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Transformation of a femtosecond pulse upon focusing

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***Abstract*.** **Diffraction of a converging spherical wave from a circular aperture is studied by the delta-pulse method. Simple expressions are obtained which describe the influence of an edge wave on the signal shape at the symmetry axis of the system. The spatiotemporal structure of the signal is studied in the vicinity of the focal point. It is found that, irrespective of the value of the solid angle of the illuminated sector, a signal proportional to the first time derivative of the input-pulse amplitude is formed at the focus.**

Keywords: *femtosecond pulse, diffraction, focusing, pulse differentiation.*

The calculation of the field-amplitude distribution in the vicinity of an objective focus is the initial stage of solving any problem of interaction of femtosecond pulses with a linear or nonlinear physical system. The problem of transformation of the spatiotemporal pulse shape having virtually an arbitrary structure can be solved by the method proposed in our papers [1 – 3]. This method can be used to study diffraction of a scalar wave perturbation having delta-like temporal shape. In particular, in this paper we use Kirchhoff's approximation to analyse the field produced by a scalar, amplitude-uniform spherical wave limited by a circular aperture in the vicinity of the focal point. The picture of the field development obtained by this method can be much simpler and clearly interpreted than pictures obtained by other methods. The study is performed in a linear approximation, and the interaction with matter is not considered. The conclusions of this paper correspond to the classical diffraction theory; however, the expressions derived by us are more convenient for the description of diffraction of namely ultrashort (in particular, extremely short) pulses. The obtained results exactly coincide with those derived in [4] for the region in front of the focal point. We calculated signals not only in the illuminated (in the geometric optics approximation) region but also in the shadow region. The analytic solution for the focal point itself has a new form. Our results are in qualitative agreement with data [5] obtained by a different method and exactly correspond to the conclusions of the classical diffraction theory of converging spherical monochromatic waves [6] in Kirchhoff's approximation.

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Let us use the expression describing the diffraction of a delta-like plane wave from an elementary part of the aperture at the coordinate origin in an absorbing screen. We assume that the screen is located in the xy plane and the wave propagates along the z axis, i.e., the wave perturbation has the form $f(t,z)=\delta(t-z/c)$. According to [3], the field amplitude at an arbitrary observation point P behind the screen with coordinates x_0, z_0 (in the spherical coordinate system with the origin coinciding with the aperture, the coordinates of this point are r,ψ,φ) is described by the expression

$$h_0(r,t)=\frac{z_0}{r^3}\delta(t-r/c)+\frac{1}{cr}(1+z_0/r)\delta_t(t-r/c), \qquad (1)$$

where

$$z_0=r\cos\psi; \quad \delta_t(t-r/c)\equiv\frac{\partial}{\partial t}\delta(t-r/c).$$

Relation (1) is the exact solution of the diffraction problem in Kirchhoff's approximation.

By using (1), we can calculate the response of an aperture of any shape illuminated by a signal of an arbitrary temporal shape (in particular, monochromatic wave). For this purpose, it is necessary to integrate (1) over the aperture area taking into account the initial propagation direction of the wave and its amplitude at a given point, and then to calculate the convolution of the real signal with the pulsed response obtained at the first stage. This procedure can be easily realised numerically, and in some cases simple analytic relations can be derived and general conclusions can be made on the field structure.

In particular, the solution of several diffraction problems for a plane wave showed [7 – 12] that, in the approximation of a delta wave incident on a finite-size aperture, there exist behind the screen a part of the initial wave whose shape and size coincide with those of the aperture and are independent of the distance to the screen, as well as a wave that can be treated, based on geometrical considerations, as the wave emitted by the aperture edge. This conclusion completely coincides with results of the classical diffraction theory [6] and was used in calculations of the interaction of ultrashort pulses with optical systems in studies of Bor and co-workers ([5] and references therein). Below, we will call this wave the edge wave.

Consider now a spherical delta wave, which produces the field

$$V(\rho, t) = \frac{1}{\rho}\delta\left(t - \frac{R-\rho}{c}\right), \quad (2)$$

where ρ is the current radius of the wave and R is its initial radius. For $t < R/c$, expression (2) describes a converging wave with a centre at the focal point F, which is located in our case at the coordinate origin. Diffraction of this wave from a circular aperture is the object of our study.

