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Polarisation dependences of harmonic generation in the plasma produced in the ionisation of excited-state hydrogen-like atoms

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***Abstract*. An analytic theory of harmonic generation in the plasma produced from the gas of hydrogen-like atoms in excited states is considered for relatively intense radiation. The consideration of *l*-degeneracy of the electrons in these excited states allowed deriving the dependence of generation efficiency on the principal quantum number. In the context of the Bethe model of gas ionisation, we revealed the threshold nonlinear dependence of the maximum generation efficiency on the degree of circular polarisation of the pump field for its given intensity. Analytic calculations were performed for the fifth and seventh harmonics. The results of these calculations allowed generalising to the case of excited atoms the previously obtained results for the third harmonic in the plasma arising from hydrogen-like atoms in the ground state.**

Keywords: *harmonic generation, plasma, hydrogen-like atoms.*

1. In Ref. [1] it was theoretically shown that the preliminary population of excited states of the atoms being ionised plays an important role in a photoionised plasma produced upon the suppression of the ionisation barrier in the Bethe regime [2]. Although it is not evident that the Bethe regime was realised under the experimental conditions of Ref. [3], it should be noted that the authors of paper [3] discovered an extremely strong increase in the intensity of the third harmonic generation in the photoionised plasma arising from the gas of excited atoms in comparison with the intensity of the third harmonic generation in the photoionised plasma produced from the gas of unexcited atoms. In Ref. [1], an analytically solvable model of a hydrogen-like atom was used, which made it possible to obtain exhaustive information on the generation of laser radiation harmonics in a photoionised plasma. In this case, the effect of excited-state prepopulation was established neglecting the possibility of the Coulomb *l*-degeneracy of the energy levels. The role of this degeneracy was discussed in Ref. [4], where approximate scaling dependences were obtained for the generation efficiency of the first five odd harmonics of the pump field on their number and the field intensity. The pump field was assumed to be linearly polarised. Unlike Ref. [4], we considered below the case of elliptically polarised pump field, for which we determine the behaviour of the efficiency of the fifth and seventh harmonic generation.

Of significance for the subsequent treatment is the form of the distribution function of electrons of a hydrogen-like atom which are in the nth energy level

$$f_n(V) = \frac{1}{\pi^2 (V_Z/n)^3 [1 + (nV/V_Z)^2]^4}, \tag{1}$$

which was obtained in Ref. [4]. Here, n is the principal quantum number and $V_Z = Ze^2/\hbar$ is the Coulomb velocity unit [5]. Function (1) depends on the magnitude of velocity V of a photoionised plasma electron. In the Bethe photoionisation regime this function repeats the electron distribution in the hydrogen-like atom. However, it is important that this function is written in the coordinate system oscillating with the electron.

Here, we make several remarks significant for the understanding of our model. According to [2], for the Bethe regime to be realised, the intensity E of the ionising electric field should satisfy the condition

$$E > \frac{I_Z^2}{4Z|e|^3}, \tag{2}$$

where I_Z is the ionisation potential. For a hydrogen-like atom with electrons in the nth energy level, inequality (2) takes the form

$$E > \frac{1}{4Ze^3}\left(\frac{Z^2 m_e e^4}{2n^2\hbar^2}\right)^2, \tag{3}$$

where m_e is the electron mass. We emphasise that the gas of excited atoms is devoid of bound electron states when condition (3) is fulfilled, i.e., turns into a plasma. Apart from strong-field condition (2), for the practical realisation of the Bethe regime in the atomic ionisation it is necessary that the strong field should appear quickly enough. In other words, the field buildup time should be shorter than

$$\tau = \frac{n^2\hbar^3}{m_e e^4} = n^2 \times 2.5 \times 10^{-17}\ \text{s}, \tag{4}$$

which corresponds to the time the electron takes to go around its excited-state orbit. For excited states, the time τ (4) is in the femtosecond range.

2. We briefly formulate some basic principles of the theory of the harmonic generation of the pump field in a weakly collisional plasma. First of all, for the electric component of the pump field we assume that

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$$E_x = e_x E\cos(\omega t - kz), \quad E_y = -e_y E\sin(\omega t - kz). \tag{5}$$

Here, the transverse plane wave of the pump field depends on the coordinate z. The coordinate dependence of the amplitude E of the electric field intensity is assumed to be weak and is neglected in the derivation of the subsequent formulas, although this dependence can be parametrically described in our treatment. The quantities e_x and e_y in expressions (5) are the components of the polarisation vector, for which $e_x^2 + e_y^2 = 1$. These components determine the maximum degree of linear polarisation ρ:

$$\rho = (e_x^2 - e_y^2)^{1/2}. \tag{6}$$

Accordingly, the degree of circular polarisation of the pump field A is defined by the expression

$$A = (1 - \rho^4)^{1/2}. \tag{7}$$

The frequencies and the wave vector of the pump field are related as

$$\omega^2 = \omega_{\text{Le}}^2 + c^2k^2, \tag{8}$$

where c is the velocity of light; $\omega_{\text{Le}} = (4\pi e^2 N_{\text{e}}/m_{\text{e}})^{1/2}$ is the electron Langmuir frequency and N_{e} is the electron density in the plasma.

