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On alternative methods for measuring the radius and propagation ratio of axially symmetric laser beams

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***Abstract*. Based on the developed efficient numerical methods for calculating the propagation of light beams, the alternative methods for measuring the beam radius and propagation ratio proposed in the international standard ISO 11146 are analysed. The specific calculations of the alternative beam propagation ratios M_i^2 performed for a number of test beams with a complicated spatial structure showed that the correlation coefficients c_i used in the international standard do not establish the universal one-to-one relation between the alternative propagation ratios M_i^2 and invariant propagation ratios M_σ^2 found by the method of moments.**

Keywords: *beam radius, method of moments, beam propagation ratio, alternative measurement methods, international standard.*

1. Introduction

The main parameters characterising an axially symmetric laser beam are the beam radius $w(z)$, the radius of curvature $R(z)$ of the wave front, and the divergence angle θ. However, these parameters can be rather simply determined only for Gaussian beams [1 – 3]. In the case of beams with a complicated spatial structure, the International Standard Organization proposed to use the normalised moments of the beam power density distribution for determining the beam radius and other parameters, in particular, the propagation ratio M^2 [4]. The averaged description of wave beams in linear and nonlinear media by using the method of moments of the transverse distribution of the energy flux density was proposed as early as 1971 [5]. The main advantage of this method is that the radius of an axially symmetric beam changes along the beam propagation direction by the same law $w_\sigma^2(z) = w_{\sigma 0}^2 + \theta_{\sigma 0}^2 (z - z_{\sigma 0})^2$ as for axially symmetric Gaussian beams; here, $w_{\sigma 0}$ is the beam radius in the $z = z_{\sigma 0}$ plane of the generalised waist and $\theta_{\sigma 0}$ is half the total beam divergence [3, 4]. These parameters for Gaussian beams coincide with usual parameters, for which the relation $w_0 \theta_0 = \lambda/\pi$ is fulfilled, where λ is the wavelength [1, 2].

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In the general case, this relation takes the form $w_{\sigma 0}\theta_{\sigma 0} = M_\sigma^2 \lambda/\pi$, where $M_\sigma^2 \geqslant 1$ is the so-called beam propagation ratio [3, 4]. Note that this ratio is minimal for Gaussian beams ($M_\sigma^2 = 1$) and $M_\sigma^2 > 1$ for all other beams, the numerical value of this ratio being preserved during the beam propagation in the first-order optical systems [1 – 3]. It is this invariance property of the ratio M_σ^2 that makes it attractive and useful. Although critical comments concerning the parameter introduced in this way are well known (see, for example, [6, 7]), nevertheless the international standard ISO 11146 [4] used till recently is also recommended in the literature in Russian [8] and the beam propagation ratio M_σ^2 (or the beam quality) is now widely employed in papers.

Unfortunately, the measurement of the beam radii and propagation ratios by using the second moments requires, first, multielement detectors for measuring the transverse distribution of the laser beam intensity, which is possible not for all wavelengths. Second, the processing of experimental results obtained with CCD or CMOS cameras [9] involves certain difficulties caused by the presence of a dark inhomogeneous background and thermal noise from individual pixels in the measured intensity distribution (energy density) (see, for example, [3, 9 – 11] and references therein). In addition, high-quality CCD cameras are still quite expensive. For this reason, the applied standard [4, 8] allowed the measurement of the beam radii and propagation ratios by the so-called alternative methods based on the use of a variable circular aperture, a moving knife or a slit. The standard [4] assumes that there exists a correlation between the propagation ratios M_σ^2 and M_i^2 determined by the alternative methods, which is established by the relation $M_\sigma = c_i(M_i - 1) + 1$. This linear (in $M = \sqrt{M^2}$) relation allows one to perform simply the conversion of results if the correlation coefficients c_i are known. The coefficients c_i recommended in [4] were found in experiments with low-power gas lasers. It is pointed out in [4] that for the beams with the propagation ratios $M^2 \geqslant 4$ and lasers of other types, the values of these coefficients should be verified. However, because of their simplicity, the alternative methods are also widely used for measuring beam radii and propagation ratios for various solid-state lasers [12 – 14], including phase-conjugate lasers having a high degree of radiation coherence [13, 14].

One of the most popular alternative methods is the method for measuring the beam radius and propagation ratio by means of a variable circular aperture [15]. It was shown recently that for the incoherent superposition of two coaxial Gaussian beams, the propagation ratio M_a^2 measured by this method can be smaller than unity [16]. This means that for the positive correlation coefficient, $c_a > 0$ and the beam propagation ratio measured by the method of moments is $M_\sigma^2 < 1$. This example shows that the use of alternative methods, assuming the existence of the above-mentioned correlation, can lead to physically unsatisratioy results. Nevertheless, a new version of the standard [17] retains the possibility of using alternative measurement methods. For this reason, we analysed in this paper in detail the relation $M_\sigma = c_i(M_i - 1) + 1$ for a number of axially symmetric beams with the known propagation ratios M_σ^2 by using the efficient numerical methods developed for calculating the propagation and focusing laser beams of a complicated spatial structure [18] and showed that no universal one-to-one relation exists between M_σ^2 and the propagation ratios M_i^2 found by alternative methods.

