (downward) the modulation depth of the control parameter h (numerical simulation); $I_1(t)$ and $I_1(t+\tau)$ are given in arbitrary units.

5.4 Synchronisation of counterpropagating waves in chaotic lasing regimes

Synchronisation is one of the characteristic phenomena taking place in ring bidirectional lasers and determining to a great extent their nonlinear dynamics. Conventionally, synchronisation means the establishment of certain relations between the oscillation frequencies or phases of the interacting subsystems. Synchronisation can be both external (forced) and internal (reciprocal). In the first case, the synchronisation (locking) of oscillations is performed by an external control signal, while in the second case, it occurs due to the interaction between equivalent subsystems. The appearance of synchronisation (both external and internal) in self-oscillating systems operating in periodic and quasi-periodic regimes is characterised by the presence of self-oscillations of the radiation intensity with a constant and rational value of the Poincare rotation number $\vartheta = \omega_p/\omega = m/n$ (m and n are integers). This regime is preserved in a finite region of the system parameters, which is called the synchronisation region.

As mentioned above, in the case of chaotic regimes, the synchronisation of counterpropagating waves has a number of characteristic features, which allow one to divide dynamic chaos regimes into several subgroups. Let us illustrate by several examples the main synchronisation regimes of chaotic oscillations of counterpropagating waves appearing in solid-state lasers with periodically modulated parameters. The case of identical synchronous chaos (the DSC-1A regime) is shown in Fig. 10. Here, the phase portrait in the (I_1, I_2) plane is a straight line $I_1(t) = I_2(t)$. Identical synchronisation can occur in the DSC and CSC regimes. The identical synchronous chaos regime appears in the laser in the absence of the frequency and amplitude nonreciprocities of the ring resonator. In the presence of nonreciprocity, other synchronous chaos regimes can take place (Figs 11 and 12). One of them (the DSC-1B regime) is characterised by the antiphase dynamics of chaotic pulse $I_1(t)$ and $I_2(t)$ in counterpropagating waves (Fig. 11), which is manifested in the fact that the sum intensity $I_1 + I_2$ proves almost constant. In this case, the phase portrait in the (I_1, I_2) plane consists of two 'lobes'.

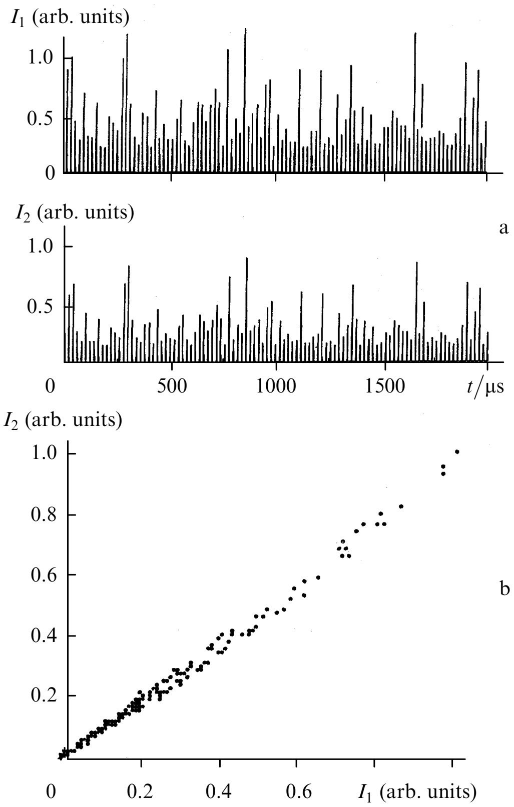

Figure 10. Intensity oscillograms for counterpropagating waves (a) and the projection of the phase portrait on the (I_1, I_2) plane (b) in the case of identical synchronous chaos (the DSC-1A regime) for $\omega_p/2\pi = 50$ kHz and $h = 0.24$ (experiment).

The generalised synchronisation regime (DSC-1C) is shown in Fig. 12. Figure 12a demonstrates the time behaviour of the intensities $I_{1,2}(t)$ of counterpropagating waves in this regime. The phase portrait (Fig. 12b) in the (I_1, I_2) plane shows that the intensities $I_1(t)$ and $I_2(t)$ are related by the dependence $I_2(t) = F(I_1(t))$ representing a closed curve.