According to Maxwell's equations, the components of the electric field of harmonics are described by the expressions

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} + \omega_{\text{Le}}^2 - c^2\frac{\partial^2}{\partial z^2}\right)E_x^{(2N+1)} \\
&\quad = 4\pi\sigma_{xx}^{(2N+1)}(2N+1)\omega e_x E\sin[(2N+1)(\omega t - kz)], \\
&\left(\frac{\partial^2}{\partial t^2} + \omega_{\text{Le}}^2 - c^2\frac{\partial^2}{\partial z^2}\right)E_y^{(2N+1)} \\
&\quad = 4\pi\sigma_{yy}^{(2N+1)}(2N+1)\omega e_y E\cos[(2N+1)(\omega t - kz)].
\end{aligned} \tag{9}$$

The nonlinear effective high-frequency conductivities σ can be represented in terms of the effective collision frequencies $\nu^{(2N+1)}$:

$$\begin{aligned}
\sigma_{xx}^{(2N+1)} &= \frac{e^2N_{\text{e}}}{m_{\text{e}}\omega^2}\nu_{xx}^{(2N+1)}(n, E, \rho), \\
\sigma_{yy}^{(2N+1)} &= \frac{e^2N_{\text{e}}}{m_{\text{e}}\omega^2}\nu_{yy}^{(2N+1)}(n, E, \rho).
\end{aligned} \tag{10}$$

By defining the $(2N+1)$th harmonic generation efficiency as the ratio between the time-average squares of the harmonic and pump field intensities,

$$\eta^{(2N+1)} = \frac{\langle[\boldsymbol{E}^{(2N+1)}]^2\rangle}{\langle\boldsymbol{E}^2\rangle}, \tag{11}$$

and using relations (9) and (11), we can write

$$\eta^{(2N+1)} = \left[\frac{2N+1}{4N(N+1)}\right]^2\left\{\left[e_x\nu_{xx}^{(2N+1)}\right]^2 + \left[e_y\nu_{yy}^{(2N+1)}\right]^2\right\}\omega^{-2}. \tag{12}$$

By describing the effective collision frequencies employing the Boltzmann equation with the Landau collision integral and using expression (1) for the distribution of electrons colliding with ions, we obtain

$$\begin{aligned}
\nu_{xx}^{(2N+1)} &= \frac{16e^4Z_{\text{eff}}N_{\text{e}}\Lambda}{\rho^3m_{\text{e}}^2V_E^3}D\alpha^{(+)}(2N+1, \alpha, \rho)\Big|_{b=1}, \\
\nu_{yy}^{(2N+1)} &= \frac{16e^4Z_{\text{eff}}N_{\text{e}}\Lambda}{\rho^3m_{\text{e}}^2V_E^3}D\alpha^{(-)}(2N+1, \alpha, \rho)\Big|_{b=1}
\end{aligned} \tag{13}$$

and accordingly

$$\begin{aligned}
\alpha^{(+)}(2N+1, \alpha, \rho) &= \frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta\,\frac{Q_{2N+1}(z) + Q_{2N-1}(z)}{z+1}, \\
\alpha^{(-)}(2N+1, \alpha, \rho) &= \frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta\,\frac{Q_{2N+1}(z) - Q_{2N-1}(z)}{z-1},
\end{aligned} \tag{14}$$

where Λ is the Coulomb logarithm of the Landau collision integral; $V_E = |eE|/(m\omega)$ is the amplitude of the electron velocity oscillation in the pump field; Z_{eff} is the effective ion charge determined by the expression

$$Z_{\text{eff}} = \frac{\sum_i \mathrm{e}_i^2 N_i}{e^2N_{\text{e}}};$$

e_i and N_i are the charge and number density of the ions of ith sort, respectively; Q_ν is the Legendre function; and

$$D = 1 - \frac{\mathrm{d}}{\mathrm{d}b} + \frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}b^2}$$

is a differential operator after the application of which in expressions (13), we set $b = 1$. Finally, θ and z in integrals (14) are related as

$$z = \frac{1}{\rho^2}\left(1 + \frac{2\alpha^2}{\sin^2\theta}\right), \tag{15}$$

where $\alpha = V_Zb/(V_En)$. Thus, the calculation of the effective collision frequencies reduces to the calculation of two integrals:

$$\begin{aligned}
&\alpha^{(\pm)}(2N+1, \alpha, \rho) = \frac{\alpha\sqrt{2}}{\pi\rho} \\
&= \int_{(1+2\alpha^2)/\rho^2}^{+\infty}\mathrm{d}z\,\frac{Q_{N+1/2}(z) \pm Q_{N-1/2}(z)}{(z\pm1)(z-1/\rho^2)[z-(1+2\alpha^2)/\rho^2]^{1/2}}.
\end{aligned} \tag{16}$$