2. Method for calculating focusing of complicated beams

For simplicity we consider axially symmetric monochromatic beams $\tilde{\boldsymbol{E}}(r,z,t) = \mathrm{Re}[\hat{e}u(r,z)\exp(\mathrm{i}kz - \mathrm{i}\omega t)]$. Here, $\hat{e}$ is the unit polarisation vector, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, ω is the cyclic frequency, and the slowly varying complex amplitude $u(r,z)$ satisfies the parabolic equation

$$\frac{\partial u}{\partial z} + \frac{\Delta_r u}{2\mathrm{i}k} = 0, \tag{1}$$

in a free space, where

$$\Delta_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

is the transverse Laplacian. It is well known [1–3] that Eqn (1) has solutions in the form of the Laguerre–Gaussian (LG) modes:

$$\tilde{u}_p(r,z) = \left[\frac{2}{\pi\tilde{w}^2(z)}\right]^{1/2} L_p\left(\frac{2r^2}{\tilde{w}^2(z)}\right)$$
$$\times \exp\left[-\frac{r^2}{\tilde{w}^2(z)} + \frac{\mathrm{i}kr^2}{2\tilde{R}(z)} - \mathrm{i}(2p+1)\arctan\frac{\lambda(z-\tilde{z}_0)}{\pi\tilde{w}_0^2}\right], \tag{2}$$

where $L_p(\cdot)$ is the Laguerre polynomial, and the beam radius $\tilde{w}(z)$ of the fundamental Gaussian mode and the radius $\tilde{R}(z)$ of the wave-front curvature are described by expressions

$$\tilde{w}(z) = \tilde{w}_0\left\{1 + \left[\frac{\lambda(z-\tilde{z}_0)}{\pi\tilde{w}_0^2}\right]^2\right\}^{1/2} \text{ and}$$
$$\tilde{R}(z) = (z-\tilde{z}_0)\left\{1 + \left[\frac{\pi\tilde{w}_0^2}{\lambda(z-\tilde{z}_0)}\right]^2\right\}. \tag{3}$$

The values of radii $\tilde{w}(z)$ and $\tilde{R}(z)$ are uniquely determined by the position of the waist plane $\tilde{z}_0$ and its radius $\tilde{w}_0$. It is obvious that the position of the waist plane $\tilde{z}_0$ and its radius $\tilde{w}_0$ can be determined from the values of radii $\tilde{w}(z)$ and $\tilde{R}(z)$.

It is well known [1–3] that any axially symmetric solution of Eqn (1) can be expanded in the eigenmodes (2)

$$u(r,z) = \sum_{p=0}^{\infty} \tilde{c}_p(z_1)\tilde{u}_p(r,z), \tag{4}$$

where the coefficients

$$\tilde{c}_p(z_1) = 2\pi\int_0^\infty u(r,z_1)\tilde{u}_p^*(z_1)r\,\mathrm{d}r$$

depend not only on the choices of the initial plane $z = z_1$ but also on the parameters determining a set of eigenmodes (2). This important circumstance is indicated by the tilde in expansion (4). Let us explain this by a simple example. The solution of Eqn (1) in the form of some LG mode $u_q(r,z)$ can be always expanded over the full system of modes (2). However, in the general case many expansion terms (4) are required. And only in the case when the parameters of expansion modes exactly coincide with those of the initial mode, only one term ($p = q$) in expansion (4) will be required. To describe the diffraction of the initial beam by an aperture, again many LG modes will be obviously required [19], the criteria for their choice being quite uncertain in the general case [20]. It seems that for this reason the efficient mode-expansion method (MEM) is used comparatively rarely in numerical calculations of diffraction problems [1–3, 18–22].

Thus, to use the MEM for numerical calculations of the propagation of beams of a complicated structure, additional considerations are required, which are obviously related to the beam propagation ratio for the problem under study. Because the determination of the beam propagation ratio for beams restricted by rigid apertures involves some difficulties [3], we will consider below only the beams that are not restricted by rigid apertures. During the propagation of such beams, the total power and the propagation ratio are preserved [23, 24]:

$$P(u) = 2\pi\int_0^\infty |u|^2 r\,\mathrm{d}r, \tag{5}$$

$$M_\sigma^2(u) = \left[\int_0^\infty\left|\frac{\partial u}{\partial r}\right|^2 r\,\mathrm{d}r\int_0^\infty r^3|u|^2\mathrm{d}r\right.$$
$$\left.-\frac{1}{4}\left|\int_0^\infty\left(u\frac{\partial u^*}{\partial r} - u^*\frac{\partial u}{\partial r}\right)r^2\mathrm{d}r\right|^2\right]^{1/2}\left(\int_0^\infty |u|^2 r\,\mathrm{d}r\right)^{-1}. \tag{6}$$

In addition, it is also convenient to introduce the beam radius and the radius of beam curvature determined by the method of moments [3, 23]:

$$w_\sigma(u) = 2\left[\frac{\pi}{P(u)}\int_0^\infty r^3|u|^2\mathrm{d}r\right]^{1/2},$$
$$\frac{1}{R_\sigma(u)} = \frac{-\mathrm{i}\lambda}{w_\sigma^2(u)P(u)}\int_0^\infty r^2\left(\frac{\partial u}{\partial r}u^* - u\frac{\partial u^*}{\partial r}\right)\mathrm{d}r, \tag{7}$$

as well as the generalised complex beam parameter

$$\frac{1}{q_\sigma} = \frac{1}{R_\sigma} + \frac{\mathrm{i}\lambda M_\sigma^2}{\pi w_\sigma^2}. \tag{8}$$

Then, the *ABCD* law [1–3] can be used for any, not only Gaussian, beams [23]. Its application shows that to any beam a Gaussian beam can be 'embedded' whose radius $\bar{w}_\mathrm{G}(z) = (w_\sigma^2(z)/M_\sigma^2)^{1/2}$ and radius of curvature $\bar{R}_\mathrm{G}(z) = R_\sigma(u)$ vary as for a usual Gaussian beam with the Rayleigh waist length $\bar{z}_\mathrm{R} = k\bar{w}_{\mathrm{G}0}^2/2$, where $\bar{w}_{\mathrm{G}0}$ is the waist radius of the embedded Gaussian beam [25, 26].

