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Complex-order fractional Fourier transforms in optical schemes with Gaussian apertures

A.A. Malyutin

***Abstract*. Several optical schemes performing the complex-order fractional Fourier transform are considered. It is shown that these schemes, containing only Gaussian apertures or their combination with lenses, have eigenbeams represented by Hermite – Gaussian modes with transverse indices m, $n \leqslant 1$ and Laguerre – Gaussian modes with $p = 0$ and $l = 1$. The wave front of the eigenbeams is, as a rule, spherical.**

Keywords: *Gaussian aperture, complex-order fractional Fourier transform, eigenbeams, Hermite – Gaussian and Laguerre – Gaussian modes.*

1. Introduction

As shown in [1], traditional schemes of optical Fourier transform (FT) can be readily transformed to perform fractional FT (FrFT). In this case, both for the integer (simple) FT and FrFT, the optical scheme is described by the $ABCD$ matrix of the form

$$M = \begin{pmatrix} AB \\ CD \end{pmatrix} = \begin{pmatrix} \cos\psi & F^* \sin\psi \\ -\sin\psi/F^* & \cos\psi \end{pmatrix}, \qquad (1)$$

where the angle ψ gives the Gouy phase, accumulated after the propagation of the eigenbeam in the system, and determines the order $a = 2\psi/\pi$ of the operation being performed (for the simple FT, $\psi = \pi/2$ and $a = 1$); $F^* = \pi w_0^2/\lambda$ is the Rayleigh length of the eigenbeam; and w_0 is the eigenbeam waist radius. The simplest eigenbeams of the real-order FrFT are Hermite – Gaussian (HG) $u_{mn}^{\rm HG}$ modes [2].

Nothing contradicts to the substitution of the complex values of ψ and F^* into (1). In this case, the main advantages of the FrFT (additivity, commutativity, etc.) are retained [3]. Gaussian apertures (GAs) play the role of an element changing the order of the optical FrFT from real to complex [4]. One of such schemes, presented in [3], is a retranslator (with the 1 : 1 mapping scale) with three GAs placed at the common focus of lenses, at the input and output.

The conditions for performing the complex-order FrFT using an optical scheme described by the $ABCD$ matrix have the general form [5]

$$\mathrm{Arg}(A) = \mathrm{Arg}(D), \qquad (2)$$

$$|\mathrm{Arg}(B) - \mathrm{Arg}(C)| = \pi\,. \qquad (3)$$

Note, however, that relation (3) is not fulfilled for the complex values of ψ and F^*. As a result, the authors of paper [5], using (3), obtain the only scheme, which is similar to the simple optical FT (of type 2 in [1]), where two GAs are set coincident with lenses. This scheme corresponds to the complex-order FrFT with the real part equal to unity.

Because the use of FrFT for signal analysis and image processing is being widely discussed, its specific applications can be expected in the near future. It is of interest in this connection to consider various optical schemes for the complex-order FrFT, including schemes employing only GAs (without lenses). This can be useful for spectral regions where the efficiency of refracting optics is low. In addition, because the complex-order FrFT results in a change in the beam energy [6], and it is useful to find radiation losses. It is also important to determine the structure of eigenbeams in optical schemes for the complex-order FrFT.

2. Properties of FrFT schemes based on Gaussian apertures

Optical transformation performed by the real-order FrFT scheme can be treated as the transfer of the eigenbeam waist from one reference plane (RP) to another. The eigenbeam in these planes has a certain waist radius and a plane wave front. Consider the properties of the eigenbeam in schemes in Figs 1a, b, which are similar to usual FrFTs where lenses are replaced by GAs.

Because the GA transmission is $T = \exp(-x^2/w_{\rm g}^2)$ (where x is the radial coordinate and $w_{\rm g}$ is the radius of a Gaussian beam formed by the aperture illuminated by a beam with an infinite diameter and a spatially uniform intensity [4]), the $ABCD$ matrix of the GA is analogous to the lens matrix

$$G = \begin{pmatrix} 1 & 0 \\ -{\rm i}/f_{\rm g} & 1 \end{pmatrix}, \qquad (4)$$

where $f_{\rm g} = \pi w_{\rm g}^2/\lambda$ is the complex focal distance or the Rayleigh length of a beam formed by the aperture. In the case of an arbitrary Gaussian beam (hereafter, simply 'beam') of radius $w_{\rm g1}$ incident on the GA without the centre