By using expression (8.712) from the reference book [6] (p. 1015),

$$Q_\nu(z) = \frac{1}{2^{\nu+1}}\int_{-1}^{+1}\frac{\mathrm{d}t}{(z-t)^{\nu+1}}(1-t^2)^\nu,$$

we obtain:

$$\begin{aligned}
&Q_{(2N+1)/2}(z) \pm Q_{(2N-1)/2}(z) \\
&\quad = \pm\frac{1}{\sqrt{2}}\int_{-1}^{+1}\frac{\mathrm{d}t}{(z-t)^{1/2}}\,\Theta_{2N+1}^{(\pm)}(t),
\end{aligned} \tag{17}$$

where, for instance,

$$\Theta_5^{(+)} = \left(\frac{1+t}{1-t}\right)^{1/2}(4t^2 - 2t - 1),$$
$$\Theta_5^{(-)} = \left(\frac{1-t}{1+t}\right)^{1/2}(4t^2 + 2t - 1), \quad (18)$$

$$\Theta_7^{(+)} = \left(\frac{1+t}{1-t}\right)^{1/2}(8t^3 - 4t^2 - 4t + 1),$$
$$\Theta_7^{(-)} = \left(\frac{1-t}{1+t}\right)^{1/2}(8t^3 + 4t^2 - 4t - 1). \quad (19)$$

Expression (17) allows writing (16) in the form

$$\alpha^{(+)}(2N+1, \alpha, \rho) = \frac{2\alpha\rho}{\pi(1+\rho^2)}$$
$$\times \int_{-1}^{+1} \mathrm{d}t\Theta_{2N+1}^{(+)} \left\{ \frac{\rho^2 \arctan[(1-\rho^2 t)/(2\alpha^2)]^{1/2}}{[2\alpha^2(1-\rho^2 t)]^{1/2}} \right.$$
$$- \frac{\rho}{2[(1+t)(1+\rho^2+2\alpha^2)]^{1/2}}$$
$$\left. \times \ln \frac{(1+\rho^2+2\alpha^2)^{1/2} + \rho(1+t)^{1/2}}{(1+\rho^2+2\alpha^2)^{1/2} - \rho(1+t)^{1/2}} \right\}, \quad (20)$$

$$\alpha^{(-)}(2N+1, \alpha, \rho) = \frac{2\alpha\rho}{\pi(1-\rho^2)}$$
$$\times \int_{-1}^{+1} \mathrm{d}t\Theta_{2N+1}^{(-)} \left\{ \frac{\rho^2 \arctan[(1-\rho^2 t)/(2\alpha^2)]^{1/2}}{[2\alpha^2(1-\rho^2 t)]^{1/2}} \right.$$
$$\left. - \frac{\rho}{2[(1-t)(1-\rho^2+2\alpha^2)]^{1/2}} \arctan \frac{\rho(1-t)^{1/2}}{(1-\rho^2+2\alpha^2)^{1/2}} \right\}. \quad (21)$$

The absence of a singularity in expression (21) for $\rho = 1$ follows from the fact that the expression in braces in the right-hand side of the integrand in expression (21) vanishes in this case. Expressions (20) and (21) permit writing explicit analytic expressions for the quantities $\alpha^{(+)}(2N+1, \alpha, \rho)$ and $\alpha^{(-)}(2N+1, \alpha, \rho)$.

3. We apply the above general relations to the description of the fifth harmonic generation ($2N+1=5$). Then, expressions (20) and (21) enable obtaining relations (22) and (23), respectively:

$$\alpha^{(+)}(5, \frac{b}{x}, \rho) = \frac{4b}{x\rho}\left[-\frac{2}{5} + \frac{8}{5\rho^2} + \frac{16b^2}{15x^2\rho^2}\right.$$
$$\left. + \left(-\frac{2}{15} - \frac{8}{5\rho^2} - \frac{16b^2}{15x^2\rho^2}\right)\left(\frac{2b^2+x^2-x^2\rho^2}{2b^2+x^2+x^2\rho^2}\right)^{1/2}\right] +$$
$$+ \frac{1}{(1+\rho^2)^{1/2}} 2^{3/2}\rho \left[\left(\frac{3}{10} - \frac{16}{15\rho^4} - \frac{2}{3\rho^2}\right)\right.$$
$$\times E\left(\arctan\left[\frac{x^2(1+\rho^2)}{2b^2}\right]^{1/2}, \left(\frac{2\rho^2}{1+\rho^2}\right)^{1/2}\right) + \left(-\frac{1}{6}\right.$$
$$\left.\left. + \frac{16}{15\rho^4} - \frac{2}{5\rho^2}\right) F\left(\arctan\left[\frac{x^2(1+\rho^2)}{2b^2}\right]^{1/2}, \left(\frac{2\rho^2}{1+\rho^2}\right)^{1/2}\right)\right], \quad (22)$$