We will solve Kirchhoff's problem by considering a sphere of radius R, coinciding with the initial position of the wave, as a closed integration surface S. The radius of a transparent aperture in the sphere is a ($a < R$). Let us draw the z axis through the centres of the sphere and aperture directed from the aperture to the focus and denote by r the distance from the current integration point at the wave front to the observation point P. According to Fig. 1, this distance is determined by the relation

$$r^2 = R^2 + (R-\zeta)^2 - 2R(R-\zeta)\cos\vartheta, \quad (3)$$

where ζ is the distance from the observation point to the nearest point on the surface S along the normal to the wave front and ϑ is the angle between the directions to the observation point and to the current integration point with the apex at the focal point F.

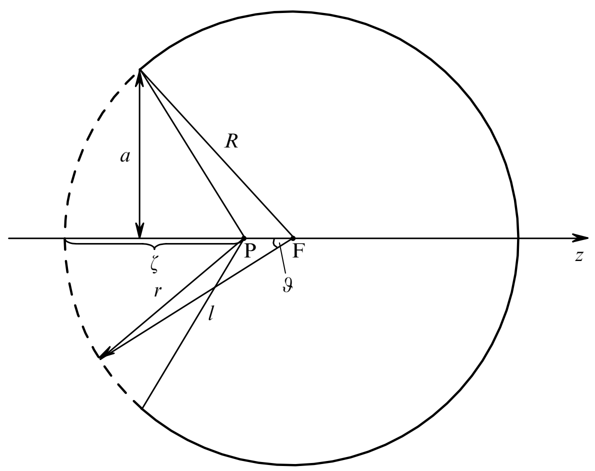

Figure 1. Coordinate system.

Expressions for the 'retarded' quantities entering Kirchhoff's integral [6] for a spherical wave have the form

$$[V] = \frac{\delta(t - r/c)}{R},$$

$$\left[\frac{\partial V}{\partial t}\right] = \frac{\delta_t(t - r/c)}{R}, \quad (4)$$

$$\left[\frac{\partial V}{\partial n}\right] = \frac{\delta(t - r/c)}{R^2} - \frac{\delta_t(t - r/c)}{cR}.$$

Let us pass to coordinates r, φ, where φ is the angle of rotation around a straight line passing through the observation point P and focal point F. Let us introduce, similarly to [2, 3, 13], the function $\Phi(r) \equiv \int_0^{2\pi} T(r,\varphi)\mathrm{d}\varphi$, where T is the amplitude transmission of the aperture in the screen. By calculating Kirchhoff's integral, we obtain

$$V(\mathrm{P}, t) = \frac{(ct + 2R - \zeta)(ct + \zeta)}{8\pi t R(R-\zeta)}\frac{\mathrm{d}\Phi(ct)}{\mathrm{d}(ct)}\{\Theta(t - \zeta/c)$$

$$- \Theta[t - (2R-\zeta)/c]\} + \frac{1}{2\pi(R-\zeta)}\Phi(\zeta)\delta(t - \zeta/c)$$

$$- \frac{1}{2\pi(R-\zeta)}\Phi(2R-\zeta)\delta[t - (2R-\zeta)/c]. \quad (5)$$

The first term in this expression describes the edge wave. The cofactor containing the difference of the Heaviside functions Θ determines the time interval within which the nonzero perturbation produced by the edge wave can exist in principle at the given observation point P. The real values of this time interval are determined not only by the distances from the point P to the nearest and farthest edges of the aperture but also by the form of the function $\mathrm{d}\Phi(ct)/\mathrm{d}(ct)$.

The physical sense of the second and third terms can be easily established by assuming that there exists a uniformly transmitting circular aperture, i.e., $T = 1$ within the aperture and $T = 0$ on the rest screen surface. We divide the inner space of the sphere into two halves by a plane passing through the sphere centre normally to the z axis. Let the observation point P be located in the left, nearest to the aperture, hemisphere embraced by the surface S, and be located in the illuminated region (Fig. 1). A beam drawn from the focus through the observation point always intersects the initial wave front, therefore, $\Phi(\zeta) = 2\pi$. A signal at the farthest from P point on the sphere is zero, $\Phi(2R - \zeta) = 0$, and the third term in (5) vanishes. Therefore, the second term describes a part of the initial wave transmitted by the screen converging to the focus.

