*LASER BEAMS*

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# Calculation of the laser-beam $M^2$ factor by the method of moments

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***Abstract*. The method is proposed for calculating the $M^2$ factor by using the averaged description of wave beams (the method of moments). The values of $M^2$ are calculated for the super-Gaussian intensity distribution with phase distortions caused by the electron and thermal self-focusing and spherical aberration.**

***Keywords***: *method of moments, super-Gaussian beams, laser-beam quality.*

## 1. Introduction

The $M^2$ factor introduced by Siegman and Ginzton in 1990 [1] is now widely used to describe laser beams. The nature of this parameter is closely related to the method of moments proposed by Vlasov, Petrishchev, and Talanov as early as 1971 [2]. However, this relation has not been discussed so far in the literature.

The wave front of a laser beam propagating through various optical elements undergoes distortions, resulting in an increase in the $M^2$ factor. The distortions can be caused both by variations in the refractive index produced by a high-power laser beam and the imperfection of optical elements. Therefore, the calculation of the $M^2$ factor for the beams propagated through phase aberrators, i.e., through thin layers of transparent dielectrics with the refractive index inhomogeneous over the cross section is of current interest.

In this paper, we show that the $M^2$ factor of an arbitrary axially symmetric beam can be easily calculated by using its central moments. As an example we calculated the values of $M^2$ for a super-Gaussian beam propagated through a medium with the electron self-focusing, stationary thermal self-focusing, and through the optical system introducing spherical aberration.

## 2. Definition of the $M^2$ factor

The $M^2$ factor characterising the beam quality is closely related to the propagation of a beam in a medium. Consider an axially symmetric wave beam with the complex field amplitude written in the form

$$u = E(r)\exp[\mathrm{i}\varphi(r)], \qquad (1)$$

where $E$ is the real field amplitude and $\varphi$ is the field phase. Let us use the coordinate whose origin coincides with the beam-waist centre (Fig. 1a). Then, the $M^2$ factor is defined by the expression [1]

$$M^2 = 2\pi a_{\mathrm{w}}\sigma_{\mathrm{f}}, \qquad (2)$$

where

$$a_{\mathrm{w}}^2 = \frac{\int_0^\infty |E(r,z=0)|^2 r^3 \mathrm{d}r}{\int_0^\infty |E(r,z=0)|^2 r \mathrm{d}r}, \quad \sigma_{\mathrm{f}}^2 = \frac{\int_0^\infty |\tilde{u}|^2 v^3 \mathrm{d}v}{\int_0^\infty |\tilde{u}|^2 v \mathrm{d}v}, \qquad (3)$$

are the second moments of the radial intensity distribution of the beam $|E|^2$ and its spatial spectrum $|\tilde{u}(v)|^2$, respectively; and $v$ is the spatial frequency.

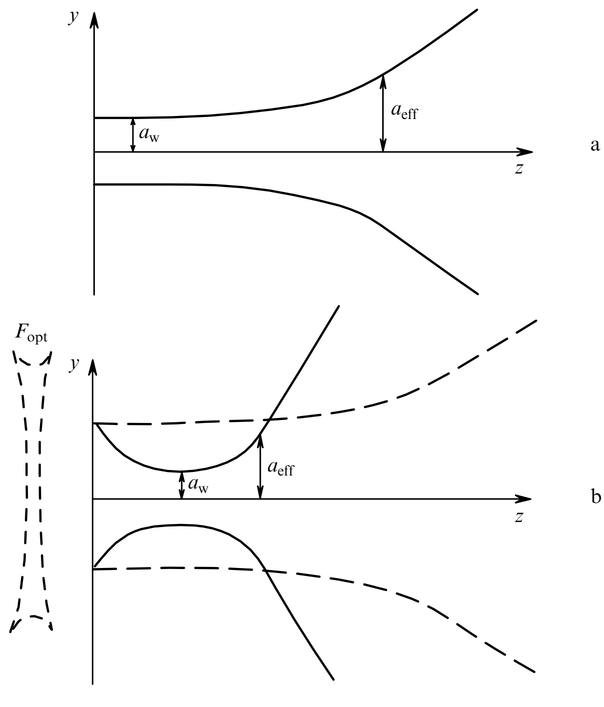


**Figure 1.** Propagation of a beam with a waist in the $z = 0$ plane (a) and a beam that experienced aberration (b) with compensation (dashed curves) and without compensation (solid curves) by a lens.