Synchronous chaos regimes exist within certain regions of laser parameters (synchronisation regions of chaotic oscillations), which a part of the region of existence of dynamic chaos. Outside synchronisation regions of counter-

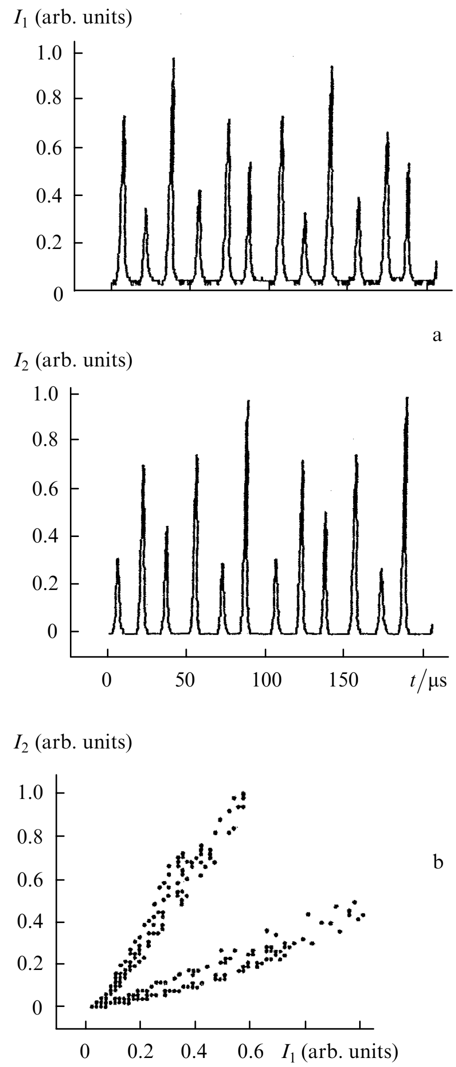

Figure 11. Oscillograms of the intensities $I_1(t)$ and $I_2(t)$ (a) and the projection of the phase portrait on the (I_1, I_2) plane (b) in the synchronous chaos regime with the antiphase dynamics of chaotic pulses in counterpropagating waves (the DSC-1B regime).

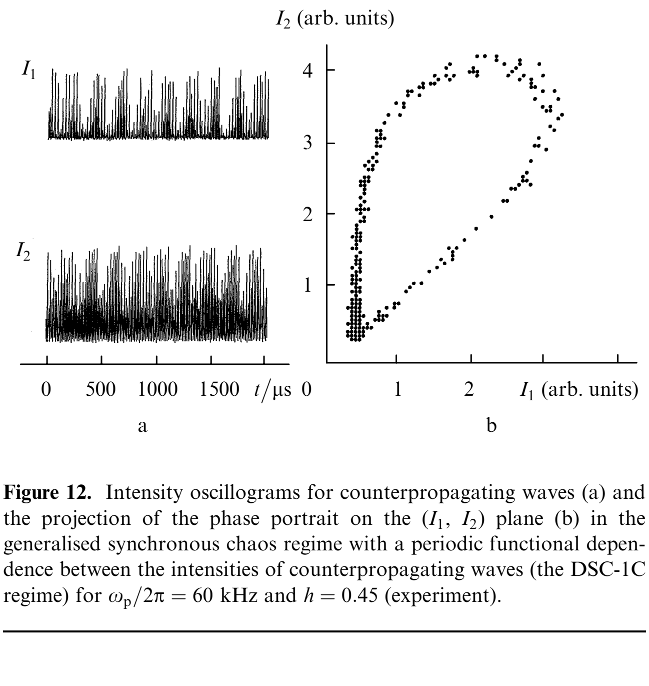

Figure 12. Intensity oscillograms for counterpropagating waves (a) and the projection of the phase portrait on the (I_1, I_2) plane (b) in the generalised synchronous chaos regime with a periodic functional dependence between the intensities of counterpropagating waves (the DSC-1C regime) for $\omega_p/2\pi = 60$ kHz and $h = 0.45$ (experiment).

propagating waves, the IC regimes and non-synchronous chaos regimes can be observed. Note that in the nonsynchronous chaos regime the spectral nonreciprocity can appear. In this case, the lasing spectra of counterpropagating waves prove to be different. Figure 6 shows the time evolution of the intensities of counterpropagating waves and the phase portrait of the nonsynchronous chaos regime (CSC-2).

The synchronisation of intensities of counterpropagating waves is also absent in the strange-nonchaotic-attractor (SNA) regime, which appears under certain conditions. The oscillograms of intensities of counterpropagating waves and their spectra in this regime are shown in Fig. 13. It is interesting that spectral nonreciprocity (the difference between the envelope spectra for counterpropagating waves) can appear in the SNA regime.