The concept of an embedded Gaussian beam is often useful for optimisation of the expansion in the LG modes

$$u_{\rm ap}(r,z) \approx \sum_{p=0}^{N_p} \tilde{c}_p \tilde{u}_p(r,z) \tag{9}$$

to minimise the approximation error [25–28]. By using the energy norm, it was shown in [28] that for beams with a plane wave front in the input plane $[R_\sigma(u) = \infty]$, the relation $\varepsilon \leqslant (M_\sigma^2 - 1)/2N_p$ for the relative error

$$\varepsilon = \frac{\int_0^\infty |u - u_{\rm ap}|^2 r\,{\rm d}r}{\int_0^\infty |u|^2 r\,{\rm d}r}$$

is valid if the LG modes with $\tilde{w}_0 = \bar{w}_{\rm G0}$ in the input plane are used in expansion (7).

Unfortunately, such simple estimates cannot be obtained in the general case, the more so for other norms [18]. Therefore, we used here the following procedure. First the LG mode parameters and their number were selected so that errors in the specified norm would be minimal for the start plane (usually, the $z_1 = 0$ plane). Note that in the general case, to achieve this goal, the radii of the beam $[\tilde{w}(z_1)]$ and curvature $[\tilde{R}(z_1)]$ of the fundamental LG mode may not be coincident generally speaking with the corresponding radii of the initial (or embedded Gaussian) beam determined by the method of moments.

Having received minimal errors in the initial plane, we can hope that rather small errors in the specified norm will be also obtained in other planes. To control additionally the accuracy of expansion of the beams in the LG modes, we calculated first from expressions (5) and (6) the power and the beam propagation ratio and compared them with the corresponding expressions [25] for the power and propagation ratio of the coherent superposition of the LG modes:

$$P(u) \cong \sum_{p=0}^{N_p} \tilde{c}_p^* \tilde{c}_p, \quad M_\sigma^2(u) \cong \left\{ \left[\sum_{p=0}^{N_p} (2p+1) R_{pp} \right]^2 - \left[2 \sum_{p=0}^{N_p} (p+1) R_{p\,p+1} \right]^2 \right\}^{1/2}, \tag{10}$$

where $R_{nm} = \tilde{c}_n^* \tilde{c}_m / P(u)$ are the normalised expansion coefficients and the condition

$$\sum_{p=0}^{N_p} (p+1) {\rm Im} R_{p\,p+1} \cong 0$$

should be fulfilled for collimated beams in the initial plane. If the LG modes with the radius of curvature obtained from the relation

$$\frac{1}{\tilde{R}} = \frac{1}{f} - \frac{1}{R_\sigma} \tag{11}$$

are used for the initial beams with aberrations and the wave-front curvature R_σ after a focusing lens with the focal distance f, relation (10) is again valid and can be once more used to control the expansion accuracy.

To reduce the number $N_{\rm r}$ of nods of a radial network, two independent transformations were normally used. First, calculations for the specified plane z were performed only within an aperture dependent on z, which was varied according to a certain law, for example, like the initial beam radius: $A(z) \sim w_\sigma(z)$. Second, the transformation of the transverse coordinate was used, which allowed one to obtain from the homogeneous grid $r_j(z) = jA(z)/N_{\rm r}$ the inhomogeneous network with the near-axis condensation of points ($j = 0, 1, 2, ...$). Taking into account that the radius of the higher LG mode determined by the method of moments is $w_{p\sigma}(z) = w_0(z)(2p+1)^{1/2}$ [3], the aperture radius was matched to the number N_p of expansion modes. Therefore, the size of the initial aperture was chosen to provide the fulfilment of the inequality $A(0) > \tilde{w}_0(2N_p + 1)^{1/2}$. In this case, the number of radial points was chosen sufficiently large ($N_{\rm r} \sim 2000$) in order that the expansion accuracy would be determined only by the restriction of the number N_p of expansion terms.

To verify the assumptions presented above, we performed the corresponding calculations of the propagation and focusing of the beams for which exact analytic solutions are known. First of all, the coherent superposition of coaxial Gaussian beams, which can give the zero intensity at the beam axis [29], belongs to the beams of this type. The exact solution is also known for the propagation of a Bessel–Gaussian (BG) beam [30]. For a BG beam collimated in the initial plane, analytic expressions were also found for the expansion coefficients in LG modes [31]. Integral expressions for the expansion coefficients of a Gaussian beam with spherical aberration were obtained in [26].

The exact and approximate (by the given method) transverse field distributions were calculated for these beams behind a focusing lens (the calculations were performed for definiteness for a spherical lens with the focal distance $f = 100$ cm and beams at a wavelength of 1.06 μm) and the calculation errors were found for the given norms. The additional control was performed by calculating the beam propagation ratios in these planes from (6), which were compared with the known values. Our calculations showed that for different beams with close values of M_σ^2, the numbers of expansion terms required to obtain the given accuracy could significantly differ depending on the selected beam radius and the radius of curvature of the LG modes.

3. Calculation of the alternative beam radii and beam propagation ratios

The main advantage of the method of expansion in the LG modes over the diffraction integral method [1–3] is that, by calculating once expansion coefficients $\tilde{c}_p$, it is easy to calculate the field distribution $u_{\rm ap}(r,z)$ in any plane because the propagation law of LG modes (2) is known. It is for this reason that the MEM is convenient for the numerical analysis of alternative methods for measuring the beam propagation ratio, when it is necessary to calculate the field distribution simultaneously in many planes.