A.A. Malyutin A.M. Prokhorov General Physics Institute, Russian Academy of Sciences, ul. Vavilova 38, 119991 Moscow, Russia; Tel.: (095) 135 03 27; Fax: (095) 135 20 55; e-mail: amal@kapella.gpi.ru

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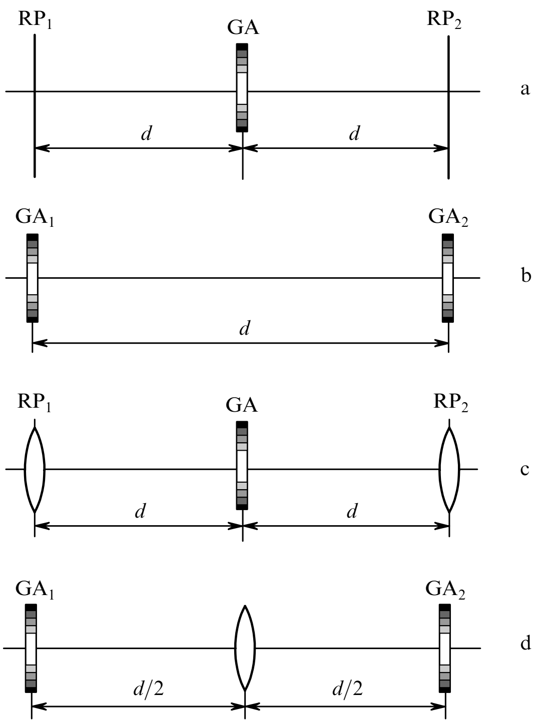

Figure 1. Optical schemes for the complex-order fractional Fourier transform using only Gaussian apertures (a, b) and in combination with lenses (c, d).

displacement, a beam of radius w_{g2} is obtained at the output, which is related to w_g by the expression [7]

$$w_{g2}^{-2} = w_{g1}^{-2} + w_g^{-2}. \tag{5}$$

In this case, the curvature of the wave front of the input beam is not transformed.

By using (4) and the propagation matrix in a free space of length d for schemes 1a, b, we have

$$M_a = \begin{pmatrix} 1 - id/f_g & 2d - id^2/f_g \\ -i/f_g & 1 - id/f_g \end{pmatrix},$$
$$M_b = \begin{pmatrix} 1 - id/f_g & d \\ -d/f_g^2 - i2/f_g & 1 - id/f_g \end{pmatrix}. \tag{6}$$

The correspondence of these matrices to the complex-order FrFT schemes assumes obviously that the replacement of real lenses in FrFT schemes by GAs admits the representation of (6) in form (1). To do this, it is necessary that $A = \cos\psi$ and $-C/B = (1/F^*)^2$ in (6), which, as can be easily shown, can be fulfilled if ψ and F^* are complex quantities. Although the form of matrices (6) is different, their eigenvalues are specified by the same equation

$$\Lambda^2 - 2\Lambda\left(1 - \frac{id}{f_g}\right) + 1 = 0, \tag{7}$$

with the roots

$$\Lambda_{1,2} = 1 - \frac{id}{f_g} \pm \left(-\frac{d^2}{f_g^2} - \frac{i2d}{f_g}\right)^{1/2}$$
$$= \cos\psi \pm i\sin\psi = \exp(\pm i\psi). \tag{8}$$

By using the notation $\psi = \psi_{re} + i\psi_{im}$, we obtain

$$\Lambda_{1,2} = \exp(\pm i\psi_{re})\exp(\mp\psi_{im}). \tag{9}$$

Because the '+' sign in the second exponential in (9) means an increase in the field (which is excluded in our case), by omitting the second solution, we obtain losses in the system

$$\mathscr{L} = 1 - |\Lambda|^2 = 1 - \exp(-2\psi_{im}). \tag{10}$$

Here, ψ_{re} is the accumulated Gouy phase (AGP) of the eigenbeam: $\theta = \psi_{re}$.