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A remarkable property of the effective radius of the beam, determined from the expression

$$a_{\text{eff}}^2(z) = \frac{\int_0^\infty |E(r,z)|^2 r^3 \text{d}r}{\int_0^\infty |E(r,z)|^2 r \text{d}r}, \quad (4)$$

is its simple dependence on the coordinate $z$ for any distributions of the field amplitude and phase [1, 2]:

$$a_{\text{eff}}^2(z) = a_{\text{w}}^2 + \vartheta^2 z^2, \quad (5)$$

where

$$\vartheta = \lambda\sigma_{\text{f}} = \frac{M^2}{ka_{\text{w}}} \quad (6)$$

is the beam divergence and $k$ is the wave number. For a Gaussian beam with a plane phase, the factor $M^2 = 1$ and expression (5) describes the well-known spread of a Gaussian beam in the paraxial approximation. The physical meaning of the $M^2$ factor follows from (6): this factor shows the excess of the divergence $\vartheta$ of a real beam over the divergence $1/(ka_{\text{w}})$ of a Gaussian beam if these beams have the same waist size $a_{\text{w}}$ determined from (3).

## 3. Calculation of the $M^2$ factor by the method of moments

By calculating the $M^2$ factor from expressions (2) and (3), it is necessary to find the second moments of the intensity in the beam waist and intensity of the spectrum (far-field intensities). This can be easily done for beams with a plane phase [3–5]. In a more general case, especially if the phase is such that the beam waist is already located not at $z = 0$, the calculation of $M^2$ is not simple because integration in the first expression in (3) should be performed in the beam waist, whose position should be preliminarily determined.

The $M^2$ factor can be determined much simply by using the averaged description of wave beams – the method of moments [2]. Let us calculate $M^2$ for a beam with the field distribution (1). According to the method of moments, the dependence of the square of the effective radius (4) on $z$ has the form [2]

$$a_{\text{eff}}^2(z) = Az^2 + bz + a_{\text{eff}}^2(z=0), \quad (7)$$

where

$$A = \frac{1}{k^2}\frac{\int_0^\infty (\nabla E)^2 r\text{d}r}{\int_0^\infty E^2 r\text{d}r} + \frac{\int_0^\infty (\nabla\varphi/k)^2 E^2 r\text{d}r}{\int_0^\infty E^2 r\text{d}r},$$
$$b = \frac{2}{k}\frac{\int_0^\infty (r\nabla\varphi) E^2 r\text{d}r}{\int_0^\infty E^2 r\text{d}r}. \quad (8)$$

One can easily see from (7) that the square of the beam divergence is $\vartheta^2 = A$, i.e., taking (6) into account, we have $\sigma_{\text{f}}^2 = A/\lambda^2$. Therefore, if the beam phase $\varphi(r)$ is such that the waist is located at $z = 0$, i.e., $a_{\text{w}} = a_{\text{eff}}$, we obtain from (2) that

$$M^2 = ka_{\text{eff}}\sqrt{A}. \quad (9)$$

In the general case, when the waist position is arbitrary, we will use the fact that the $M^2$ factor does not change when the beam propagates through an aberration-free lens [1]. Let us place in the $z = 0$ plane an imaginary aberration-free lens with the focal length $F_{\text{opt}}$ providing the location of the beam waist in this plane, i.e., the minimal divergence of the beam (Fig. 1b) [6]. As shown in [2], an aberration-free lens with the focal length $F$ does not change $a_{\text{eff}}$, and the parameters $A'$ and $b'$ at the lens output are expressed in terms of the input parameters $A$ and $b$ as

$$b' = b - \frac{2a_{\text{eff}}^2}{F}, \quad A' = A - \frac{b}{F} + \frac{a_{\text{eff}}^2}{F^2}. \quad (10)$$

