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Eigenfrequencies of an inhomogeneously filled ring optical resonator

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***Abstract*. A ring optical resonator with the simplest type of an inhomogeneity causing backscattering of waves is considered. A boundary problem of the periodic type is formulated in terms of the reflection and transmission coefficients. In the case of a small reflection coefficient, analytic expressions for eigenfrequencies are found in the limiting cases of their degeneracy and maximum splitting.**

Keywords: *ring resonator, resonator eigenfrequencies, inhomogeneous resonator filling*.

1. The problem of studying the eigenfrequency spectrum of an inhomogeneously filled resonator, in particular, a ring optical resonator (ROR) is traditional. It has been discussed many times in physical and mathematical papers (see, for example, monograph [1]). However, in the general formulation of the problem, only general, although important results were obtained. At the same time, a number of applications require more detailed information (see, for example, review [2]). In this case, the type of inhomogeneities should be specified. The results can be obtained most simply when the perturbation theory can be used in one or another treatment. Thus, the perturbation theory was phenomenologically applied in [3] and other papers for studying small local inhomogeneities of simple structures (a plate, a small sphere, etc.). This allows one, in principle, to estimate the asymptotics of the spectrum, although even this problem cannot be finally solved analytically.

In this paper, in some sense the opposite case is considered: two simple inhomogeneities (jumps of the refractive index) are separated over the ROR perimeter by a maximum distance. The problem is formulated and solved consistently electrodynamically without any artificial assumptions (as in [3]). The transcendent equation, from which the eigenfrequency spectrum of the resonator is determined, is obtained and can be solved (for example, numerically or graphically) not using the perturbation theory. The analytic results are illustrated in the case of relatively small reflection coefficients.

RORs are used in a number of optical schemes, for example, in a ring laser. An important problem both in the ROR theory and its applications is the consideration of the mode coupling due to the inhomogeneity of the resonator filling, which results in mode locking. In particular, a great attention was devoted to the determination of the mode-locking region [2, 4, 5], which is usually expressed directly in terms of the mode-coupling parameter within the framework of the nonlinear theory of a laser. However, these questions can be also studied indirectly using the linear theory of the resonator because its eigenfrequency spectrum also contains information on the properties of mode coupling (through the influence of an inhomogeneity on the spectrum splitting). In this way, the results of this paper can be used in the theory of ring lasers.

2. The eigenfrequencies of a homogeneously filled resonator are equidistant (the interval between them is inversely proportional to the resonator perimeter) and are doubly degenerate. Two counterpropagating travelling waves or any their linear combination can be chosen as the natural waves for each frequency. In the presence of backscattering in the resonator, which can be caused by any inhomogeneity of the resonator medium, travelling waves cannot be the eigenfunctions. The eigenfrequency spectrum of a ROR also changes.

If the dielectric constant of the medium is described by a sufficiently smooth function of coordinates, the asymptotic behaviour of eigenfrequencies and eigenfunctions can be determined quite easily by the Wentzel – Kramers – Brillouin method. If this function is not continuous, another method should be used. In this paper, the problem of determining the eigenfrequencies of a resonator is solved for the simplest situation when the function describing the optical properties of the medium is a step function. The problem is formulated in more detail below.

3. We assume that a ROR is one-dimensional [z is the coordinate of a point on the axis and L is the resonator perimeter ($0 \leqslant z \leqslant L$)]. The refractive index $n(z)$ of the medium filling the resonator has discontinuities at the points $z = L/2$ and $z = L$: $n(z) = n_1$ for $0 \leqslant z < L/2$ and $n(z) = n_2$ for $L/2 \leqslant z \leqslant L$. Stationary waves with the frequency ω in such a structure satisfy the homogeneous wave equation in each interval where the refractive index is constant and also the continuity conditions for the field and its derivative at the discontinuity points.

The continuity condition can be written in terms of the reflection and transmission (refraction) coefficients. We denote these coefficients as $R_1 = (n_1 - n_2)/(n_1 + n_2)$ and $T_1 = 2n_1/(n_1 + n_2)$, for the waves propagating in the direction of increasing z and as $R_2 = (n_2 - n_1)/(n_1 + n_2) = -R_1$ and $T_2 = 2n_2/(n_1 + n_2)$ for counterpropagating waves. The ROR field can be represented as a sum of the two waves:

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$$u_1(z) = A_1 \exp(-ik_1 z) + B_1 \exp(ik_1 z) \text{ for } 0 \leqslant z \leqslant L/2,$$

$$u_2(z) = A_2 \exp(-ik_2 z) + B_2 \exp(ik_2 z) \text{ for } L/2 \leqslant z \leqslant L.$$

Here, $k_1 = kn_1$, $k_2 = kn_2$ and $k = \omega/c$. We will call the waves with amplitudes A_1 and B_1 the forward waves and the waves with amplitudes A_2 and B_1 the backward waves. Let us introduce the reduced amplitudes of the forward and backward waves $A_1^{(0)} = A_1 \exp(-ik_1 L/2)$, $B_1^{(0)} = B_1 \exp(ik_1 \times L/2)$, $A_2^{(0)} = A_2 \exp(-ik_2 L/2)$, and $B_2^{(0)} = B_2 \exp(ik_2 L/2)$.