A fraction of the beam energy propagating through a centred circular aperture of radius a in a given plane can be calculated most conveniently from the known field distribution $u_{\rm ap}(r,z)$. For a Gaussian beam, this fraction is $\eta_{\rm a\,G}(a,z) = 1 - \exp[-2a^2/w_{\rm G}^2(z)]$, where $w_{\rm G}(z)$ is the beam radius in the z plane. The calculation of the beam energy fraction propagating through a slit of width $2d$ whose middle is located at a distance of c from the beam axis is somewhat more complicated. For a Gaussian beam, this fraction is

$$\eta_{sG}(c,z) = \frac{1}{2}\left[\operatorname{erf}\left(\sqrt{2}\,\frac{c+d}{w_G(z)}\right) + \operatorname{erf}\left(\sqrt{2}\,\frac{-c+d}{w_G(z)}\right)\right],$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\mathrm{d}t$$

is the error function [12]. Obviously, this fraction will be small for a thin slit ($d \to 0$). Therefore, it is more convenient to use the energy fraction transmitted through a narrow slit normalised to a maximum. For a Gaussian beam, this fraction is $\bar{\eta}_{sG}(c,z) = \exp(-2c^2/w_G^2(z))$. Note that the bar over η_s in figures below is omitted for simplicity. Calculations of the energy fractions transmitted through a slit and a moving knife are closely interrelated. If the fraction of energy propagated through a slit of width $2b$ centred with the beam axis ($c = 0$) is known, the standard energy fractions behind a moving knife at any specified level can be easily calculated [3, 4, 8, 17]. For a Gaussian beam, the fraction of energy propagated through such a slit is $\eta_{kG}(b,z) = \operatorname{erf}(\sqrt{2}b/w_G(z))$. Note that the transmission curves for a Gaussian beam are self-similar at any cross section of the beam.

The radius of an arbitrary axially symmetric beam is measured according to the international standard ISO 11146 [4, 8] and its new modification [17] as follows. The aperture radius $a_{0.86}$ through which 86.5 % of the total beam power propagates is found by the linear interpolation method. This radius is taken as the beam radius $w_{a\,0.86}$ determined by the variable aperture method from the fraction of transmitted energy $\eta_a = 0.865$. To measure the aperture radius a_η for a different fraction (η) of transmitted energy, it is necessary to find the relation between this radius and the standard (at the 86.5 % level of the total power) radius of the equivalent Gaussian beam $w_{a\eta} = k_a(\eta)a_\eta$.

The expression for the correspondence coefficient $k_a(\eta) = \{2[-\ln(1-\eta)]^{-1}\}^{1/2}$ can be easily obtained from the above expressions. The equivalent radii for any specified level can be obtained from the calculated transmission curve $\eta_a(a,z)$ monotonically increasing with the aperture radius a. Below, we present the calculated equivalent radii for transmission of the energy fractions 0.80, **0.86**, and 0.95 for which the coefficients k_a are equal to 1.12, **1.00**, and 0.82, respectively. (The transmission levels and corresponding coefficients for determining alternative beam radii recommended by the standard ISO 11146 [17] are indicated in bold.) The relation $w_{k\eta} = k_k(\eta)b_\eta$ is found by the same method from the transmission fraction $\eta_k(b)$. The coefficients k_k equal to **2.00**, 1.56, and 1.33 correspond to the energy fractions **0.68**, 0.80, and 0.86. Recall that to the radii b_η, the positions of a moving sharp edge (knife) correspond for which the fractions of transmitted power (energy) are equal to **84** % and **16** %, 90 % and 10 %, 93 % and 7 %, respectively [3, 4, 8, 17]. In the case of a slit, the corresponding coefficients k_s are **1.00**, 1.11, and 1.29 for the normalised transmission levels equal to **0.135**, 0.20, and 0.30, respectively.

The specific calculations of the alternative propagation ratios are presented for a number of beams for which the standard beam propagation ratios M_σ^2 are known. Then, by using the expression [2, 3]

$$w_\sigma^2(z) = w_\sigma^2(0)\left[\left(A + \frac{B}{R_\sigma(0)}\right)^2 + \left(\frac{\lambda M_\sigma^2 B}{\pi w_\sigma^2(0)}\right)^2\right] \tag{12}$$

we can find the beam radius at any plane z behind a spherical lens. Here, the corresponding $ABCD$ matrix has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - z/f & z \\ -1/f & 1 \end{pmatrix}.$$

This analytic expression for the beam radius allows us to verify additionally the accuracy of numerical calculations because, aside from the alternative radii $w_{a,k,s}(z)$, we also found the radii $w_\sigma(z)$ in each numerical experiment directly from expression (7), which were compared with those calculated by a simple expression (12).

For a coherent superposition of two Gaussian beams

$$u_{GG}(r, z=0) = A_1\exp\left[-\left(\frac{r}{w_1}\right)^2\right] + A_2\exp\left[-\left(\frac{r}{w_2}\right)^2\right] \tag{13}$$

with the common waist in the $z = 0$ plane, the beam propagation ratio can be easily found as $M_{GG\sigma}^2 = (I_2 I_3)^{1/2}/I_1$. Here, $I_1 = \frac{1}{4}[A_1^2 w_1^2 + A_2^2 w_2^2 + 2A_1A_2 w_{12}^2]$; $I_2 = \frac{1}{8}\times[A_1^2 w_1^4 + A_2^2 w_2^4 + 2A_1A_2 w_{12}^4]$; $I_3 = \frac{1}{2}[A_1^2 + A_2^2 + 2A_1A_2 w_{12}^4(w_1\times w_2)^{-2}]$ and $w_{12}^2 = 2(1/w_1^2 + 1/w_2^2)^{-1}$. By knowing $M_{GG\sigma}^2$ and the initial radius $w_{GG\sigma}(z=0) = (2I_2/I_1)^{1/2}$ of such a beam, we can easily find from (12) the beam radius in any plane behind the lens.