It follows from (10) that the beam divergence behind the lens (the minimal value of $A'$) will be minimal for the lens focal length

$$F_{\text{opt}} = \frac{2a_{\text{eff}}^2}{b}, \quad (11)$$

and is equal to

$$A_{\min} = A'(F_{\text{opt}}) = A - \frac{b^2}{4a_{\text{eff}}^2}. \quad (12)$$

When condition (11) is fulfilled, we have $b' = 0$, i.e., the beam waist will be located at the lens output and, hence, $a_{\text{w}} = a_{\text{eff}}$. By substituting (12) into (9), we obtain

$$M^2 = k(Aa_{\text{eff}}^2 - b^2/4)^{1/2}, \quad (13)$$

where $a_{\text{eff}}$, $A$, and $b$ are determined by integrals in (4) and (8). Unlike (2) and (3), expression (13) does not require the calculation of the beam spectrum and the determination of the beam waist location.

## 4. Examples of calculation of the $M^2$ factor

Consider the propagation of a super-Gaussian beam with the amplitude

$$u = E_0 \exp\left[-\frac{1}{2}\left(\frac{r}{w}\right)^{2m}\right]$$

through phase aberrators characterised by parameters $\varphi_{\text{e}}$, $\varphi_{\text{t}}$, and $\varphi_{\text{s}}$ and appearing upon electron self-focusing determined by the $B$ integral [7], stationary thermal self-focusing determined by the dimensionless parameter $p_{\text{i}}$ [8], and spherical aberration of a telescope consisting of two centred coaxial and confocal lenses determined by the parameter $S$, respectively [9, 10]. Then,

$$\varphi_{\text{e}}(r) = B\exp\left(-\frac{r^{2m}}{w^{2m}}\right),$$
$$\varphi_{\text{t}}(r) = -p_{\text{i}}\frac{m}{2\Gamma(1/m)}\int_0^{r^2/w^2}\left[\int_0^y \exp(-y^m)\text{d}y\right]\frac{\text{d}r}{r}, \quad (14)$$
$$\varphi_{\text{s}}(r) = -S\frac{r^4}{w^4},$$

where

$$B = \frac{263}{\lambda}\int_0^L \frac{n_2}{n} I_0(z)\mathrm{d}z; \quad p_\mathrm{i} = \frac{PP_\mathrm{L}\alpha L}{\lambda\kappa};$$
$$S = \frac{kw^4}{2f_2^4}(G_1 f_1 + G_2 f_2); \tag{15}$$

$P_\mathrm{L}$ is the average radiation power; $I_0$ is the radiation intensity at the beam axis; $n$, $n_2$, $\alpha$, $\kappa$, $L$ and $P$ are the linear and nonlinear refraction indices, the absorption coefficient, the heat conduction and length of the medium, and the thermooptic constant [11, 12] determining the thermal lens of an optical element, respectively; $f_{1,2}$ and $G_{1,2}$ are the focal lengths and the Zeidel sums of lenses, respectively [9, 10]; and $\Gamma(x)$ is the Euler gamma function.

Expressions for the focal length $F_\mathrm{opt}$ of a correcting lens and the $M^2$ factor for these three aberrators can be easily calculated by substituting (4), (8), and (14) into (11) and (13):

$$F_\mathrm{opt\,e} = -\frac{kw^2}{B}2^{1/m}\frac{\Gamma(2/m)}{\Gamma(1/m)}, \quad F_\mathrm{opt\,t} = -\frac{kw^2}{p_\mathrm{i}}\frac{2\Gamma(2/m)}{\Gamma(1/m)},$$
$$F_\mathrm{opt\,s} = -\frac{kw^2}{S}\frac{\Gamma(2/m)}{4\Gamma(3/m)}, \tag{16}$$