One can see from Fig. 1 that $\Phi(\zeta) = 2\pi$ then and only then when the observation point is located within a cone with an apex at the point F, which rests on the aperture edges, i.e., lies within the region illuminated by the initial converging delta wave.

The total shape of the pulsed response at this point is determined by the sum of the initial and edge waves. If the point P is located in the right hemisphere, one can easily see that the situation is opposite: the factor at the second term vanishes, and $\Phi(2R - \zeta) = 2\pi$. Therefore, the third term describes the diverging delta wave, which has already passed through the focus. One can see from (5) that the sign of the field produced by this wave is opposite to that of the incident wave. Such a change in the sign coincides with data [5] and corresponds to the change in the phase of a monochromatic wave by π after propagation through the focal point [6]. In a free space, $\Phi(ct) = 2\pi$ and $\mathrm{d}\Phi(ct)/\mathrm{d}(ct) = 0$ for a homogeneous closed spherical wave for all $ct \in [\zeta, 2R - \zeta]$. Only the second and third terms in (5) are nonzero, while the edge wave is absent.

The solutions obtained for the left hemisphere are qualitatively similar to solutions for diffraction of a plane wave from a circular aperture: behind the screen, a part of the incident wave 'cut off' by the aperture propagates (in our case, this is a converging spherical wave). This wave propagates inside a cone with an apex at the point F resting on the aperture edges. The cone plays the role of the light – shadow interface and, in the case of a plane wave, it represents a cylinder resting on the aperture edges. Simultaneously, another wave scattered from the aperture edge propagates in the space. Figure 2 shows the amplitude distributions for this wave calculated for several instants. As in the case of a plane wave considered above, the duration of the response of the edge wave at any observation point is determined by the distances from this point to the nearest and farthest aperture edges. A more detailed study shows that the sign of the field changes at the light – shadow interface.

Figure 2 clearly demonstrates the singularity of the edge-wave field at the symmetry axis of the system, where the spatial region of existence of the corresponding signal turns to a point. This point moves at the velocity exceeding c [1, 9], by 'running down' the initial wave in front of the focus and 'leaving behind' the initial wave behind the focus. The expression for the pulse response of a circular aperture at the z axis has the form

$$V(\zeta, t) = \frac{1}{R-\zeta}\delta(t-\zeta/c) - \frac{1}{2(R-\zeta)}$$

$$\times\left[1+\frac{R}{l}-\frac{(R^2-a^2)^{1/2}}{l}+\frac{(R^2-a^2)^{1/2}}{Rl}\zeta\right]\delta(t-l/c), \quad (6)$$

where l is the distance from the observation point to the aperture edge. Again, an analogy exists with the case of diffraction of a plane wave [1, 2, 3]: the signal consists of two delta functions of opposite signs, corresponding to the transmitted and edge waves.

One can easily see that, if the observation point P is located on the z axis closer to the screen than the focus, first a segment of the incident converging spherical wave cut off by the aperture (in our case, the positive delta pulse) comes to it, and then the negative delta pulse of the edge wave. In the right hemisphere (behind the focal point), the distance from the aperture edge to the observation point lying on the z axis becomes smaller than the distance along the axis, and the picture changes to the opposite one. However, the total pulsed response again represents a sequence of the positive and negative delta functions. Therefore, both the transmitted and edge waves change their sign after propagation through the centre of curvature.

Near the focus, the sum in square brackets in (6) tends to 2, and

$$V(\zeta, t) \underset{\zeta\to \mathrm{F}}{\longrightarrow} \frac{a^2}{2cR^2}\delta_t(t-R/c). \quad (7)$$

The action of the operator δ_t on a real input signal corresponds to its differentiation with respect to time, i.e., the first time derivative of the initial signal should be observed at the focal point. This property of the temporal structure of the signal at large distances from the observation point to the aperture is well known in the theory of diffraction of plane waves of complicated shape with a restricted spectrum [14]. The same (qualitatively) conclusion can be obtained by using results [1 – 3] and taking into account that the lens focus is the image of an infinitely remote point.