It follows from the continuity condition at the point $z = L/2$ that

$$B_1^{(0)} = R_1 A_1^{(0)} + T_2 B_2^{(0)}, \qquad A_2^{(0)} = T_1 A_1^{(0)} + R_2 B_2^{(0)}. \tag{1}$$

From the continuity condition at the point $z = L$, we obtain

$$B_2 \exp(ik_2 L) = R_2 A_2 \exp(-ik_2 L) + T_1 B_1, \qquad A_1 = R_1 B_1 + T_2 A_2 \exp(-ik_2 L). \tag{2}$$

From (1) and (2), we obtain the coupling equations for the reduced amplitudes:

$$\begin{aligned} B_2{}^{(0)} &= \frac{R_2}{T_2} \exp\left[i(k_1 - k_2)\frac{L}{2}\right] A_1^{(0)} + \frac{1}{T_2} \exp\left[-i(k_1 + k_2)\frac{L}{2}\right] B_1^{(0)}, \\ A_2{}^{(0)} &= \frac{1}{T_2} \exp\left[i(k_1 + k_2)\frac{L}{2}\right] A_1^{(0)} - \frac{R_1}{T_2} \exp\left[-i(k_1 - k_2)\frac{L}{2}\right] B_1^{(0)}. \end{aligned} \tag{3}$$

By combining (1) and (3), we obtain the homogeneous vector equation

$$(H - E) \begin{bmatrix} A_1^{(0)} \\ B_1^{(0)} \end{bmatrix} = 0, \tag{4}$$

where E is the unit matrix and

$$H = \begin{bmatrix} \dfrac{1}{T_1} & -\dfrac{R_2}{T_1} \\ \dfrac{R_1}{T_1} & \dfrac{1}{T_1} \end{bmatrix} \times \begin{bmatrix} \dfrac{1}{T_2} \exp\left[i(k_1 + k_2)\dfrac{L}{2}\right] & -\dfrac{R_1}{T_2} \exp\left[-i(k_1 - k_2)\dfrac{L}{2}\right] \\ \dfrac{R_2}{T_2} \exp\left[i(k_1 - k_2)\dfrac{L}{2}\right] & \dfrac{1}{T_2} \exp\left[i(k_1 + k_2)\dfrac{L}{2}\right] \end{bmatrix}. \tag{5}$$

4. The nonzero solutions of Eqn (4) exist if $\mathrm{Det}(H - E) = 0$. It is obvious that $\mathrm{Det}(H - E) = 1 - \mathrm{Sp}\,H + \mathrm{Det}\,H$. The diagonal elements of the matrix H and, hence, expressions for $\mathrm{Sp}H$ and $\mathrm{Det}H$ can be easily obtained from (5):

$$\mathrm{Sp}\,H = 2\frac{1}{T_1}\frac{1}{T_2}\left[\cos(k_1 + k_2)\frac{L}{2} - R_1^2 \cos(k_1 - k_2)\frac{L}{2}\right],$$

$$\mathrm{Det}\,H = \mathrm{Det}\begin{bmatrix} \dfrac{1}{T_1} & -\dfrac{R_2}{T_1} \\ \dfrac{R_1}{T_1} & \dfrac{1}{T_1} \end{bmatrix} \times \mathrm{Det}\begin{bmatrix} \dfrac{1}{T_2} \exp\left[i(k_1 + k_2)\dfrac{L}{2}\right] & -\dfrac{R_1}{T_2} \exp\left[-i(k_1 - k_2)\dfrac{L}{2}\right] \\ \dfrac{R_2}{T_2} \exp\left[i(k_1 - k_2)\dfrac{L}{2}\right] & \dfrac{1}{T_2} \exp\left[i(k_1 + k_2)\dfrac{L}{2}\right] \end{bmatrix}.$$

The calculation of determinants entering the latter expression shows that the matrix H is unimodular: $\mathrm{Det}\,H = 1$. This allows us to represent the equation $1 - \mathrm{Sp}\,H + \mathrm{Det}\,H = 0$ in the form

$$1 - \frac{1}{T_1}\frac{1}{T_2}\left[\cos(k_1 + k_2)\frac{L}{2} - R_1^2 \cos(k_1 - k_2)\frac{L}{2}\right] = 0,$$

or in the equivalent form

$$\cos\phi - 1 = R_1^2(\cos\delta\phi - 1), \tag{6}$$

where new variables

$$\phi = \frac{n_1 + n_2}{2} kL, \quad \delta = \frac{n_1 - n_2}{n_1 + n_2} = R_1, \quad \delta\phi = \frac{n_1 - n_2}{2} kL \tag{6a}$$

are introduced.