$$\alpha^{(-)}\left(5, \frac{b}{x}, \rho\right) = \frac{1}{15\rho^3(1-\rho^2)}\left\{\frac{8b}{x}(1-\rho^2)\right.$$
$$\times \left(12 + 3\rho^2 + \frac{8b^2}{x^2}\right) + \left[\frac{6b}{x}\left(\frac{2b^2+x^2+x^2\rho^2}{2b^2+x^2-x^2\rho^2}\right)^{1/2}\right.$$
$$\times \left(4 - 2\rho^2 - \rho^4 - \frac{4b^2\rho^2}{x^2} + \frac{16b^2}{x^2} + \frac{16b^4}{x^4}\right)$$
$$- \frac{2b}{x}\left(\frac{2b^2+x^2-x^2\rho^2}{2b^2+x^2+x^2\rho^2}\right)^{1/2}\left(60 - 2\rho^2 - 19\rho^4 + \frac{80b^2}{x^2}\right.$$
$$\left.\left. + \frac{4b^2\rho^2}{x^2} + \frac{48b^4}{x^4}\right)\right] + \sqrt{2}(1+\rho^2)^{1/2}(-32 + 20\rho^2 + 9\rho^4)$$
$$\times E\left(\arctan\left[\frac{x^2(1+\rho^2)}{2b^2}\right]^{1/2}, \left(\frac{2\rho^2}{1+\rho^2}\right)^{1/2}\right)$$
$$- \frac{\sqrt{2}(1-\rho^2)}{(1+\rho^2)^{1/2}}(-32 - 12\rho^2 + 5\rho^4)$$
$$\left. \times F\left(\arctan\left[\frac{x^2(1+\rho^2)}{2b^2}\right]^{1/2}, \left(\frac{2\rho^2}{1+\rho^2}\right)^{1/2}\right)\right\}. \quad (23)$$

Here, $E(\varphi, k)$ and $F(\varphi, k)$ are elliptical functions defined according to Ref. [6] and $x = nV_E/V_Z$.

By substituting relations (22) and (23) into (13), we obtain, according to (12), the expression

$$\eta^{(5)}(x, A) = \left(\frac{10e^4 Z_{\mathrm{eff}} N_{\mathrm{e}} \Lambda}{3m_{\mathrm{e}}^2 V_Z^3 \omega}\right)^2 n^6 \Phi\left(5, \frac{nV_E}{V_Z}, A\right), \quad (24)$$

where

$$\Phi(5, x, A) = \frac{1}{2\rho^6 x^6}\left\{(1+\rho^2)\left[D\alpha^{(+)}\left(5, \frac{b}{x}, \rho\right)\right]^2\Big|_{b=1}\right.$$
$$\left. + (1-\rho^2)\left[D\alpha^{(-)}\left(5, \frac{b}{x}, \rho\right)\right]^2\Big|_{b=1}\right\}. \quad (25)$$

In this case, the maximum degree of linear polarisation ρ in the right-hand side of relation (25) is expressed in terms of the degree of circular polarisation A according to (7). Explicit expression (25) is rather cumbersome, and therefore the dependences it describes are illustrated in Figs 1 and 2. It follows from Fig. 1 that the efficiency of the fifth harmonic generation initially builds up rapidly with increasing the electric intensity of the pump field in accordance with the growth of the scaling argument $x = nV_E/V_Z$ and then decreases drastically as its sixth power. Note that expression (24) gives the efficiency of the fifth harmonic as a function of principal quantum number for a given value of x. One can easily see from Fig. 1 that the generation efficiency decreases rather fast with increasing the degree of circular polarisation, which corresponds to the general case of the absence of harmonic generation for a circularly polarised pump field. However, the peculiarity of the dependence of the fifth harmonic efficiency on the degree of circular polarisation is more clearly manifested in Fig. 2, where the ratio

$$\frac{\eta^{(5)}(x,A)}{\eta^{(5)}(x,0)}=\frac{\Phi(5,nV_E/V_Z,A)}{\Phi(5,nV_E/V_Z,0)} \tag{26}$$