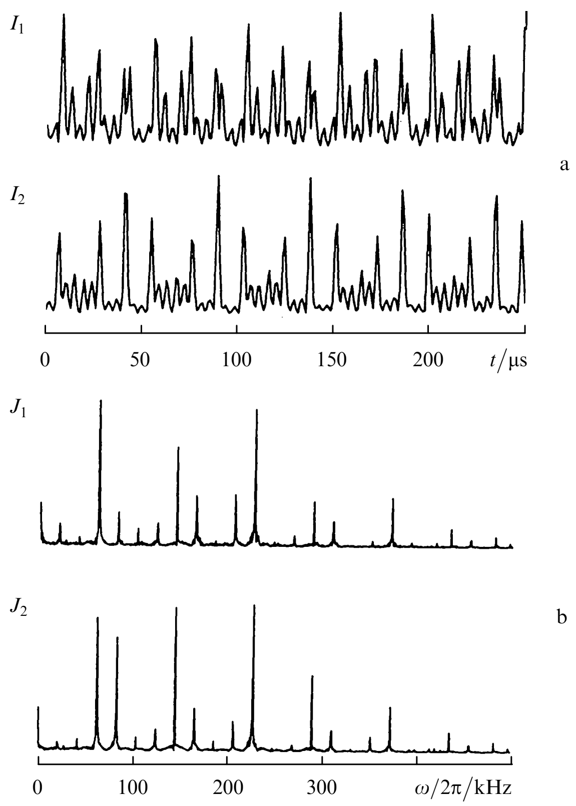

Figure 13. Intensity oscillograms for counterpropagating waves (a) and their spectra (b) in the SNA regime for $\omega_p/2\pi = 60$ kHz and $h = 0.5$ (experiment).

5.5 Phase dynamics in chaotic lasing regimes

Important information on the characteristics of quasi-periodic and chaotic lasing regimes of bidirectional SRLs can be obtained by studying the dynamics of optical phases of counterpropagating waves. The features of phase synchronisation of chaotic oscillations have been recently widely discussed in the literature [31–34, 74–76]. However, theoretical studies in this field cannot predict yet the properties of the phase dynamics for specific nonlinear systems and, therefore, the experimental study of the phase dynamics in chaotic lasing regimes is undoubtedly of great interest.

The phase dynamics of ring lasers can be studied by the interferometric method (Fig. 14) by detecting a mixing signal of counterpropagating waves. The intensity of this signal can be written in the form

$$I_{pm} = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos\varphi, \quad (17)$$

where $I_{1,2}$ is the radiation intensities of counterpropagating waves of the same polarisation and φ is the optical phase difference for interfering waves.

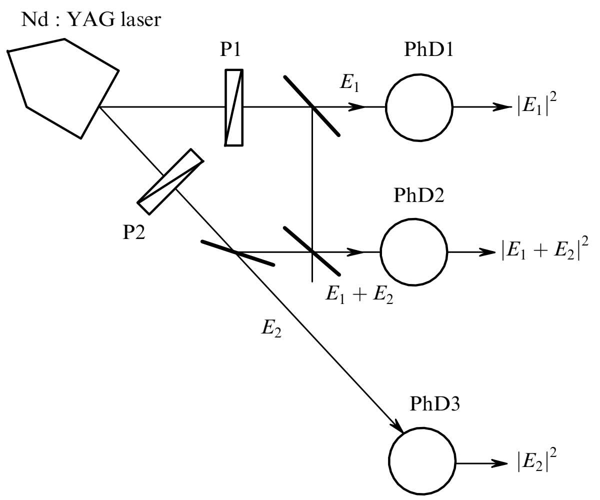

Figure 14. Principal scheme of the setup for studying phase effects: P1 and P2 are polarisers; PhD1–PhD3 are photodetectors.

We used in our experiments a laser with the self-modulation frequency $\omega_m/2\pi = 230$ kHz, the relaxation oscillation frequency $\omega_r/2\pi = 53$ kHz, and the relative excess over the threshold pump power $\eta = 0.08$. Depending on the pump modulation frequency and depth, different lasing regimes (periodic, quasi-periodic, and dynamic chaos regimes) appeared in the laser. We found that the identical chaos regime exists within a limited region of pump modulation depths $h_{min} < h < h_{max}$, the width of this region being dependent on the modulation frequency, achieving the maximum at $\omega_p/2\pi = 29$ kHz.

Figure 15 shows the typical oscillograms of intensities of counterpropagating waves and optical mixing signal. One can see that pulses of two types are present in radiation. For pulses of the first type, the peak intensity of the optical mixing signal is approximately four times greater than that of the interfering waves. Pulses of the second type in the signal I_{pm} have a very low intensity. This means that optical oscillations in the counterpropagating pulses of the first type occur in phase and in the pulses of the second type – out of phase, i.e., periodic jumps in the optical phase difference by π occur in the interval between adjacent chaotic pulses [31–33].

The numerical simulation of the radiation dynamics of a ring laser with parameters close to those of the laser studied in experiments, taking into account the influence of the spontaneous radiation noise, showed that the model of a bidirectional SRL, which takes into account the influence of the spontaneous radiation noise, also well describes in this case the temporal characteristics of the ring chip laser and its phase dynamics.