Let us compare the result obtained above with the results of the diffraction theory for a monochromatic spherical wave [6]. We simplify expression (6) for the response of a circular aperture to a delta wave in the vicinity of the focal point. Let us introduce the linear coordinate z' measured from the point F and make the change of the time variable, by measuring the time τ from $t_0 = (t_1 + t_2)/2$ – the middle of the interval between the instants t_1 and t_2 of the successive appearance of the two delta functions. Then, $\tau = t - t_0$ and instead of (6) we obtain the expression

$$V(z', \tau) \approx \frac{1}{x}\left[\delta\left(\tau+\frac{z'}{c}\sin^2\frac{\alpha}{2}\right)-\delta\left(\tau-\frac{z'}{c}\sin^2\frac{\alpha}{2}\right)\right],$$

$$-\frac{t_2-t_1}{2} < \tau < +\frac{t_2-t_1}{2}, \quad (8)$$

where α the aperture angle between the z axis and the aperture edge.

By calculating the Fourier transform of (8) and omitting the phase factor describing the delay of the wave with respect to its initial position, we have

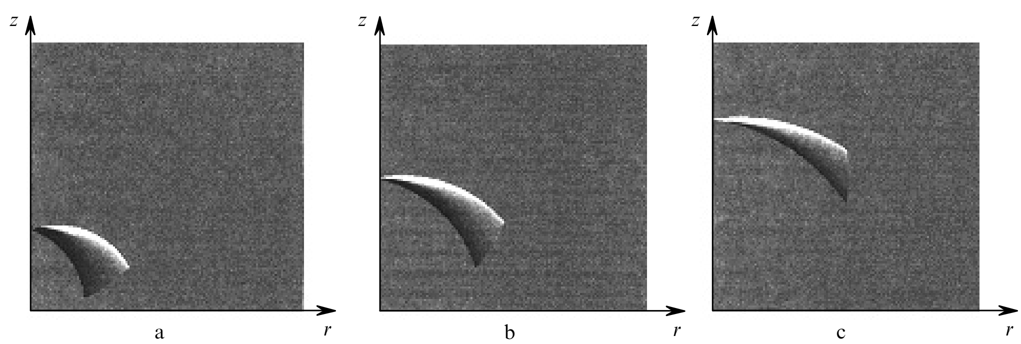

Figure 2. Time development of the edge wave upon illumination of a circular aperture by a converging spherical delta wave: the field distribution in the plane passing through the sphere centre (grey background corresponds to the zero amplitude; the lighter and darker pixels correspond to positive and negative amplitudes, respectively; the field exists only inside the sphere) at instants when the exciting wave is in front of the focus (a), at the focus (b), and behind it (c).

$$A(z', \lambda) = 4\pi \mathrm{i} \frac{\sin^2(\alpha/2)}{\lambda}$$

$$\times \left[\sin\left(2\pi \frac{z'}{\lambda} \sin^2 \frac{\alpha}{2}\right) \Big/ \left(2\pi \frac{z'}{\lambda} \sin^2 \frac{\alpha}{2}\right) \right], \quad (9)$$

where $A(z', \lambda)$ is the spectral density of the wave amplitude. Expression (9) coincides with the known amplitude distribution near the focus of a monochromatic spherical wave [6].

Therefore, we have found by the delta-pulse method that the reasons for formation of two delta pulses of the same amplitude on the symmetry axis are different. In the left hemisphere (in front of the focus), the first pulse is the element of the incident wave, while the second pulse is the element of the diffracted wave whose structure corresponds to emission of the aperture edge and somewhat 'delayed' with respect to the incident wave. In the right hemisphere (behind the focus), both pulses change their signs; now the edge wave leaves behind the incident wave by propagating everywhere along the z axis except the focal point at the velocity exceeding c. This looks like the preservation of the shape of the total pulsed response. The time dependences of amplitudes of these two pulses become different only in the case of nonuniform transmission of the aperture (for example, for diffraction from a Gaussian aperture [8]).

The correctness of the result obtained is also confirmed by the numerical calculations of the spatiotemporal structure of the signal. To simplify calculations, the velocity c of light was set equal to unity ($c = 1$) and the pulse shape was described by the function $f(t) = \exp[-(t/0.3)^2] \cos 2(t/0.3)$. It corresponds approximately to the ultrashort-pulse model used in [15] and experimental results presented in review [16]. The effective spatial length of such a signal is much smaller than the sphere radius, i.e., the signal can be treated as a correct delta-pulse model. The signal was specified on the sphere of radius $R = 10c = 10$ at the instant $t = 0$ and was directed to the sphere centre. The relative aperture of a circular hole was taken to be 0.2.