Because the elements B and C of matrices (6) in schemes in Figs 1a, b are different, one can expect that the radii and curvatures of the wave fronts of eigenbeams in these schemes will be different, since

$$F^* = \left(-\frac{B}{C}\right)^{1/2} = \frac{B}{\sin\psi}. \tag{11}$$

Furthermore, because $A = D$, we obtain the expressions

$$w_e = |F^*|\left(\frac{\lambda}{\pi|\mathrm{Re}F^*|}\right)^{1/2}, \quad \rho_e = -\frac{\mathrm{Im}F^*}{|F^*|^2} \tag{12}$$

for the beam radius and wave-front curvature. Therefore, the problem of finding the eigenvalues and eigenfunctions in schemes for the complex-order FrFT in Figs 1a, b can be solved rather formally.

Note that in the general case (for $\mathrm{Im}F^* \neq 0$), beams with a spherical wave front are eigenbeams for the complex-order FrFT, and due to the equality of losses and the same AGP value, the FrFT order for both schemes in Figs 1a, b is the same,

$$a^* = \frac{2\psi}{\pi}. \tag{13}$$

However, for simple schemes in Fig. 1, a different approach is more descriptive. Consider first the scheme in Fig. 1a.

Let us assume that a required beam with the waist radius w_0 located at a distance z from the RP_1 has the radius w_e in the RP_1 and the wave-front curvature ρ_e. We denote the beam radius and its curvature at the GA input as w_{g1} and ρ_g, respectively. The wave-front curvature of the beam behind the GA will not change, and its radius w_{g2} is specified by relation (5).

We should find a beam with the radius w_0 whose radius and curvature in the reference plane RP_2 will be w_e and ρ_e as well, i.e., we should find the beam (Fig. 2a) satisfying the relations

$$w_e = w(z + d), \quad w_{g1} = w(z + 2d), \quad w_{g2} = w(z),$$
$$\rho_e = \rho(z + d), \quad \rho_g = \rho(z) = \rho(z + 2d), \tag{14}$$

which, in combination with (5), is reduced to a simple system of equations, having the solution

$$w_0 = \left\{\frac{\lambda d}{\pi\sqrt{2}}\left[\left(1 + \frac{4f_g^2}{d^2}\right)^{1/2} - 1\right]^{1/2}\right\}^{1/2},$$
$$z = d\left\{\left[\left(1 + \frac{4f_g^2}{d^2}\right)^{1/2} + 1\right]^{1/2} - 1\right\}. \tag{15}$$

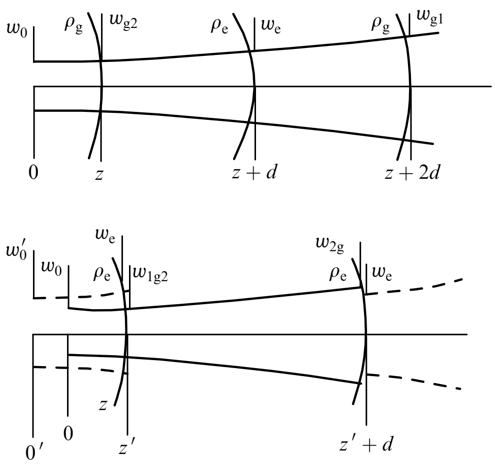

Figure 2. Diagrams for calculating the radii of eigenbeams and their wave-front curvature for the fractional Fourier transform schemes with the use of Gaussian apertures (Figs 1a, b).

The further calculations are trivial. Note only that points with coordinates z and $z+2d$ are located on both sides of the point $z_0 = \pi w_0^2/\lambda$, where the wave-front curvature of the eigenbeam is maximal.

For the scheme in Fig. 1b, the eigenbeam with the parameters w_e and ρ_e should exist at the GA_1 input and the GA_2 output. Behind the GA_1 and in front of the GA_2, the beam should have the same curvature (ρ_e) but obviously different radii. Therefore, we can write

$$w_{1g2}^{-2} = w_e^{-2} + w_g^{-2}, \quad w_e^{-2} = w_{2g1}^{-2} + w_g^{-2}, \tag{16}$$

where, as above, figures 1 and 2 after the index 'g' denote the input and output from the GA, and the figures in front of the index denote the GA number. Consider now Fig. 2b, in which w_0 (and other non-primed quantities) is related only to the beam existing between the GA_1 and GA_2. Taking the above-said into account, we can write

$$w_{1g2} = w(z), \quad w_{2g1} = w(z+d),$$
$$\rho_e = \rho(z) = \rho(z+d), \tag{17}$$

which, together with (16), gives a complete set of equations for finding the parameters of the beam between the GA and ρ_e. The rest of the parameters will be different for the scheme in Fig. 1b because the elements B and C of matrices (6) are different. For example,

$$w_0 = \left\{ \frac{\lambda d}{\pi 2\sqrt{2}} \left[\left(1 + \frac{4f_g^2}{d^2} \right)^{1/2} - 1 \right]^{1/2} \right\}^{1/2},$$
$$z = \frac{d}{2\sqrt{2}} \left\{ \left[\left(1 + \frac{4f_g^2}{d^2} \right)^{1/2} + 1 \right]^{1/2} - \sqrt{2} \right\}. \tag{18}$$