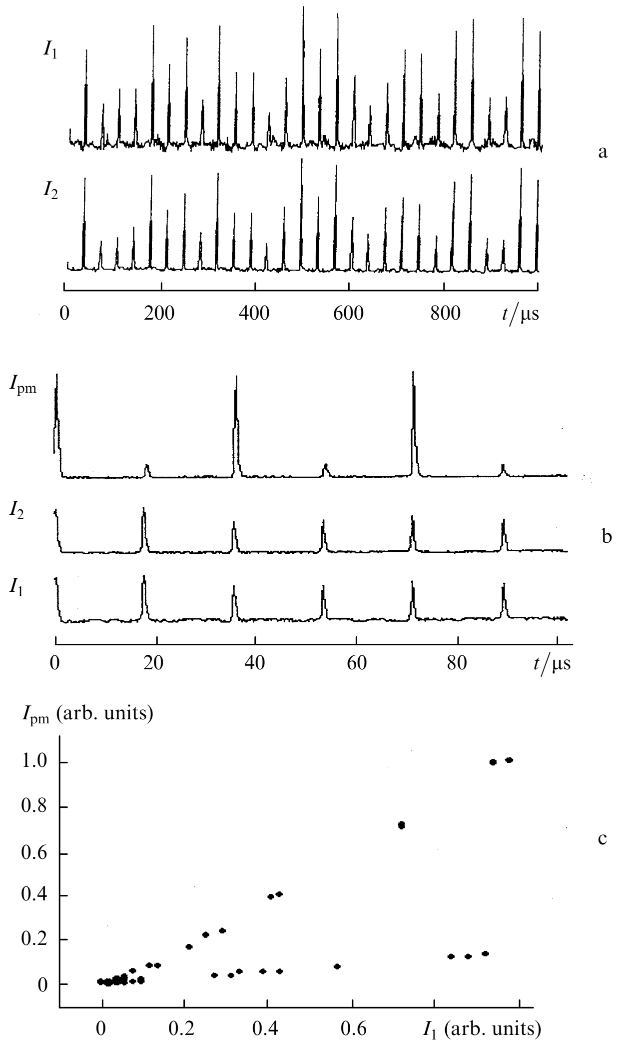

Figure 15. Intensity oscillograms for counterpropagating waves (a), fragments of these oscillograms and the photomixing signal (b), and the corresponding projection of the phase portrait (c) in the idengtical synchronous chaos regime (the DSC regime) for $\omega_p/2\pi = 29$ kHz and $h = 0.5$ (experiment).

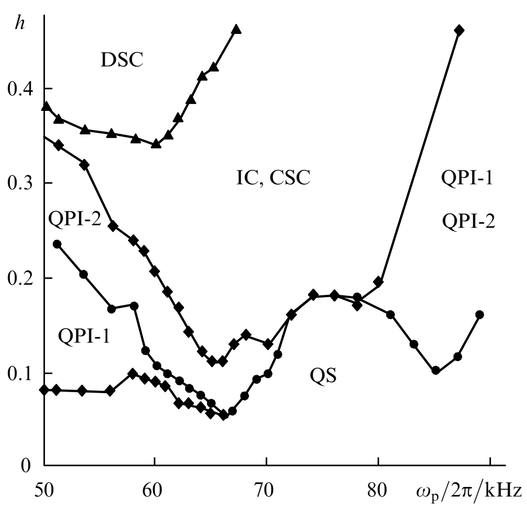

Figure 16. Stability regions for different lasing regimes in the ring laser in the $(\omega_p/2\pi, h)$ plane for the self-modulaton frequency $\omega_m/2\pi = 170$ kHz, the fundamental relaxation frequency $\omega_r/2\pi = 66$ kHz, and $\eta = 0.21$ (experiment).

5.6 Regions of existence of different lasing regimes

As was mentioned repeatedly, the basic characteristics of quasi-periodic and chaotic regimes and regions of their existence depend on many parameters at small values of h: the pump excess over the lasing threshold, the relation between the relaxation, self-modulation frequencies and modulation frequency of the control parameter, etc. Therefore, it is impossible to represent the regions of existence of different lasing regimes, for example, in the (h, ω_p) plane for arbitrary parameters of the laser. Figure 16 shows the example of a mutual arrangement of the regions of existence of different lasing regimes in a ring chip laser. These results were obtained experimentally in the absence of the frequency and amplitude nonreciprociies of the resonator, for the self-modulation frequency $\omega_m/2\pi =$ 170 kHz, the fundamental relaxation frequency $\omega_r/2\pi =$ 66 kHz, and the excess over the threshold pump level $\eta = 0.21$ [68].

When the modulation depth is small, both types of the above-described QS regimes are excited (the QS region). As h is increased, periodic and quasi-periodic regimes of pulsed modulation of counterpropagating waves appear (the QPI regions). In the IC and CSC regions, the intermittent and quasi-continuous-spectrum chaotic regimes are observed. In the DSC region, the quasi-discrete-spectrum synchronous chaotic regime is observed.

The presence of the frequency nonreciprocity Ω of the resonator narrows down the region of parameters where the CSC regime is observed. As Ω increases in a SRL operating in the DSC regime, small frequency nonreciprocity leads to the appearance of the antiphase dynamics in counter-propagating waves. As Ω further increases (at constant values of h and $\omega_p/2\pi$), the dynamic chaos regime passes to the quasi-periodic antiphase pulse-modulation regime, i.e., the region of existence of pulsed modulation is extended at the expense of synchronous and nonsynchronous chaotic regimes.