Of great interest are super-Gaussian beams

$$u_{SG}^{(L,NL)}(r, z=0) = u_{SG}(r)\exp(\mathrm{i}\Phi_{L,NL}(r)) \tag{14}$$

with linear $[\Phi_L(r,z) = \alpha_m(r/w_0)^m]$ and nonlinear $[\Phi_{NL}(r) = \alpha_{NL}u_{SG}^2(r)]$ aberrations in the initial plane. Here, $u_{SG}(r) = \exp[-(r/w_0)^n]$ and the coefficients α_m and α_{NL} determine the value and sign of aberrations. The power

$$P(u_{SG}^{(L)}) = 2\pi w_0^2 \frac{1}{n}\left(\frac{1}{2}\right)^{2/n}\Gamma\left(\frac{2}{n}\right)$$

and initial radius

$$w_\sigma(u_{SG}^{(L,NL)}) = \sqrt{2}w_0^2\left(\frac{1}{2}\right)^{2/n}\left[\Gamma\left(\frac{4}{n}\right)\Big/\Gamma\left(\frac{2}{n}\right)\right]^{1/2}$$

of such beams [where $\Gamma(z)$ is the gamma function] are well known. Also, the analytic expressions are known [23, 32] for the radius of curvature

$$R_\sigma(u_{SG}^{(L)}) = 2\pi w_0^2\Gamma\left(\frac{4}{n}\right)\left[m\lambda\alpha_m\left(\frac{1}{2}\right)^{(m-2)/n}\Gamma\left(\frac{m+2}{n}\right)\right]^{-1}, \tag{15}$$

$$R_\sigma(u_{SG}^{(NL)}) = \frac{2^{4/n}\sqrt{\pi}w_0^2}{\lambda\alpha_{NL}}\Gamma\left(\frac{2}{n}+\frac{1}{2}\right) \tag{16}$$

and the beam propagation ratios

$$M_\sigma^2(u_{SG}^{(L)}) = \left\{\frac{n^2}{4}\Gamma\left(\frac{4}{n}\right) + \alpha_m^2 m^2\left(\frac{1}{2}\right)^{2m/n}\right.$$

$$\left.\times\left[\Gamma\left(\frac{2m}{n}\right)\Gamma\left(\frac{4}{n}\right) - \left[\Gamma\left(\frac{m+2}{n}\right)\right]^2\right]\right\}^{1/2}\Big/\Gamma\left(\frac{2}{n}\right), \tag{17}$$

$$M_\sigma^2(u_{\rm SG}^{(\rm NL)}) = M^2(u_{\rm SG})\left\{1+\left(\frac{2\alpha_{\rm NL}}{3}\right)^2 \times\left[1-\left(\frac{3}{2^{2/n}n}\right)^2\Gamma^2\left(\frac{2}{n}\right)\Big/\Gamma\left(\frac{4}{n}\right)\right]\right\}^{1/2}. \quad (18)$$

In a particular case of a Gaussian beam with a spherical aberration, expressions (15) and (17) coincide with the expressions known earlier (see [26] and references therein).

Note that the calculations of Gaussian beams with aberrations in this method require a great number N_p of expansion terms compared to super-Gaussian beams for the same aberration coefficients. This is explained by the fact that in the case of a Gaussian beam, a comparatively large fraction of the beam energy $\eta \approx 0.135$ with rapidly increasing linear aberrations and slowly decreasing intensity distribution remains beyond the radius $r = w_0$ at which the aberration value is α_m. In addition, to reduce the number of expansion terms in the case of beams with aberrations, expression (11) should be used for calculating the optimal radius of curvature of the LG modes.

The beam [29]

$$u_{{\rm d}l}(r, z=0) = \left(\frac{r}{w_0}\right)^l \exp\left(-\frac{r^2}{w_0^2}\right) \quad (19)$$

has a circular structure in the input plane with the zero intensity on the beam axis. In this case, the beam propagation coefficient is described by a simple expression $M_\sigma^2 = (l+1)^{1/2}$.

The Bessel – Gaussian beam

$$u_{\rm BG}(r, z=0) = J_0(\beta r)\exp\left[-\left(\frac{r}{w_0}\right)^2\right] \quad (20)$$

(β is the Bessel beam parameter) has a complicated circular structure in the input plane. However, the propagation law of this beam through the *ABCD* system [30] and the propagation coefficient $M_{\rm BG}^2 = \{[1+\mu I_1(\mu)/I_0(\mu)]^2-\mu^2\}^{1/2}$, are known, where $\mu = \beta^2 w_0^2/4$, $I_{0\,1}(\mu)$ are the modified Bessel functions [33]. Therefore, by using expansion (9), we can quite simply calculate the distributions of the field and propagation ratios in any plane (Figs 1 and 2). One can see that the dependence $\eta_{\rm s}(c)$ either oscillates in the input plane (Fig. 1c) or has a nonmonotonic character in the focal plane (Fig. 2c). Note also that similar dependences $\eta_{\rm s}(c)$ are also typical for higher LG modes (Fig. 3c).