The value of w_e can be calculated from any of the relations in (16), and then, knowing w_e and ρ_e and using the expression

$$\tan\psi(z) = \frac{\pi w^2(z)\rho(z)}{\lambda} = \frac{z\lambda}{\pi w_0^2}, \tag{19}$$

we can calculate the waist radius of the beam entering the GA_1 and the waist position (the quantities primed in Fig. 2b).

Recall that, although the scheme in Fig. 1b uses two GAs rather than one, the losses for the eigenbeam entering the GA_1 in this scheme are, according to (6)–(10), the same as for the scheme with one GA in Fig. 1a, i.e.,

$$\mathscr{L} = 1 - |\Lambda|^2 = 1 - \exp(-2\psi_{im})$$
$$= 1 - \left(\frac{w_{g2}}{w_{g1}}\right)^2 = 1 - \left(\frac{w_{1g2}}{w_{2g1}}\right)^2. \tag{20}$$

The AGP of the eigenbeam and the complex FrFT order, which can be expressed in terms of the AGP and losses, also remain unchanged. This is fulfilled obviously under the condition that both schemes use the same wavelength, identical GAs, and the optical length of the scheme in Fig. 1b is half that of the scheme in Fig. 1a.

The normalised radii of the eigenbeams, their wave-front curvatures, and dependences of losses and AGP on the ratio f_g/d are presented for schemes of Figs 1a, b (marked by the symbol ∞) in Figs 3 and 4, respectively.

3. Properties of some FrFT schemes based on Gaussian apertures and lenses

Consider now optical schemes in Figs 1c, d representing combinations of the complex-order FrFT schemes (Figs 1a, b) with the real-order FrFT schemes (based on lenses). Note that schemes in Figs 1c, d differ from optical schemes considered in [3, 5].

Calculations of the elements of the $ABCD$ matrix for schemes in Figs 1c, d give

$$A_c = D_c = \left(1 - \frac{2d}{f}\right) - \frac{id}{f_g}\left(1 - \frac{d}{f}\right), \quad B_c = 2d - \frac{id^2}{f_g},$$
$$C_c = -\frac{2}{f}\left(1 - \frac{d}{f}\right) - \frac{i}{f_g}\left(1 - \frac{d}{f}\right)^2 \tag{21}$$

and

$$A_d = D_d = \left(1 - \frac{d}{2f}\right) - \frac{id}{f_g}\left(1 - \frac{d}{4f}\right), \quad B_d = d\left(1 - \frac{d}{4f}\right),$$
$$C_d = -\frac{1}{f} - \frac{d}{f_g^2}\left(1 - \frac{d}{4f}\right) - \frac{2i}{f_g}\left(1 - \frac{d}{2f}\right), \tag{22}$$

which corresponds for $f \to \infty$ to the transition to matrices (6). One can see that, unlike (6), the elements of the principal diagonal of matrices (21) and (22) do not coincide. Therefore, it is reasonable to expect that, in the case of lenses with identical parameters, the eigenbeams and losses in schemes in Figs 1c, d will be different.

The normalised radii of the eigenbeams and their wave-front curvatures for the scheme in Fig. 1c, as well as the dependences of losses and the AGP on the ratio f_g/d are presented in Fig. 3 [calculations were performed using (1) and (9)–(12)]. Note that the matrix with elements (21) has two singularities. Thus, for $f = d$, this matrix degenerates to the matrix