5. The left and right-hand sides of Eqn (6) are shown by two curves in Fig. 1. The reflection coefficient is usually $R_1 \ll 1$. Therefore, the right-hand side of (6) has much lower frequency and amplitude than the left-hand side. This cannot be displayed at the real scale in the plot, so that the curves in Fig. 1 serve only as an illustration.

Assuming that the reflection coefficient is small, we can find approximately the roots of Eqn (6). They form two discrete series $\phi_q^{(+)}$ and $\phi_q^{(-)}$ of positive numbers: the roots of one of the series are somewhat higher than $2q\pi$, while the roots of another series are somewhat lower than these numbers. In the general case, the roots are described by the expressions

$$\phi_q^{(+)} = 2q\pi + R_1\sigma_q^{(+)}, \quad \phi_q^{(-)} = 2q\pi - R_1\sigma_q^{(-)}, \tag{7}$$

where the influence coefficients $\sigma_q^{(\pm)}$ are of the order of unity and depend on the root number q. They can be found, for example, graphically (see Fig. 1).

By using (7) and (6a), we find the eigenfrequency spectrum of the ROR:

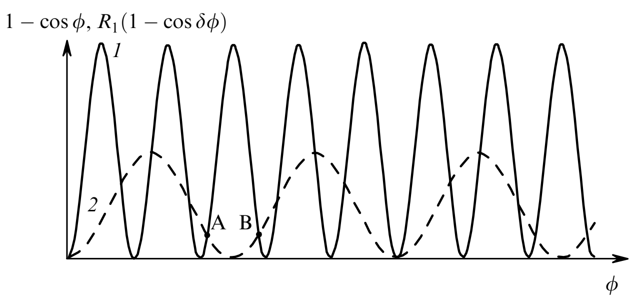

Figure 1. Graphic representation of Eqn (6): $1 - \cos\phi$ [curve (*1*)], $R_1(1 - \cos\delta\phi)$ [curve (*2*)].

$$\omega_q^{(\pm)} = \phi_q^{(\pm)} \left(\frac{2}{n_1 + n_2} \right) \frac{c}{L}. \tag{8}$$

Some eigenfrequencies can be doubly degenerate.

6. Consider a degenerate frequency spectrum. Let $\delta = n/m$, where m and n are the natural numbers. Let us assume that the roots of Eqn (6) have the form $\phi = 2m\pi + \Phi$. Then, $\delta\phi = 2n\pi + \delta\Phi$. For small Φ, we have

$$1 - \cos\phi \approx \frac{\Phi^2}{2} \text{ and } 1 - \cos\delta\phi = 1 - \cos\delta\Phi \approx \frac{\delta^2\Phi^2}{2}.$$

By using these approximations, we can represent (6) in the form of the equation

$$\frac{\Phi^2}{2} - \frac{R_1^2\delta^2\Phi^2}{2} = 0,$$

which has the doubly degenerate root $\Phi^{(\pm)} = 0$. Therefore, for each integer m, there exist the multiple root $\phi_m^{(\pm)} = 2m\pi$ and the multiple eigenfrequency

$$\omega_{m0}^{(\pm)} = \omega_{m0} = 2\pi m \left(\frac{2}{n_1 + n_2} \right) \frac{c}{L}.$$

Therefore, the degenerate frequency spectrum in the presence of backscattering coincides with the spectrum of the equivalent ROR homogeneously filled with a medium with the refractive index equal to $(n_1 + n_2)/2$ (i.e., without backscattering).