is given. Here, one can clearly see the nonmonotonic dependence, when expression (26) achieves its maximum for a finite magnitude of the degree of circular polarisation ($A \neq 0$). This effect was experimentally observed in the harmonic generation in a gas [7]. The authors of Ref. [7] pointed out that this effect contradicted to the then existing opinion that the harmonic generation efficiency is highest when produced by the electrons returning to the atom from which they were removed by the pump field. For a plasma model of harmonic generation in the case of preformed plasma with the Maxwell distribution, the effect similar to the effect observed in Ref. [7] was theoretically obtained in Ref. [8]. As a result, a viewpoint appeared that the maximum harmonic generation efficiency arises not for individual harmonics, as in the experiment of Ref. [7], but for all harmonics, this effect having a threshold nature. The thresholds will be different for different harmonics, and therefore failures to experimentally detect this effect for some harmonics are caused by the insufficient radiation intensity of the laser being employed. The fact that the effect under discussion is a threshold effect is clearly seen from Fig. 3: the maximum appears, as in Ref. [7], for $x = nV_E/V_Z > 2.65$. However, the special feature of this effect, as is evident from Fig. 3, also consists in the fact that the degree A of circular polarisation at which the efficiency of fifth harmonic generation has a clearly defined maximum decreases with increasing the scaling parameter x. For this reason, this effect may escape detection when the accuracy of determination of the parameter A is not high enough. In this case, this effect [7] may be experimentally observed in the range $x_{\rm th} < x < x_{\rm error}$.

4. Consider now the efficiency of the seventh harmonic generation ($2N+1=7$). By using expressions (20) and (21) we find

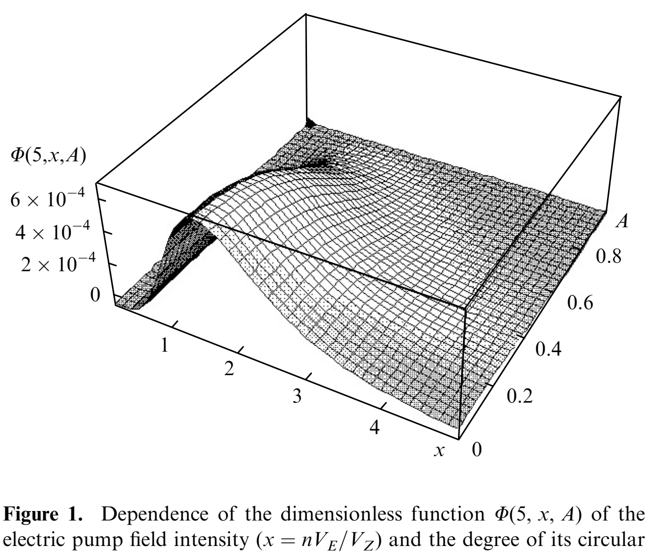

Figure 1. Dependence of the dimensionless function $\Phi(5, x, A)$ of the electric pump field intensity ($x = nV_E/V_Z$) and the degree of its circular polarisation.

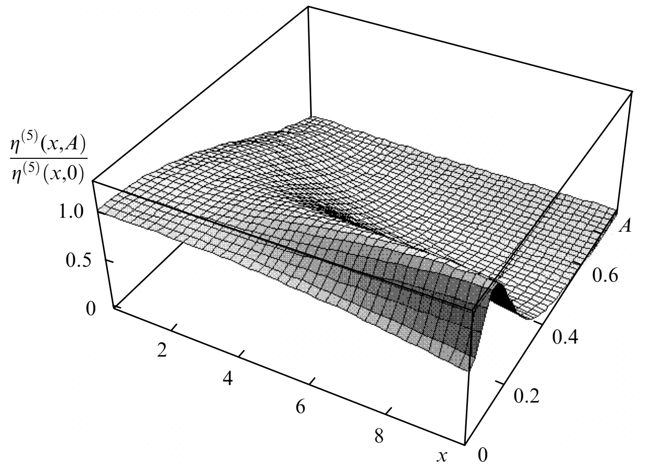

Figure 2. Relative efficiency of the fifth harmonic generation $\eta^{(5)}(x,A)/\eta^{(5)}(x,0)$ as a function of the degree of circular polarisation and pump field intensity.

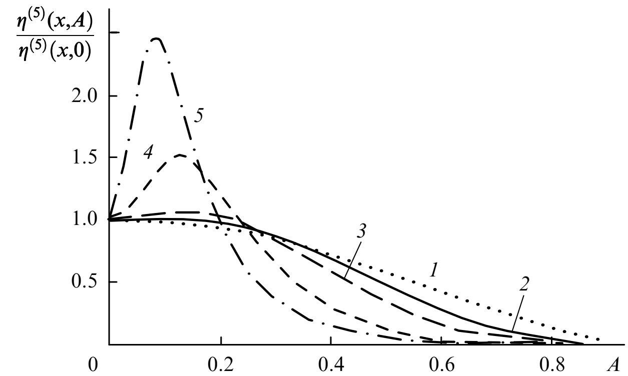

Figure 3. Dependences of the normalisation ratio (26) on the degree of circular polarisation for different pump fields. Curve (*1*) corresponds to $x = 0.5$, which is realised below the threshold of displacement of the generation efficiency peak from the value $A = 0$ (plane polarisation), curve (*2*) corresponds to the threshold case, when $x = 2.65$, and curves (*3*), (*4*), and (*5*) to the above-threshold cases for $x = 4$ (*3*), 8 (*4*), and 11 (*5*).