To reduce the calculation time, $N = 2500$ random point sources emitting a wave with the amplitude described by expression (1) were located in the spherical sector. Note that already for $N > 2000$, the shape of the calculated signal near the sphere centre ceased to depend on N. Figure 3 shows the initial function $f(t)$, its first derivative $\partial f/\partial t$, and the signal at the focus. Comparison of the results of numerical experiment with the derivative $\partial f/\partial t$ confirms the correctness of the transformation of the initial pulse shaped that we have found. The numerical calculation also showed the coincidence of the structure of the pulsed response at points symmetrically located with respect to the focus at the z axis, as follows from (8). Therefore, the simulations completely confirm the conclusion about a change in the sign of amplitudes of the transmitted and edge waves after their propagation through the focal point.

Then we simulated the propagation of a closed spherical wave (obviously, the edge wave is absent in this case). Random point sources, whose number was increased up to 10 000, were again located on the surface of the sphere of radius $R = 10$. The initial shape of the pulse was now described by the Gaussian $f(t) = \exp[-(t/0.3)^2]$. The absence of negative extrema simplifies the qualitative interpretation of the calculation results. The velocity of light is again set to be unity. The pulse is emitted from the points on the sphere surface at the instant $t = 0$ toward the sphere centre.

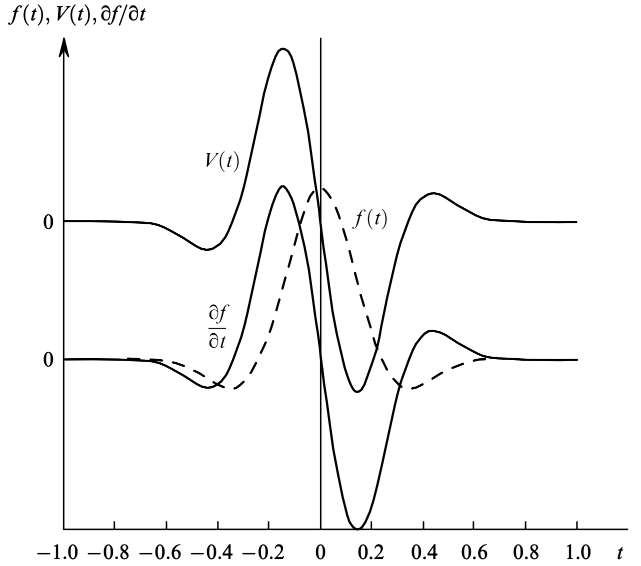

Figure 3. Pulse shape (signal amplitudes are normalised to the maximum value) on the sphere surface $f(t)$ and at the sphere centre (shifted along the ordinate for convenience) $V(t)$ and the shape of the first derivative $\partial f/\partial t$ of the signal.

The time dependence of the field distribution in the plane containing the sphere centre is demonstrated by a series of 'film frames' presented in Fig. 4. The formation of a converging spherical wave is clearly observed. This wave becomes diverging behind the focus and the sign of its amplitude changes simultaneously. The results of calculations again confirm the conclusion about the change in the pulse polarity after passage through the focus and about the transformation of its shape to the first time derivative directly at the focal point. The wave amplitude, as expected, is inversely proportional to the time of its propagation to the centre (Fig. 5). The coordinates of emitting points on the surface were randomly changed in time, which allowed us to estimate the accuracy of the calculation of the pulse amplitude; the number of emitting points was increased up to 180000.

When the centre of a Gaussian pulse propagated through the focal point, the peculiarity was observed – the field amplitude vanished, as is demonstrated in Fig. 5. The same effect is observed within the entire sphere volume (fluctuations of the field amplitude, which are distinctly observed in Fig. 4 at this instant, are caused by a finite density of emitting points on the surface). This circumstance is indicative of the problems that can appear if the term determined by the time derivative of the amplitude is excluded from the study of questions related to the energy conservation law [17].