$$M_c' = \begin{pmatrix} 1 & 2f - if^2/f_g \\ 0 & 1 \end{pmatrix}, \tag{23}$$

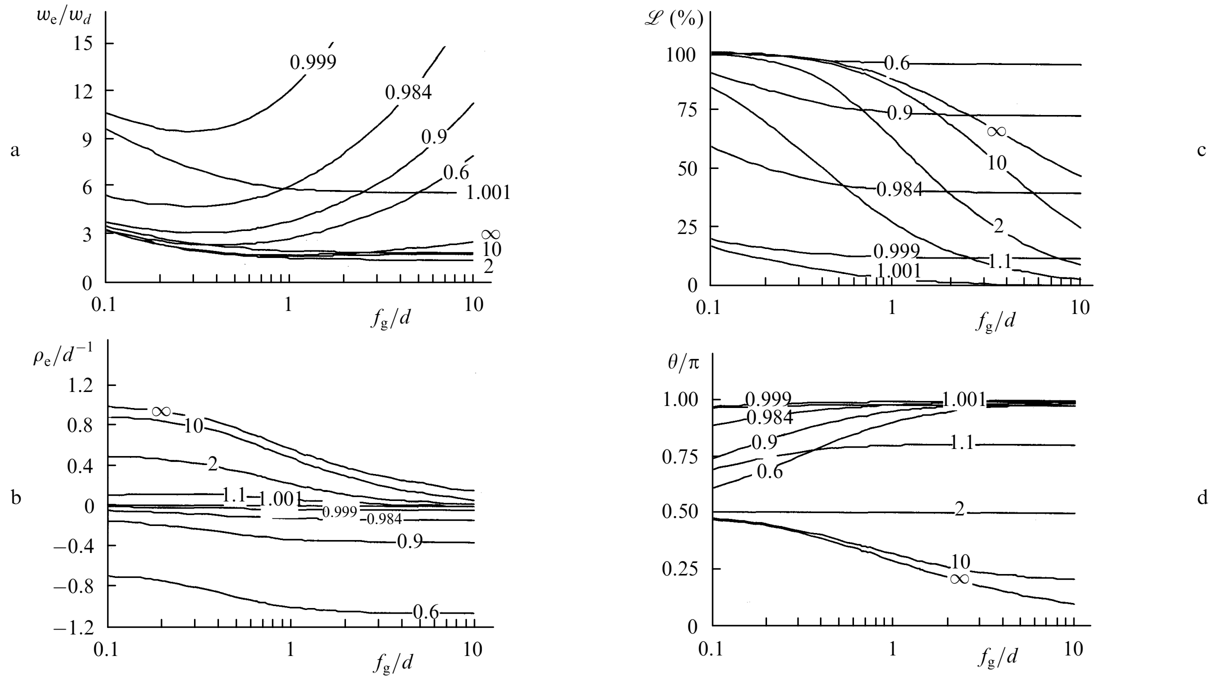

Figure 3. Normalised radii of the eigenbeams (a), their radii of curvature (b), radiation losses (c), and accumulated Gouy phase (d) as functions of the Gaussian aperture parameter f_g/d for different focal lengths f/d of lenses in the scheme in Fig. 1c and Fig. 1a (indicated by the symbol ∞). The number at the curves indicate the normalised focal distance f/d of lenses; $\omega_d = (\lambda d/\pi)^{1/2}$.

corresponding to the complex segment and has no eigenbeams, i.e., for $f = d$, the optical system in Fig. 1c is not the FrFT. For $f = 2d$, we have

$$M_c'' = \begin{pmatrix} -\mathrm{i}f/4f_g & f - \mathrm{i}f^2/4f_g \\ -1/f - \mathrm{i}/4f_g & -\mathrm{i}f/4f_g \end{pmatrix}. \qquad (24)$$

The eigenbeam of matrix (24) has the parameters

$$w_{ce}'' = \frac{1}{2}\left[\frac{\lambda f\left(16f_g^2 + f^2\right)^{1/2}}{\pi f_g}\right]^{1/2}, \qquad (25)$$

$$\rho_{ce}'' = \left(16f_g^2 + f^2\right)^{-1/2}. \qquad (26)$$

The AGP for the beam with such parameters is independent of the ratio f_g/d (Fig. 3d).