$$\alpha^{(+)}\left(7,\frac{b}{x},\rho\right)=\frac{1}{1+\rho^2}\Bigg\{-\frac{2b\rho}{7x}\Bigg\{1-\frac{4}{\rho^2}-\frac{4}{\rho^4}+\frac{8}{\rho^6}$$

$$+\frac{b^2}{x^2\rho^2}\left(-8-\frac{16}{\rho^2}+\frac{48}{\rho^4}\right)+\frac{b^4}{x^4\rho^4}\left(-16+\frac{96}{\rho^2}\right)$$

$$+64\frac{b^6}{x^6\rho^6}+\left(\frac{x^2+2b^2-x^2\rho^2}{x^2+2b^2+x^2\rho^2}\right)^{1/2}\Bigg[1+\frac{4}{\rho^2}-\frac{4}{\rho^4}$$

$$-\frac{8}{\rho^6}+\frac{b^2}{x^2\rho^2}\left(8-\frac{16}{\rho^2}-\frac{48}{\rho^4}\right)+\frac{b^4}{x^4\rho^4}\left(-16-\frac{96}{\rho^2}\right)-$$

$$-64\frac{b^6}{x^6\rho^6}\Bigg]\Bigg\}+\Bigg\{2\rho\Bigg[\Bigg(\frac{64b^7}{7\rho^6x^7}+\frac{96b^5}{5\rho^6x^5}+\frac{16b^5}{5\rho^4x^5}$$

$$+\frac{48b^3}{\rho^6x^3}+\frac{16b^3}{3\rho^4x^3}-\frac{8b^3}{3\rho^2x^3}+\frac{8b}{\rho^6x}+\frac{4b}{\rho^4x}-\frac{4b}{\rho^2x}-\frac{b}{x}\Bigg)+$$

$$+\frac{1}{105x^7}\Big\{b\Big(\frac{x^2+2b^2-x^2\rho^2}{x^2+2b^2+x^2\rho^2}\Big)^{1/2}\Big[-\frac{960b^6}{\rho^6}+x^6$$

$$\times\Big(143+\frac{92}{\rho^2}-\frac{876}{\rho^4}-\frac{840}{\rho^6}\Big)-48b^4x^2\Big(\frac{42}{\rho^6}+\frac{1}{\rho^4}\Big)$$

$$-8b^2x^4\Big(\frac{1}{\rho^2}+\frac{210}{\rho^6}+\frac{166}{\rho^4}\Big)\Big]\Big\}+\Big(63+\frac{208}{\rho^2}$$

$$-\frac{224}{\rho^4}-\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times E\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)$$

$$+\Big(25-\frac{144}{\rho^2}-\frac{160}{\rho^4}+\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times F\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big]\Big\}\Big\}, \qquad (27)$$

$$\alpha^{(-)}\Big(7,\frac{b}{x},\rho\Big)=\frac{1}{1-\rho^2}\Big\{-\frac{2b\rho}{7x}\Big\{-1-\frac{4}{\rho^2}+\frac{4}{\rho^4}+\frac{8}{\rho^6}$$

$$+\frac{b^2}{x^2\rho^2}\Big(-8+\frac{16}{\rho^2}+\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(16+\frac{96}{\rho^2}\Big)$$

$$+64\frac{b^6}{x^6\rho^6}-\Big(\frac{x^2+2b^2+x^2\rho^2}{x^2+2b^2-x^2\rho^2}\Big)^{1/2}\Big[1-\frac{4}{\rho^2}-\frac{4}{\rho^4}$$

$$+\frac{8}{\rho^6}+\frac{b^2}{x^2\rho^2}\Big(-8-\frac{16}{\rho^2}+\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(-16+\frac{96}{\rho^2}\Big)$$

$$+64\frac{b^6}{x^6\rho^6}\Big]\Big\}+\Big\{2\rho\Big[\Big(\frac{64b^7}{7\rho^6x^7}+\frac{96b^5}{5\rho^6x^5}-\frac{16b^5}{5\rho^4x^5}$$

$$+\frac{16b^3}{\rho^6x^3}-\frac{16b^3}{3\rho^4x^3}-\frac{8b^3}{3\rho^2x^3}+\frac{8b}{\rho^6x}-\frac{4b}{\rho^4x}-\frac{4b}{\rho^2x}+\frac{b}{x}\Big)$$

$$+\frac{1}{105x^7}\Big\{b\Big(\frac{x^2+2b^2-x^2\rho^2}{x^2+2b^2+x^2\rho^2}\Big)^{1/2}\Big[-\frac{960b^6}{\rho^6}+x^6$$