5.7 Effect of the frequency and amplitude nonreciprocities on nonlinear dynamics

The experimental and theoretical studies have shown that the use of the frequency or amplitude nonreciprocity of the ring laser resonator is one of the possible ways to control the nonlinear dynamics of ring lasers [72, 77]. This attracts interest because it is impossible to introduce any control elements into monolithic ring chip lasers, whereas the frequency and/or amplitude nonreciprocity can be rather simply produced using a magnetic field.

The possibility of using the frequency nonreciprocity to control lasing regimes is based on two physical mechanisms. One of them is a change in the self-modulation frequency produced by a magnetic field applied to an active element, which provides the conditions of parametric resonances both in autonomous and nonautonomous ring lasers. The second mechanism is related to the frequency degeneracy of counterpropagating waves in ring lasers and to the appearance a new degree of freedom (or a control parameter) in them.

The use of a magnetic field to control lasing regimes was demonstrated in Ref. [72], where it was shown that a constant magnetic field applied to the active element of a monolithic ring laser can substantially change the lasing dynamics. For example, a ring laser with the parameters $\omega_m/2\pi = 230$ kHz, $\omega_r/2\pi = 60$ kHz, $\eta = 0.21$ and $\omega_p/2\pi =$

29.5 kHz in the absence of a magnetic field (and, hence, in the absence of the frequency nonreciprocity) operates in the identical synchronous-chaos regime (the CSC regime). The application of a magnetic field of a strength of several tens of oersted leads to the passage of this lasing regime to the antiphase synchronous QPI regime. Figure 17 shows the typical intensity oscillograms for counterpropagating waves and the emission spectrum of one of them in the absence and in the presence of a magnetic field.

The use of the amplitude nonreciprocity is mainly based on creation of asymmetry in the laser as a nonlinear system: the presence of certain asymmetry is necessary for the appearance of dynamic chaos in the case of parametric resonances. The amplitude nonreciprocity of the resonator also leads to the appearance of the unequal fields of counterpropagating waves, resulting in a change in the resonator dynamics. In particular, the introduction of the amplitude nonreciprocity of the ring resonator is necessary for the appearance of the 'single-lobe' generalised-synchronous-chaos regime. The example of such an effect is also a change in the relaxation frequencies during the passage from unidirectional to bidirectional lasing in a ring laser. In principle, a change in the amplitude nonreciprocity can be also used to change the laser parameters to obtain parametric-resonance conditions.

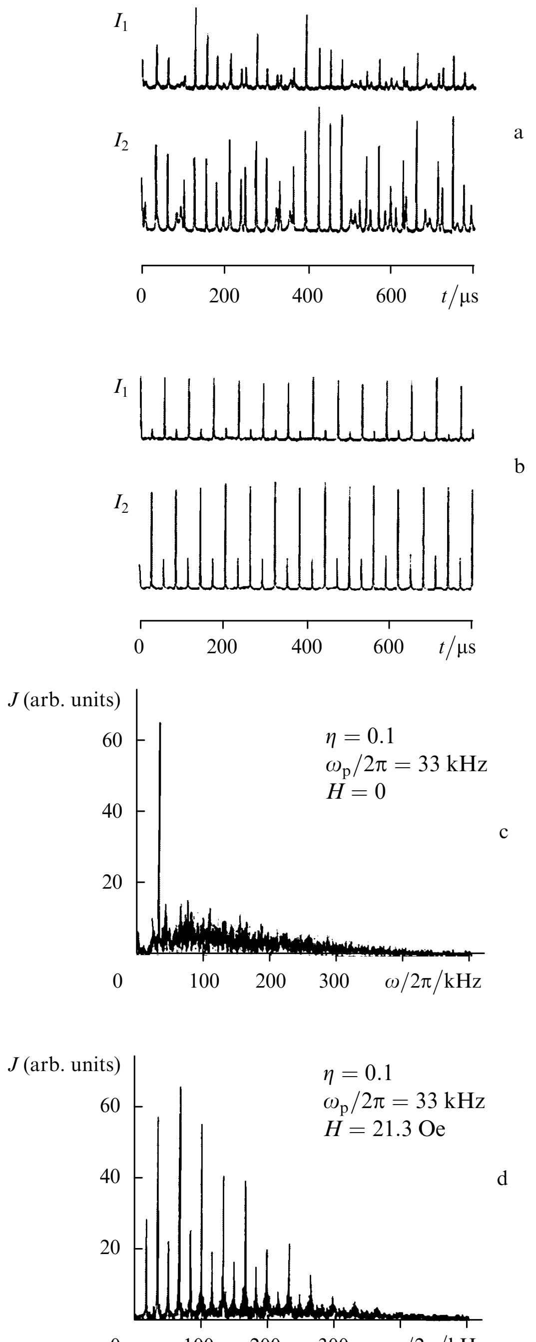

Figure 17. Oscillograms (a, b) and emission spectra (c, d) in the identical synchronous chaos regime (the CSC regime in the absence of a magnetic field) (a, c) and in the antiphase QPI regime in the external magnetic field $H = 21.3$ Oe (b, d).