Note in conclusion that the delta-like spherical wave considered in this paper (as the delta-like plane wave that we introduced earlier), like a harmonic wave, does not exist in nature and is an ideal mathematical representation, which is convenient for studying the propagation of pulsed waves in the linear approximation. Usually, by describing a complicated wave process with the help of monochromatic waves, one should consider the interaction of each of its harmonics with the optical system, take into account the spatiotem-

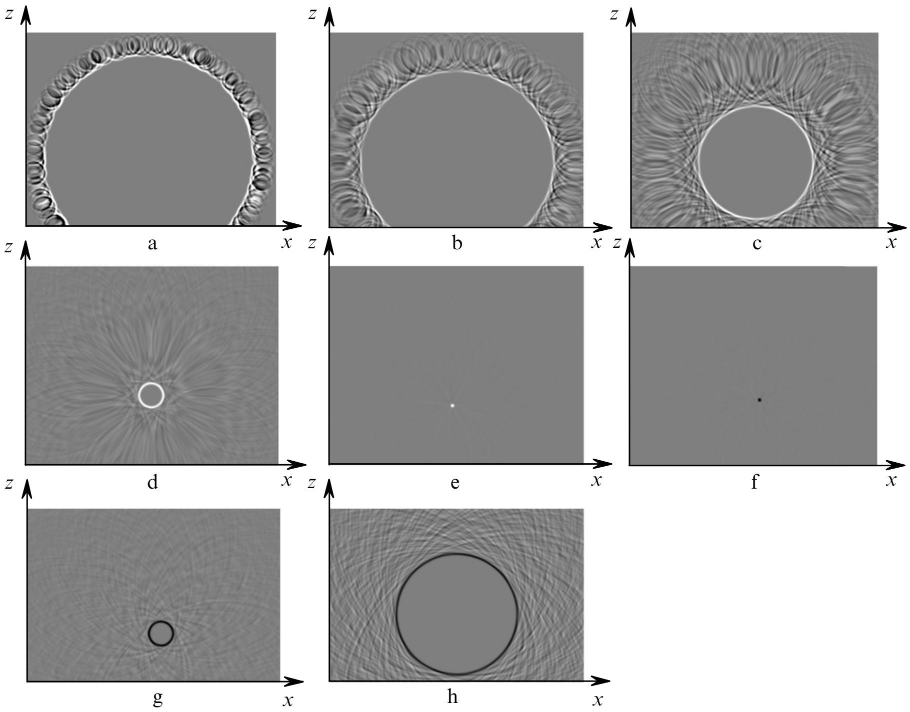

Figure 4. Simulations of the pulsed-wave field inside the sphere at instants $t = 1$ (a), 2 (b), 5 (c), 9 (d), 9.92 (e), 10.08 (f), 11 (g), and 15 (h).

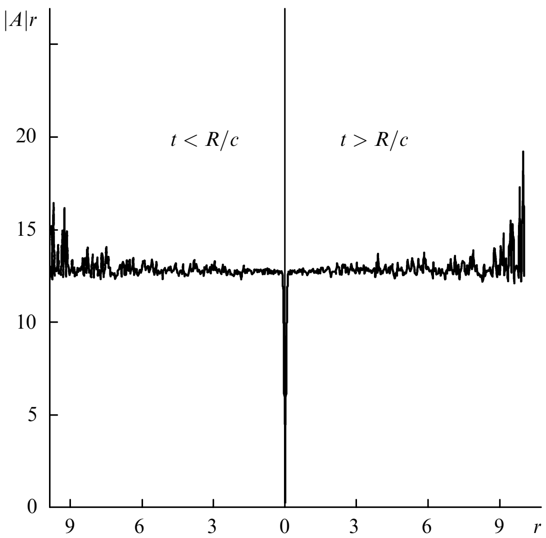

Figure 5. Dependence of the pulse amplitude $|A|r$ on its position inside the sphere.

poral filtration of waves by the system and then perform the inverse Fourier transformation. In our case, its is necessary to find the pulsed response at the first stage, which is done by integrating expression (1) over the aperture area, and to calculate the convolution of the response with the real input signal. The simplicity of final expressions obtained in our papers shows that our approach is preferable for application to femtosecond and any other extremely short pulses as compared to the traditional method.

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