$$\times\Big(17+\frac{484}{\rho^2}-\frac{36}{\rho^4}-\frac{840}{\rho^6}\Big)+576b^4x^2\Big(-\frac{7}{2\rho^6}-\frac{1}{\rho^4}\Big)$$

$$-16b^2x^4\Big(-\frac{41}{2\rho^2}+\frac{105}{\rho^6}+\frac{13}{\rho^4}\Big)\Big]\Big\}+\Big(-63+\frac{208}{\rho^2}$$

$$+\frac{224}{\rho^4}-\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}\times$$

$$\times E\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)$$

$$+\Big(-185+\frac{210\rho^2}{1+\rho^2}+\frac{304}{\rho^2}-\frac{608}{\rho^4}+\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times F\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big]\Big\}\Big\}, \qquad (28)$$

respectively. For small arguments x, functions (27) and (28) have the form:

$$\alpha^{(+)}\Big(7,\frac{b}{x},\rho\Big)\approx\frac{\rho^9x^9}{1008b^9}+O(x^{10}),$$

$$\alpha^{(-)}\Big(7,\frac{b}{x},\rho\Big)\approx-\frac{\rho^9x^9}{1008b^9}+O(x^{10}),$$

respectively. By substituting (27) and (28) into (13), we obtain, according to (12), the efficiency of the seventh harmonic generation

$$\eta^{(7)}(x,A)=\Big(\frac{7e^4Z_{\rm eff}N_{\rm e}A}{3m_{\rm e}^2V_Z^2\omega}\Big)^2n^6\Phi\Big(7,\frac{nV_E}{V_Z},A\Big), \qquad (29)$$

where

$$\Phi(7,x,A)=\frac{1}{2\rho^6x^6}\Big\{(1+\rho^2)\Big[D\alpha^{(+)}\Big(7,\frac{b}{x},\rho\Big)\Big]^2\Big|_{b=1}$$

$$+(1-\rho^2)\Big[D\alpha^{(-)}\Big(7,\frac{b}{x},\rho\Big)\Big]^2\Big|_{b=1}\Big\}. \qquad (30)$$

Here, as in expression (25), the maximum degree ρ of linear polarisation is expressed in terms of the degree A of circular polarisation from (7). Explicit expression (30) is, like expression (25), relatively cumbersome, so that the dependences described by expressions (29) and (30) are illustrated by Figs 4 and 5. Figure 4 shows the dependence of the harmonic generation efficiency on x: a rapid rise with increasing x up to a maximum, and the subsequent rather rapid decrease. The scaling parameter of this dependence is $x=nV_E/V_Z$ for any principal quantum number of the excited state. The efficiency of the seventh harmonic generation is proportional to the sixth power of the principal quantum number, as for the fifth harmonic. We can claim that this proportionality takes place for any harmonics. As in Fig. 1 for the fifth harmonic, Fig. 4 clearly demonstrates a relatively rapid decrease in the harmonic generation efficiency with increasing the degree A of circular polarisation. One can see from Fig. 5 that the relative harmonic generation efficiency depends on the degree of circular polarisation in a peculiar way, which is similar to the dependence demonstrated for the fifth harmonic in Fig. 2. We present in Fig. 6 the dependence

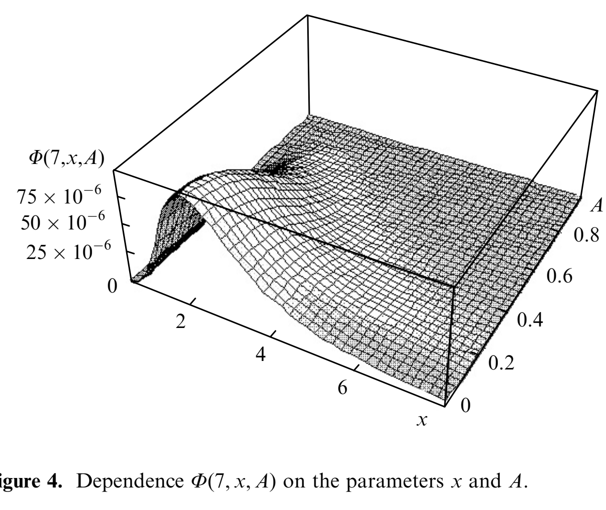

Figure 4. Dependence $\Phi(7,x,A)$ on the parameters x and A.

$$\frac{\eta^{(7)}(x,A)}{\eta^{(7)}(x,0)} = \frac{\Phi(7, nV_E/V_Z, A)}{\Phi(7, nV_E/V_Z, 0)} \tag{31}$$

for illustration. Like ratio (26), the dependence on the degree of polarisation A is nonmonotonic with a maximum at a nonzero value of A, which depends on the scaling parameter x. This nonmonotonic dependence has a threshold and for the seventh harmonic appears for $x \approx 4$. With increasing the parameter x, the range of nonmonotonicity first broadens and then becomes narrower, making difficult the observation of this range for large x.