6. Conclusions

We have presented the results of theoretical and experimental studies of single-mode SRLs with modulated parameters. Such SRLs can operate in a variety of non-stationary lasing regimes, including deterministic (dynamic) chaotic regimes. We have studied the temporal, spectral, and phase characteristics of counterpropagating waves in different non-stationary lasing regimes. The dependence of lasing regimes on the control parameters was investigated and the regions of the laser parameters were found in which quasi-periodic and chaotic regimes were established.

It has been shown that parametric processes and synchronisation of counterpropagating waves play a substantial role in excitation of non-stationary lasing regimes in SRLs. The characteristic features of the nonlinear dynamics of SRLs were compared in different regions of parametric resonances. The amplitude and phase synchronisation of counterpropagating waves in chaotic lasing regimes has been investigated.

Non-stationary lasing regimes have been classified based on the difference in the temporal and spectral characteristics of radiation of counterpropagating waves. This classification can be also useful for studying lasing regimes in other lasers (for example, lasers with two orthogonally polarised radiation components, coupled lasers, etc.).

The studies performed have shown that solid-state ring lasers, being complex nonlinear systems, are convenient objects for studying the general properties of the nonlinear dynamics of a variety of nonlinear self-oscillating systems. It has been shown that almost all the results of experimental studies of the nonlinear dynamics of solid-state ring lasers are well described by a standard mathematical model.

Acknowledgements. This work was supported by the Russian Foundation for Basic Research (Grant Nos 02-02-16391, 04-02-16532). The authors thank I.G. Zubarev for constructive comments.

References

1. Grasiuk A.Z., Oraevsky A.N. *Proc. IV Int. Congress on Microwave Tubes* (Sheveningen, Holland, 1962) p. 446; Zubarev I.G. *Diploma* (Department of Physics, Moscow State University, 1962).
2. Grasiuk A.Z., Oraevsky A.N. *Radiotekhn. Elektron.*, **9**, 524 (1964).
3. Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **15**, 1993 (1988) [*Sov. J. Quantum Electron.*, **18**, 1248 (1988)].