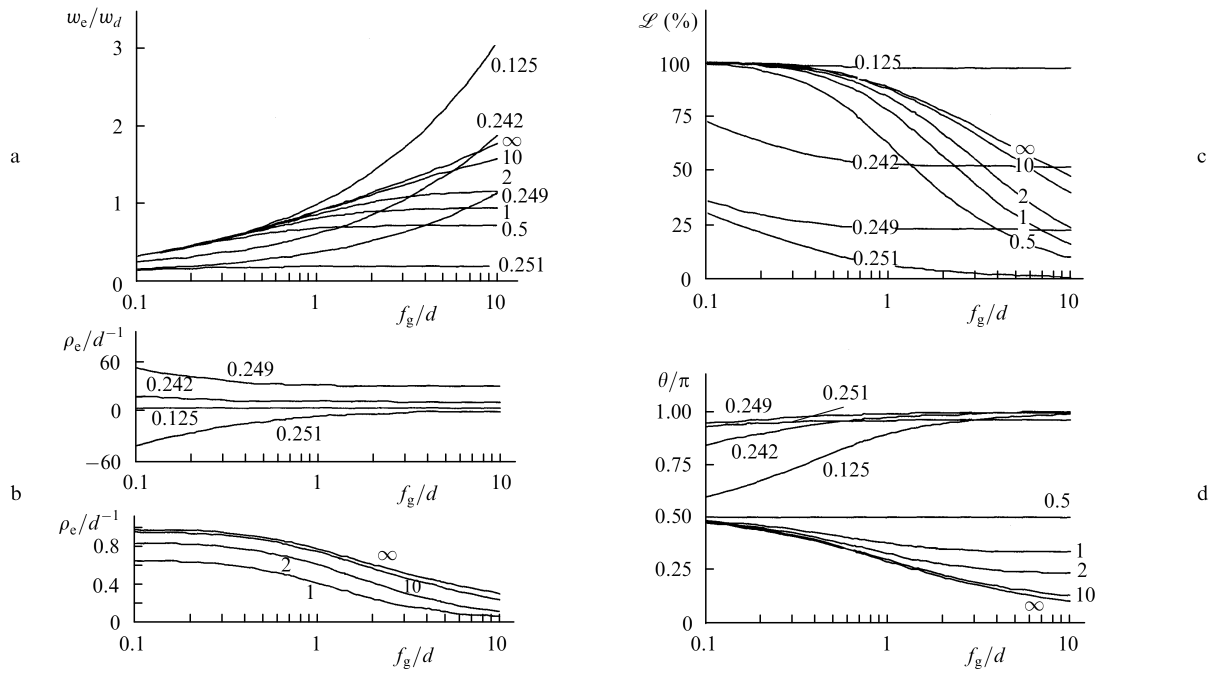

Figure 4. Same as in Fig. 3 for the scheme in Fig. 1d and 1a (indicated by the symbol ∞). The curve for $f/d = 0.5$ is absent in Fig. 4b because the beam curvature is arbitrary.

Figure 4 shows the eigenbeam parameters, losses and AGP as functions of the ratio f_g/d for the scheme in Fig. 1d. As for the scheme in Fig. 1c, the optical scheme in Fig. 1d also has two singularities. For $f = d/2$, we have

$$M_d' = \begin{pmatrix} -\mathrm{i}f/f_g & f \\ -1/f - f/f_g^2 & -\mathrm{i}f/f_g \end{pmatrix}, \tag{27}$$

which gives the eigenbeam radius

$$w_d' = \left[\frac{\lambda f_g f}{\pi (f_g^2 + f^2)^{1/2}}\right]^{1/2}$$

and an arbitrary curvature of the wave front. In this case, as mentioned in [5], the scheme in Fig. 1d represents the imaginary-order FrFT. For a beam with a plane wave front, the order is

$$a_d' = \mathrm{i}\left\{1 - \frac{2}{\pi}\ln\left[\frac{(f_g^2 + f^2)^{1/2} - f}{(f_g^2 + f^2)^{1/2} + f}\right]^{1/2}\right\}. \tag{28}$$

For $f = d/4$, a lens project the RP_1 on the RP_2, and matrix (22) degenerates to the matrix of the lens with the complex focal distance

$$M_d'' = \begin{pmatrix} -1 & 0 \\ -1/f + 2\mathrm{i}/f_g & -1 \end{pmatrix}, \tag{29}$$

i.e., the optical scheme in Fig. 1d is no longer the FrFT.

4. Structure of the eigenbeams of the complex-order FrFT

The above discussion of the complex-order FrFT schemes and the expressions that we obtained are obviously valid for a Gaussian beam, i.e., for the zero-order Hermite–Gaussian u_{00}^{HG} mode. What about the higher-order modes?