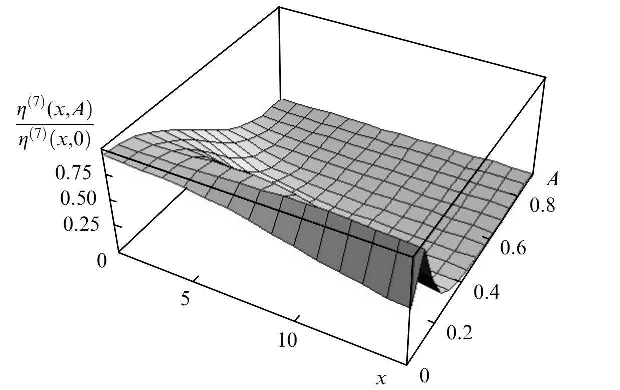

Figure 5. Two-dimensional distribution pattern of the function $\eta^{(7)}(x, A)[\eta^{(7)}(x, 0)]^{-1}$.

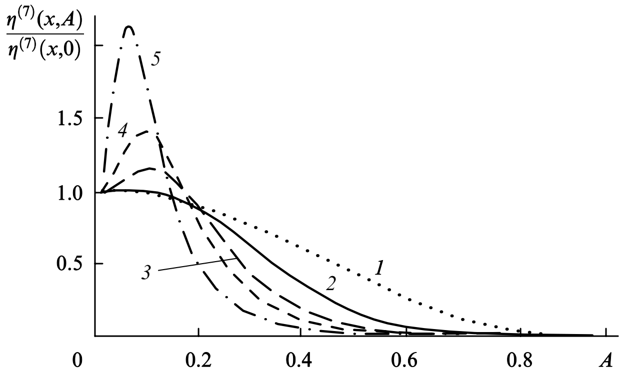

Figure 6. Dependences of ratio (31) on the parameter A for $x = 1$ (*1*), $x \approx 4$ (*2*) (the threshold dependence), $x = 7$ (*3*), 10 (*4*), and 18 (*5*).

5. For a photoionised plasma in the Bethe regime, the effect of polarisation ellipticity of the pump field on the efficiency of the third harmonic generation was considered in Ref. [9]. However, this work was concerned with the investigation of the plasma emerging from the gas of atoms with electrons in the 1s ground state.

First of all we show how the results of Ref. [9] can, on the basis of our present work, be extended to the plasma arising from the gas of atoms with electrons that are in the energy states with a principal quantum number n. This is easily derived from the fact that expression (1) for $n = 1$ applies to the 1s state. For this reason all the results of Ref. [9] are directly extended to excited states by replacing V_Z in the expressions from Ref. [9] with the combination V_Z/n. In particular, $\alpha = b(V_Z/V_E)$ should be replaced with $\alpha = b[V_Z/(nV_E)]$ in expressions (19), (20), and (24) from Ref. [9]. Expression (23) from Ref. [9] for the efficiency of the third harmonic generation takes the form

$$\eta^{(3)}(\rho) = \left(6\frac{e^4 Z_{\rm eff} N_{\rm e} A}{m_{\rm e}^2 V_Z^3 \omega}\right)^2 n^6 F^{(3)}\left(\frac{nV_E}{V_Z}, \rho\right). \tag{32}$$

This relation corresponds to the scaling of the harmonic generation efficiency $\sim n^6$ and the scaling of the function $F^{(3)}$ of the argument nV_E/V_Z for excited states which is used in Ref. [9]. Accordingly, in Figs 1–3 from Ref. [9], nV_E/V_Z should be employed instead of V_E/V_Z in the case of excited states.

All this allows applying the results of Ref. [9] not only to a plasma produced from the gas of atoms with the 1s ground-state electrons, but also in the case of excited states corresponding to principal quantum numbers $n \neq 1$ concerned in our present work.

In summary, we can assert, first, that the efficiencies of the third, fifth, and seventh harmonic generation as functions of pump field intensity achieve their maxima for the values of nV_E/V_Z of the order of magnitude of unity, according to Figs 1 and 4 and the generalisation of the results of Ref. [9]. Second, for a given value of nV_E/V_Z, the harmonic generation efficiency has a maximum in the case of a plane polarisation, i.e., for $A = 0$, when the pump field intensity is below the threshold intensity. When it exceeds the threshold value (specific for each harmonic), the maximum of the harmonic generation efficiency takes place for a nonzero degree of circular polarisation, which depends on the radiation intensity. We can assume that these properties are inherent in higher-order harmonics as well. The basic notions are thereby formed about the optimal conditions for maximising the harmonic generation efficiency for the rapidly switched high-intensity pump radiation.

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