The problem of ambiguity of measuring the beam radius in the standard method of a narrow moving slit can be solved in different ways. In the simplest method, the minimal radius is taken as the beam radius with a given transmission level. In the calculations presented below, we used this method. Then, by increasing the slit width, we can smooth the transmission curve, which sometimes eliminates its oscillations in the case of a narrow slit.

The most efficient and, generally speaking, applied to any alternative methods for measuring the beam radius is the fitting of the above transmission functions $\eta_{(\rm a,k,s)G}$ for Gaussian beams to any other beams by the method of least squares. Such a fitting gives one certain value of the equivalent radius $w_{(\rm a,k,s)G}$ in each plane (already without indication of the transmission fraction). This method for measuring the beam radius by using a scanning slit was employed in [12]. However, taking into account the complicated behaviour of the transmission curves demonstrated in Figs 1 and 2, it is difficult to assert that this method has the advantage. In the general case, the radii found by this method can differ significantly from those measured by other methods and are not necessarily closer to the values of radii determined by the method of moments.

Figure 3 presents the fitting curves and radii for the higher $\rm LG_{02}$ mode obtained by different methods. One can see that the obtained values of radii strongly differ in the general case from $w_\sigma = \sqrt{5}w_0 \approx 0.45$ cm, while some values that are close to 0.45 cm seem accidental. As expected, the radius $w_{\rm sG} \approx 0.34$ cm found by fitting the transmission

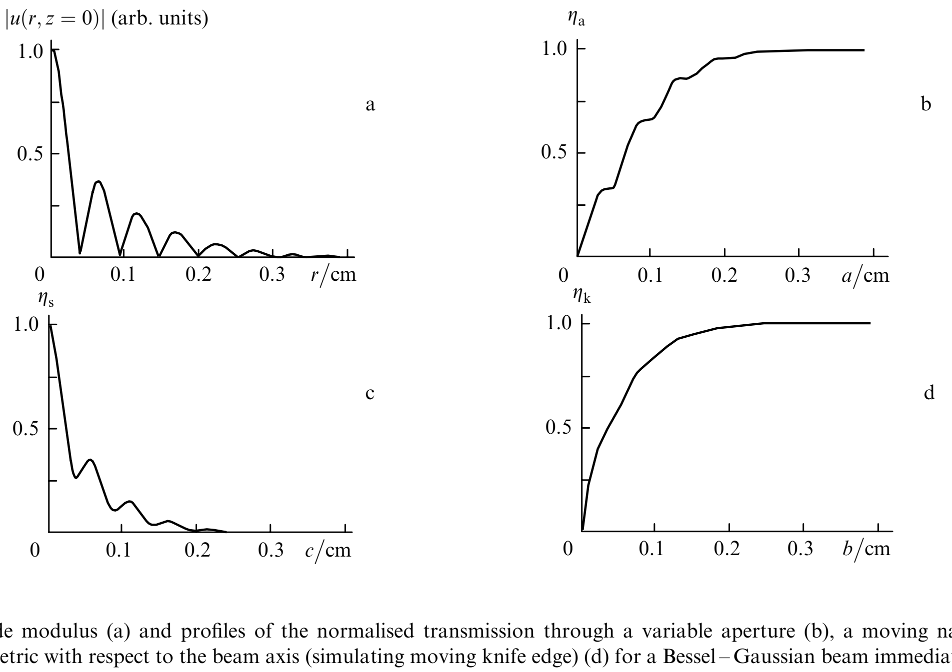

Figure 1. Amplitude modulus (a) and profiles of the normalised transmission through a variable aperture (b), a moving narrow slit (c), and an increasing slit symmetric with respect to the beam axis (simulating moving knife edge) (d) for a Bessel – Gaussian beam immediately behind a focusing lens ($z = 0$).

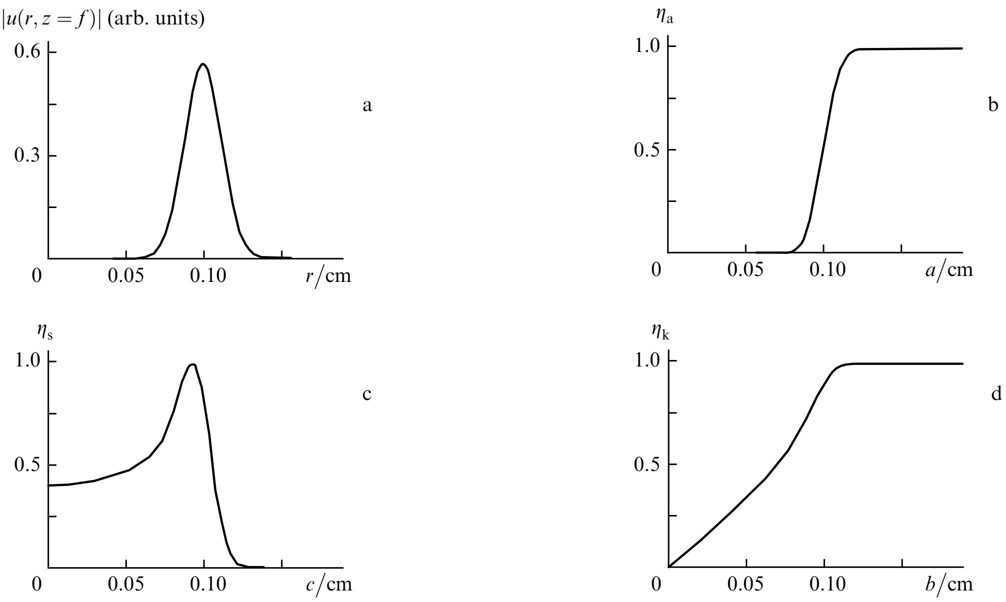

Figure 2. Amplitude modulus (a) and profiles of the normalised transmission through a variable aperture (b), a moving narrow slit (c), and an increasing slit symmetric with respect to the beam axis (d) for a Bessel–Gaussian beam in the focal plane of a lens $(z = f)$.