$$M_\mathrm{e}^4 = \frac{m^2\Gamma(2/m)}{\Gamma^2(1/m)} + VB^2, \quad M_\mathrm{t}^4 = \frac{m^2\Gamma(2/m)}{\Gamma^2(1/m)} + Up_\mathrm{i}^2,$$
$$M_\mathrm{s}^4 = \frac{m^2\Gamma(2/m)}{\Gamma^2(1/m)} + WS^2, \tag{17}$$

where

$$V = \frac{4m^2\Gamma(2/m)}{9\Gamma^2(1/m)^2} - 2^{-2/m}; \quad U = \frac{m^3\Gamma(2/m)}{\Gamma^4(1/m)}$$
$$\times \int_0^\infty \left[\int_0^t \exp(-y^m)\mathrm{d}y\right]^2 \exp(-t^m)\frac{\mathrm{d}t}{t} - \frac{1}{4}; \tag{18}$$

$$W = \frac{16}{\Gamma^2(1/m)}[\Gamma(2/m)\Gamma(4/m) - \Gamma^2(3/m)].$$

Note that the value of $M^4$ always quadratically depends on the parameters $B$, $p_\mathrm{i}$, and $S$ characterising the aberration strength. In the absence of aberrations ($B = 0$, $p_\mathrm{i} = 0$, $S = 0$), expressions (17) transfer to the known expressions for the $M^2$ factor of a super-Gaussian beam [3]. The minimal value of $M^2$ is achieved for $m = 1$ (i.e., for a Gaussian beam), and this value shifts to larger $m$ with increasing the aberration strength (Fig. 2).

Curves in Fig. 2 are constructed for the beam radius $w$ independent of $m$. In practice, beams with large values of $m$ and the same aperture of optical elements have usually a greater radius and, hence, a lower axial intensity $I_0$. In this case, the minimal values of $M^2$ shift to the right upon electron self-focusing ($B$ decreases with increasing $m$), remain constant upon thermal self-focusing ($p_\mathrm{i}$ does not change with increasing $m$), and significantly (to $m < 1$) shift to the left upon spherical aberration ($S$ increases with increasing $m$).

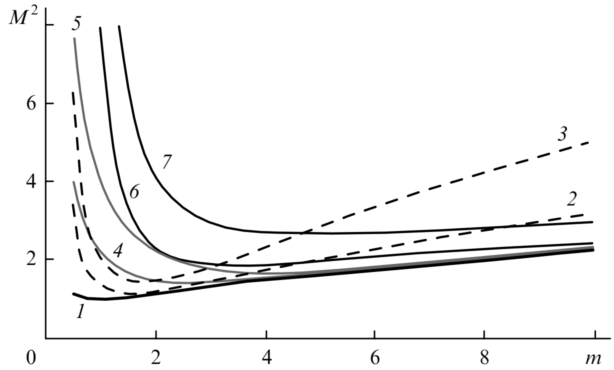


**Figure 2.** Dependences $M^2(m)$ in the absence of aberrations (*1*), upon electron self-focusing with $B = 2$ (*2*) and 4 (*3*), thermal self-focusing with $p_\mathrm{i} = 10$ (*4*) and 20 (*5*), and upon spherical aberration with $S = 2$ (*6*) and 4 (*7*).

## 5. Conclusions

By using the method of moments [2], we have obtained expressions for the $M^2$ factor of an arbitrary axially symmetric laser beam. The values of $M^2$ were calculated for a super-Gaussian beam with phase distortions caused by the electron and thermal self-focusing and spherical aberration.

The possibilities of the method of moments are not restricted by the cases considered here. Analytic expressions for the $M^2$ factor can be similarly obtained in the case of a simultaneous action of self-focusing, spherical and other types of aberration. The method of moments can be also applied to nonlinear media, so that the $M^2$ factor can be calculated for a beam propagated through a thick nonlinear layer, i.e., the layer in which phase distortions caused by self-focusing change the intensity distribution over the medium length.

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