By omitting the insignificant coefficients and phase factors and considering only the one-dimensional case, we can write the field of the Hermite–Gaussian mode behind the GA in the form

$$u_n^{HG}(x') = u_n^{HG}(x, w_{g1})\exp\left(-\frac{x^2}{w_g^2}\right)$$

$$\sim \exp\left(-\frac{x^2}{w_{g1}^2}\right)H_n\left(\frac{x\sqrt{2}}{w_{g1}}\right)\exp\left(-\frac{x^2}{w_g^2}\right), \tag{30}$$

where w_{g1} is the radius of the Gaussian envelope of the beam in front of the GA; H_n is the Hermite polynomial of order n; x' is the radial coordinate at the output of the GA of radius w_g. One can easily show that the right-hand side of expression (30) can be written in the form $\sim u_n^{HG}(x', w_{g2})$, where w_{g2} is specified by expression (5), only when $n = 0, 1$. In all other cases, the Hermite polynomial contains more than one term. Therefore, the eigenbeams of the complex-order FrFT (13) can have only the structure of the u_{mn}^{HG} modes with $m, n \leqslant 1$ (or Laguerre–Gaussian u_{01}^{LG} modes). In this case, because we have for the FrFT [8]

$$\mathscr{F}^a[u_{mn}^{HG}] = \exp\left[-\mathrm{i}a(N+1)\frac{\pi}{2}\right]u_{mn}^{HG}, \tag{31}$$

where $N = m + n = 2p + l$, the losses for the eigenmodes u_{01}^{HG}, u_{11}^{HG}, and u_{01}^{LG} for the FrFT of the complex order $a^* = 2(\psi_{re} + \mathrm{i}\psi_{im})/\pi$ are

$$\mathscr{L}_{mn,pl} = 1 - \exp[-2(N+1)\psi_{im}] \tag{32}$$

or

$$\mathscr{L}_{01} = 1 - (1 - \mathscr{L})^2, \quad \mathscr{L}_{11} = 1 - (1 - \mathscr{L})^3, \tag{33}$$

where $\mathscr{L}$ is calculated for the Gaussian eigenbeam (the u_{00}^{HG} mode) according to (10) or (20). According to (31), the AGP for the eigenmodes u_{01}^{HG}, u_{11}^{HG}, and u_{01}^{LG}, determined by the real part of the complex order a^*, also changes.

5. Conclusions

We have considered the complex-order FrFT performed with the help of several optical schemes, which are either similar to known FrFT schemes with lenses replaced by Gaussian apertures, or to their combination with traditional lens FrFT schemes. The radii of the eigenbeams, their radii of curvature, radiation losses, and accumulated Gouy phases have been calculated in a broad range of the GA and lens parameters. We have found that, for certain parameters of lenses in combined schemes, there exists in one case the degeneration to the optical scheme equivalent to the complex segment, and in the other – to the scheme equivalent to a lens with the complex focal length. We have also shown that the eigenbeams of the complex-order FrFT in the schemes considered in the paper are only the Hermite–Gaussian u_{mn}^{HG} modes with $m, n \leqslant 1$ (or the Laguerre–Gaussian u_{01}^{LG} modes).

Note that, upon the longitudinal diode pumping of active media, the gain can be approximated by the Gaussian distribution $G = (G_0 - 1)\exp(-x^2 w_g^{-2}) + 1$ (G_0 is the gain at the beam axis), which is analogous to the GA; therefore, the matrix method used in the paper can be extended to this case as well.

References

1. Lohmann A.W. *J. Opt. Soc. Am. A*, **10**, 2181 (1993).
2. Mendlovic D., Ozaktas H.M. *J. Opt. Soc. Am. A*, **10**, 1875 (1993).
3. Shih Ch.-Ch. *Opt. Lett.*, **20**, 1178 (1995).
4. Yariv A., Yeh P. *Opt. Commun.*, **13**, 370 (1975).
5. Wang C., Lu B. *Opt. Commun.*, **203**, 61 (2002).
6. Dragoman D. *J. Mod. Opt.*, **45**, 2117 (1998).
7. Gerrard A., Burch J.M. *Introduction to Matrix Methods in Optics* (New York: Wiley, 1975; Moscow: Mir, 1978).
8. Alieva T., Bastiaans M.J., Calvo M.L. *Recent Res. Developments Opt.*, **1**, 105 (2001).