curve $\eta_{sG}(c)$ to the data obtained by the moving slit method strongly differs from the radius w_σ of the LG_{02} mode. The radius $w_G \approx 0.38$ cm obtained by fitting the Gaussian distribution of the amplitude to the modulus of the amplitude of the LG_{02} mode by the method of least squares also strongly differs from the exact value (Fig. 3a). Note that the values of the beam radius found by the variable aperture and moving slit methods at the transmission levels recommended by the standard ISO 11146 are most close to the exact value w_σ.

As mentioned above, the beam radii were calculated by different methods usually for beams with $w_0 = 0.2$ cm and the focal distance of a focusing lens $f = 100$ cm in planes with coordinates from $z = 0$ to $z = 200$ cm with a step of 2 cm. The superposition of Gaussian beams was calculated for the amplitudes $A_1 = -A_2 = 1$ and radii $w_1 = 0.2$ cm and $w_2 = 0.1$ cm. Other required parameters are indicated in Table 1. The alternative values of M_i^2 can be found from the radii determined by the methods described above by using the standard fitting of the hyperbola [17]

$$w_{i\,\mathrm{fit}} = w_{i0}\left[1 + \frac{\lambda^2 (M_i^2)^2 (z - z_{i0})^2}{\pi^2 w_{i0}^4}\right]^{1/2} \tag{21}$$

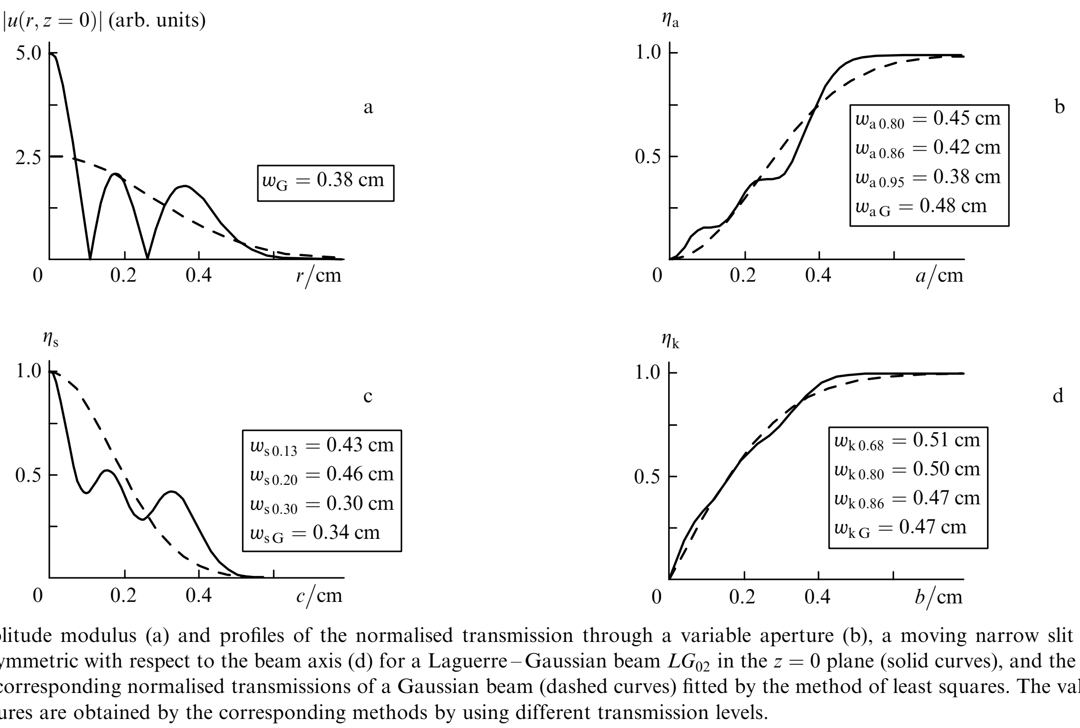

Figure 3. Amplitude modulus (a) and profiles of the normalised transmission through a variable aperture (b), a moving narrow slit (c), and an increasing slit symmetric with respect to the beam axis (d) for a Laguerre–Gaussian beam LG_{02} in the $z = 0$ plane (solid curves), and the plots of the amplitude and corresponding normalised transmissions of a Gaussian beam (dashed curves) fitted by the method of least squares. The values of radii presented in figures are obtained by the corresponding methods by using different transmission levels.

by the method of least squares. The fitting is performed by varying the values of the radius w_{i0} and positions z_{i0} of the waist, as well as the beam propagation ratio M_i^2.

Having found the corresponding values of the propagation ratios M_i^2 and assuming the presence of correlation with M_σ^2 [4, 8, 17], we can find the correlation coefficients

$$c_i = \frac{M_\sigma - 1}{M_i - 1}. \qquad (22)$$

The correlation coefficients obtained in this way for different beams are presented in Table 1. Unfortunately, the results presented in Table 1 contradict to the conclusion [4, 17] that there exists a certain one-to-one correlation between the beam propagation ratios M_i^2 and M_σ^2 for coherent axially symmetric beams. Because the value of the propagation ratio M_i^2 in alternative methods is not principally limited (like $M_\sigma^2 \geqslant 1$), the correlation coefficients c_i found by the method described above can have negative values, which can be large in modulus (in the case of $M_i^2 \sim 1$).