4. Zolotoverkh I.I., Kravtsov N.V., Lariontsev E.G., Makarov A.A., Firsov V.V. *Kvantovaya Elektron.*, **22**, 213 (1995) [*Quantum Electron.*, **25**, 197 (1995)].
5. Zolotoverkh I.I., Kravtsov N.V., Kravtsov N.N., Lariontsev E.G., Makarov A.A. *Kvantovaya Elektron.*, **24**, 638 (1997) [*Quantum Electron.*, **27**, 621 (1997)].
6. Ivanov D.V., Khanin Ya.I., Matorin I.I., Pikovsky A.S. *Phys. Lett.*, **89**, 229 (1982).
7. Uchida A., Ogawa T., Shinozuka M., Kannari F. *Phys. Rev. E*, **62**, 1960 (2000).
8. De Shazer D., Breban R., Ott E., Roy R. *Phys. Rev. Lett.*, **87**, 044101 (4) (2001).
9. Krauskopf B., Wieczorek S., Lenstra D. *Appl. Phys. Lett.*, **77**, 1611 (2000).
10. Roy R., Thornburg K.S. *Phys. Rev. Lett.*, **72**, 2009 (1994).
11. Thornburg K.S., Moller M., Roy R. *Phys. Rev. E*, **55**, 3865 (1997).
12. Terry J., Thornburg K.S., De Shazer D., Van Wiggeren G., Zhu Sh., Ashwin P., Roy R. *Phys. Rev. E*, **59**, 4036 (1999).
13. Moller M., Forsmann B., Jansen M.J. *Opt. B: Quantum Semiclass. Opt.*, **2**, 371 (2000).
14. Lariontsev E.G. *Kvantovaya Elektron.*, **25**, 405 (1998) [*Quantum Electron.*, **28**, 392 (1998)].
15. Oraevsky A.N. *Izv. Vyssh. Uchebn. Zaved. Ser. Probl. Nelin. Dinam.*, (4), 3 (1996).
16. Kane T.J., Byer R.L. *Opt. Lett.*, **10**, 65 (1985).
17. Kravtsov N.V., Nanii O.E. *Kvantovaya Elektron.*, **20**, 322 (1993) [*Quantum Electron.*, **23**, 272 (1993)].
18. Kravtsov N.V. *Kvantovaya Elektron.*, **31**, 661 (2001) [*Quantum Electron.*, **31**, 661 (2001)].
19. Fan T.Y., Byer R.L. *IEEE J. Quantum Electron.*, **24**, 895 (1988).
20. Nilsson A.C., Gustafson E.K., Byer R.L. *IEEE J. Quantum Electron.*, **25**, 767 (1989).
21. Kravtsov N.V., Lariontsev E.G. *Kvantovaya Elektron.*, **21**, 903 (1994) [*Quantum Electron.*, **24**, 841 (1994)].
22. Kravtsov N.V., Lariontsev E.G. *Laser Phys.*, **7**, 196 (1997).
23. Kravtsov N.V., Lariontsev E.G. *Kvantovaya Elektron.*, **30**, 105 (2000) [*Quantum Electron.*, **30**, 105 (2000)].
24. Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Laser Physics*, **3**, 21 (1993).
25. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Pis'ma Zh. Eksp. Teor. Fiz.*, **17**, 405 (1973).
26. Golyaev Yu.D., Dedysh V.V., Dmitriev V.G., Kravtsov N.V., Lariontsev E.G., Livintsev A.L., Nanii O.E., Nadtochiev V.E., Solov'eva T.I., Firsov V.V., Veselovskaya T.V. *Izv. Ross. Akad. Nauk. Ser. Fiz.*, **56**, 163 (1992).
27. Boiko D.L., Kravtsov N.V. *Kvantovaya Elektron.*, **25**, 361 (1998) [*Quantum Electron.*, **28**, 350 (1998)].
28. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Zh. Eksp. Teor. Fiz.*, **65**, 1344 (1973).
29. Zolotoverkh I.I., Lariontsev E.G. *Kvantovaya Elektron.*, **23**, 620 (1996) [*Quantum Electron.*, **26**, 604 (1996)].
30. Zolotoverkh I.I., Kravtsov N.V., Lariontsev E.G., Makarov A.A., Firsov V.V. *Opt. Commun.*, **113**, 249 (1994).
31. Kotomtseva L.A., Kravtsov N.V., Lariontsev E.G., Chekina S.N. *Kvantovaya Elektron.*, **32**, 654 (2002) [*Quantum Electron.*, **32**, 654 (2002)].
32. Kravtsov N.V., Lariontsev E.G., Naumkin N.I., Chekina S.N., Firsov V.V. *Kvantovaya Elektron.*, **32**, 251 (2002) [*Quantum Electron.*, **32**, 251 (2002)].
33. Lariontsev E.G. *Opt. Express*, **2**, 198 (1998).
34. Kotomtseva L.A., Kravtsov N.V., Lariontsev E.G., Chekina S.N. *Chaos*, **13**, 279 (2003).
35. Anishchenko V.S. *Znakomstvo s nelineinoi dinamikoi* (Introduction to Nonlinear Dynamics) (Moscow – Izhevsk: Institute of Computer Research, 2002).
36. Schuster H.G. *Deterministic Chaos* (Weinheim: Physik-Verlag, 1984; Moscow: Mir, 1988).
37. Anishchenko V.S. *Slozhnye kolebaniya v prostykh sistemakh* (Complex Oscillations in Simple Systems) (Moscow: Nauka, 1990).
38. Haken H. *Laser Light Dynamics* (Moscow: Mir, 1988).
39. Swift J.W., Wiesenfeld K. *Phys. Rev. Lett.*, **52** (9), 705 (1984).
40. Kravtsov N.V., Lariontsev E.G. *Izv. Ross. Akad. Nauk, Ser. Fiz.*, **60**, 188 (1996).
41. Kornienko L.S., Kravtsov N.V., Shelaev A.N. *Opt. Spektrosk.*, **35**, 775 (1973).
42. Dotsenko A.V., Lariontsev E.G. *Kvantovaya Elektron.*, **4**, 1099 (1977) [*Sov. J. Quantum Electron.*, **7**, 616 (1977)].