7. Consider now the maximum splitting in the frequency spectrum. Let us assume that the reflection coefficient is approximated by the rational ratio $\delta = (2n+1)/(2m)$. In this case, along with degenerate eigenfrequencies, non-degenerate frequencies also exist. We will seek the solution of Eqn (6) in the form $\phi = 2m\pi + \Phi$ for small Φ. Because $\delta\phi = 2m\pi\delta + \delta\Phi = (2n+1)\pi + \delta\Phi$, we have

$$1 - \cos\delta\phi \approx 2 - \frac{\delta^2\Phi^2}{2}.$$

Therefore, Eqn (6) in the first approximation has the form

$$\frac{\Phi^2}{2} \approx R_1^2 \left(2 - \frac{\delta^2\Phi^2}{2} \right).$$

The roots of this equation are the approximate values of the roots of Eqn (6)

$$\Phi^{(\pm)} \approx \pm\sigma R_1, \tag{9}$$

where

$$\sigma = 2 \left(\frac{1}{1 + \delta^2 R_1^2} \right)^{1/2} \approx 2.$$

Therefore, under these conditions, the roots of Eqn (6) are located symmetrically with respect to points $2m\pi$, being removed from them by the maximum distance. In Fig. 1, these roots correspond to the points of intersection of the curves in the vicinity of the points of coincidence of the maxima of curve (*1*) and minima of curve (*2*) (for example, points A and B). It is obvious graphically that in this case the influence coefficients in (7) do coincide ($\sigma_m^{(+)} = \sigma_m^{(-)} = \sigma$) and are maximal.

The eigenfrequencies of the ROR are found from expressions (8) and (9):

$$\omega_m^{(\pm)} = \omega_{m0} \pm \sigma R_1 \left(\frac{2}{n_1 + n_2} \right) \frac{c}{L}. \tag{10}$$

In the absence of backscattering ($n_1 = n_2$), we obtain $\omega_m^{(\pm)} = \omega_{m0}$ from (10), i.e., the entire frequency spectrum is degenerate.

Because only the estimate of the maximum splitting is of interest, smaller splittings are not considered.

8. The problem of the ROR spectrum considered above is similar to the traditional problems of quantum mechanics and theory of waves, where some periodic structures are studied (see, for example, [6–9]). However, in these problems the propagating stable or unstable waves are mainly investigated. The eigenfrequencies corresponding to these waves belong to a continuous spectrum. Periodic waves with the discrete frequency spectrum are an exception in these papers and have not been adequately studied. This is explained by the fact that the waveguide treatment was used, namely, the derivation of the dispersion equation for the specified frequency. In the present paper, the resonator treatment was employed to obtain frequencies at which periodic waves exist in a periodic structure. The use of the concept of the reflection coefficient allows one to pass from the simplest model of the filling medium, which we used only to illustrate the method, to obvious generalisations.

The general approach to the solution of the formulated spectral problem is quite natural. It was used, for example, in [10, 11] for determining the frequency spectrum and polarisation eigenstates of a linear optically anisotropic resonator. However, the eigenfrequencies were not found analytically in these and other papers known to us. In the present paper, the expressions for the eigenfrequencies of a ring resonator have been obtained (in the first approximation in a small reflection coefficient). The sufficient conditions for the formation of degenerate eigenfrequencies have been also found and the maximum removal of the spectrum degeneracy has been estimated.

References

1. Babich V.M., Buldyrev V.S. *Asimtoticheskie metody v zadachakh difraktsii korotkikh voln* (Asymptotic Methods in Problems of Diffraction of Short Waves) (Moscow: Nauka, 1972) Ch. 9.
2. Kravtsov N.V., Lariontsev E.G. *Kvantovaya Elektron.*, **30**, 105 (2000) [*Quantum Electron.*, **30**, 105 (2000)].
3. Haus H., Statz H., Smith W. *IEEE J. Quantum Electron.*, **21** (1), 78 (1985).
4. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Zh. Eksp. Teor. Fiz.*, **65**, 1344 (1973).
5. Zeiger S.G., Klimontovich Yu.L., Landa P.S., Lariontsev E.G., Fradkin E.E. *Volnovye i fluktuatsionnye protsessy v lazerakh* (Wave and Fluctuation Processes in Lasers) (Moscow: Nauka, 1974) Chs 5–9.
6. Elashi Sh. *Trudy IIER*, **64**, 22 (1976).
7. Brillouin L. *Wave Propagation in Periodic Structures* (New York: Dover, 1953; Moscow: Inostrannaya Literatura, 1959) Ch. 9.
8. Louisell W.H. *Coupled Mode and Paramagnetic Electronics* (New York: Wiley, 1960); Moscow: Inostrannaya Literatura, 1963) Chs 4–9.
9. Vinogradova M.B., Rudenko O.V., Sukhorukov A.P. *Teoriya voln* (Theory of Waves) (Moscow: Nauka, 1979) pp 140–150.
10. Ivanov E.I., Chaika M.P., in *Fizika gazovykh lazerov* (Leningrad: Izd. LGU, 1969) pp 20–32.
11. Grigor'eva V.N., Rymarchuk V.A., in *Fizika gazovykh lazerov* (Leningrad: Izd. LGU, 1969) pp 33–35.