Taking into account that the total beam power

$$P(u) = 2\pi \int_0^\infty |u|^2 r\mathrm{d}r \cong \sum_{p=0}^{N_p} \tilde{c}_p^* \tilde{c}_p$$

for such variations in the coefficients $\tilde{c}_p(t)$ does not change in time, the corresponding alternative radii were calculated, as usual, by using the average value $\overline{|u|^2}$. In this case, the beam propagation ratio, determined by the method of moments, should be close to the incoherent ratio

$$M_\sigma^2(u) \cong \sum_{p=0}^{N_p} (2p+1) R_{pp},$$

where $R_{nm} = \tilde{c}_n^* \tilde{c}_m / P(u)$ [25]. A close value of the beam propagation ratio was also obtained by fitting (21) over the radius

$$w_\sigma(u) = 2\left[\frac{\pi}{P(u)} \int_0^\infty r^3 |u|^2 \mathrm{d}r\right]^{1/2}.$$

Table 1. Correlation coefficients c_i calculated for different beams.

Beam	M_σ^2	Variable aperture			Moving knife			Moving slit		
		$c_{a\,0.80}$	$c_{a\,0.86}$ **(1.14)**	$c_{a\,0.95}$	$c_{k\,0.68}$ **(0.81)**	$c_{k\,0.80}$	$c_{k\,0.86}$	$c_{s\,0.135}$ **(0.95)**	$c_{s\,0.20}$	$c_{s\,0.30}$
GG	1.63	1.30	0.83	0.74	3.38	1.15	0.84	2.47	6.86	−2.69
SG_{L^+}	1.87	1.47	0.99	0.85	5.19	2.18	1.38	27.06	−6.05	−2.97
SG_{L^-}	1.87	2.20	1.20	0.86	7.45	3.27	1.80	−3.85	−2.82	−2.98
SG_{NL^+}	1.71	2.98	1.40	0.76	−2.05	6.12	2.35	−10.10	−3.38	−3.15
SG_{NL^-}	1.71	1.72	1.14	0.82	6.69	2.49	1.57	15.69	−6.32	−2.71
LG_{02}	5.00	1.01	1.13	1.41	0.81	1.25	0.67	1.05	0.95	1.10
D_{24}	5.00	0.94	0.94	0.978	1.26	0.98	0.82	0.997	1.09	1.39
BG	5.91	1.05	1.01	1.08	1.08	1.02	1.00	1.29	1.63	1.72

Note: GG: coherent superposition of two Gaussian beams; SG_{L+}, SG_{L-}, SG_{NL+}, SG_{NL-}: super-Gaussian beams with linear and nonlinear aberrations $\alpha = \pm\pi$; LG_{02}: Laguerre–Gaussian beam; D_{24}: 'doughnut' beam ($l = 24$); BG: Bessel–Gaussian beam ($\beta w_0 = 11.8$). The values of correlation coefficients proposed by the standard ISO 1114 are indicated in bold.

Preliminary results obtained for incoherent beams also do not confirm the presence of such correlation. The corresponding calculations were performed by two methods. In the first method, an incoherent beam was obtained by varying randomly the coefficients of expansion in the LG modes. In the second method, the expansion coefficients were varied determinately, but for the time during which the phase of all modes changed more than by 2π. For example, for a super-Gaussian beam with $n = 10$, in which aberrations are assumed absent for simplicity, in the standard case ($w_0 = 0.2$ cm and $f = 100$ cm), the coefficients $\tilde{c}_p(0)$ of expansion in the LG modes found in the initial instant changed in time as $c_p(t) = c_p(0)\exp[-\mathrm{i}2\pi\Delta\nu_p t]$, where $\Delta\nu_p = (c/\lambda)(2p+1)\arccos(g_1 g_2)^{1/2}/\pi$ and $g_{1,2}$ are the parameters of a stable resonator [2]. Calculations were performed for the time interval Δt providing the fulfilment of the relation $\Delta\nu_0 \Delta t \geqslant 1$ even for the fundamental mode. By dividing this time interval into N_t intervals, we calculated for each time step $t_j = j\Delta t/N_t$ ($j = 0, 1, 2, ..., N_t - 1$), as usual $u_j = u(r, z, t_j)$. Instead of the energy density obtained by integrating the results with respect to time, we used the average power density

$$\overline{|u|^2} = \sum_{j=0}^{N_t - 1} |u_j|^2 / N_\mathrm{t}.$$

As expected, the alternative beam propagation ratios and the corresponding correlation coefficients for different realisations changed randomly, which confirms the absence of the one-to-one correlation between the invariant (M_σ^2) and alternative (M_i^2) beam propagation ratios.

4. Conclusions

By using the efficient numerical methods for calculating the propagation and focusing of axially symmetric beams of a complicated spatial structure by expanding them in the Laguerre–Gaussian modes of a free space, we calculated the alternative radii and propagation ratios for a number of beams with known M_σ^2 according to the current standard ISO 11146. The calculations of the alternative propagation ratios M_i^2 for these beams showed the absence of the universal one-to-one correlation between ratios M_σ^2 and M_i^2, which is assumed in the standard ISO 11146. Therefore, a direct use of the alternative propagation ratios, which are not propagation invariants, is unjustified. However, the data obtained by measuring the beam radii by alternative methods can be used in certain cases [34, 35] to find the invariant beam propagation ratios M_σ^2.

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