43. Klochan E.G., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Radiotekhn. Elektron.*, **19**, 2096 (1974).
44. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Dokl. Akad. Nauk SSSR*, **215**, 313 (1974).
45. Zolotoverkh I.I., Kravtsov N.N., Lariontsev E.G., Makarov A.A., Firsov V.V. *Kvantovaya Elektron.*, **21**, 5 (1994) [*Quantum Electron.*, **24**, 3 (1994)].
46. Khanin Ya.I. *Dinamika lazerov* (Laser Dynamics) (Moscow: Nauka, 1999).
47. Kravtsov N.V., Kravtsov N.N., Makarov A.A., Firsov V.V. *Kvantovaya Elektron.*, **23**, 195 (1996) [*Quantum Electron.*, **26**, 189 (1996)].
48. Kravtsov N.V., Lariontsev E.G., Sidorov S.S., Firsov V.V., Chekina S.N. *Kvantovaya Elektron.*, **31**, 189 (2001) [*Quantum Electron.*, **31**, 189 (2001).
49. Khandokhin P.A., Khanin Ya.I., Koryukin I.V. *Opt. Commun.*, **81**, 297 (1996).
50. Dotsenko A.V., Lariontsev E.G. *Kvantovaya Elektron.*, **8**, 1504 (1981) [*Sov. J. Quantum Electron.*, **11**, 907 (1981)].
51. Kravtsov N.V., Lariontsev E.G. *BRAS, Suppl. Phys. Vibration*, **58** (3), 134 (1994).
52. Parfenov V.A., Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **15**, 1985 (1988) [*Sov. J. Quantum Electron.*, **18**, 1243 (1988)].
53. Bandy D.K., Narducci L.M., Lugiato L.A., Abragam N.B. *J. Opt. Soc. Am. B*, **2**, 56 (1985).
54. Matorin I.I., Pikovskii A.S., Khanin Ya.I. *Kvantovaya Elektron.*, **11**, 2096 (1984) [*Sov. J. Quantum Electron.*, **14**, 1401 (1984)].
55. Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **23**, 36 (1996) [*Quantum Electron.*, **26**, 34 (1996)].
56. Khandohin P.A., Khanin Ya.I. *J. Opt. Soc. Am. B*, **2**, 226 (1985).
57. Hoffer L.M., Lippi G.L., Abraham N.B., Mandel P. *Opt. Commun.*, **66**, 219 (1988).
58. Kravtsov N.V., Lariontsev E.G., Pashinin P.P., Sidorov S.S., Chekina S.N. *Kvantovaya Elektron.*, **34**, 325 (2004) [*Quantum Electron.*, **34**, 325 (2004)].
59. Kravtsov N.V., Sidorov S.S., Pashinin P.P., Firsov V.V., Chekina S.N. *Kvantovaya Elektron.*, **34**, 329 (2004) [*Quantum Electron.*, **34**, 329 (2004)].
60. Lugiato L.A., Prati F., Bandy D.K., Narducci L.M., Ru P., Tredicce J.R. *Opt. Commun.*, **64**, 167 (1987).
61. Mayr M., Risken H., Vollmer H.D. *Opt. Commun.*, **36**, 480 (1981).
62. Zhang J., Haken H., Ohno H. *J. Opt. Soc. Am. B*, **2**, 141 (1985).
63. Narducci L.M., Tredicce J.R., Lugiato L.A., Abragam N.B., Bandy D.K. *Phys. Rev. A*, **33**, 1842 (1986).
64. Abraham N.B., Lugiato L.A., Narducci L.M. *J. Opt. Soc. Am. B*, **2**, 7 (1985).
65. Zolotoverkh I.I., Klimenko D.N., Lariontsev E.G. *Kvantovaya Elektron.*, **23**, 625 (1996) [*Quantum Electron.*, **26**, 609 (1996)].
66. Vladimirov A.G. *Opt. Commun.*, **149**, 67 (1998).
67. Zolotoverkh I.I., Klimenko D.N., Kravtsov N.V., Lariontsev E.G., Firsov V.V. *Kvantovaya Elektron.*, **23**, 938 (1996) [*Quantum Electron.*, **26**, 914 (1996)].
68. Klimenko D.N., Kravtsov N.V., Lariontsev E.G., Firsov V.V. *Kvantovaya Elektron.*, **24**, 649 (1997) [*Quantum Electron.*, **27**, 631 (1997)].
69. Klimenko D.N., Lariontsev E.G. *Kvantovaya Elektron.*, **25**, 369 (1998) [*Quantum Electron.*, **28**, 358 (1998)].
70. Kravtsov N.V., Pashinin P.P., Sidorov S.S. *Kvantovaya Elektron.*, **32**, 562 (2002) [*Quantum Electron.*, **32**, 562 (2002)].
71. Kravtsov N.V., Lariontsev E.G., Pachinin P.P., Sidorov S.S. *Laser Phys.*, **13**, 305 (2003).
72. Kravtsov N.V., Pashinin A.P., Sidorov S.S., Chekina S.N., Firsov V.V. *Kvantovaya Elektron.*, **33**, 321 (2003) [*Quantum Electron.*, **33**, 321 (2003)].

73. Rulkov N.F., Sushchik M.M., Tsimring L.S., Abarbanel D. *Phys. Rev. E*, **51**, 980 (1995).
74. Bogolyubov N.N., Mitropol'skii Yu.A. *Asimptoticheskie metody v teorii nelineinykh kolebanii* (Asymptotic Methods in the Theory of Nonlinear Oscillations) (Moscow: Nauka, 1974).
75. Rosenblum M.G., Pikovsky A.S., Kurths J. *Phys. Rev. Lett.*, **76**, 1804 (1996).
76. Pikovsky A.S., Rosenblum M.G., Osipov G., Kurths J. *Physica D*, **104**, 219 (1997).
77. Kravtsov N.V., Kravtsov N.N. *Kvantovaya Elektron.*, **27**, 98 (1999) [*Quantum Electron.*, **29**, 378 (1999)].