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Approximate orthogonality relation for the modes of an open cavity

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***Abstract*. A simple orthogonality relation for the modes of an open cavity is obtained in the scalar field approximation. The relation directly follows from the wave equation and boundary condition at the cavity mirrors and can be applied for a broad class of systems with a cavity that can contain a random gain medium with a weak dispersion.**

Keywords: *open cavity, cavity mode, orthogonality*.

1. Introduction

The study of the dynamics of laser systems is an actual field of modern science. In the classical approximation, the dynamics of the field in a laser cavity is described by Maxwell's equations. The direct solution of this system of equations in partial derivatives is complicated even in simple cases, so that the expansion of the field in the cavity modes is used in many problems, which reduces the problem to a system of ordinary differential equations for mode amplitudes. The essential condition of the applicability of this method is the mutual orthogonality of cavity modes. For a closed cavity with perfectly reflecting walls filled with a homogeneous medium, the orthogonality relation is commonly written in the form

$$\int U_m^*(\boldsymbol{r})U_{m'}(\boldsymbol{r})\mathrm{d}V = 0 \quad \text{for} \;\; m \neq m'. \tag{1}$$

Because in this case the cavity modes $U_m(\boldsymbol{r})$, representing standing waves, can be always selected in the form of real functions, the complex conjugation in this relation can be omitted, by writing it in the form

$$\int U_m(\boldsymbol{r})U_{m'}(\boldsymbol{r})\mathrm{d}V = 0 \quad \text{for} \;\; m \neq m' \tag{2}$$

(hereafter, we will use the scalar approximation for the field).

A laser cavity is an essentially open system; radiation is extracted from the cavity through one or two partially transmitting cavity mirrors. Relation (1) is not fulfilled for modes in such a cavity. To overcome this problem, mirror losses were commonly replaced by losses distributed uniformly over the cavity volume (see, for example, [1, 2]). However, it was shown that such an approach led to incorrect results for a number of problems when the reflectivity of cavity mirrors was substantially lower than unity. In particular, a correct consideration of losses localised on the cavity mirrors gives the expression for the natural single-frequency linewidth of the laser which differs from the classical modified Schawlow – Townes formula by the additional factor [3, 4]. For example, for a laser with mirrors having reflectivities 100 % and 1 % this factor is 4.6 [4]. Note that such a low value of the mirror reflectivity is quite real for semiconductor lasers, and this problem is of current interest mainly for lasers of this type.

The problem of mode orthogonality in an open cavity was considered by many authors. In [5, 6], the approach was used based on the introduction of a system of conjugate modes, which was biorthogonal to the initial mode system. In [7], the orthogonality relation was proposed for modes in an open cavity in the complex form, which does not require the determination of conjugate modes. For many laser systems, however, the scalar field approximation can be used with good accuracy. In this paper, we derive the orthogonality relation in the scalar approximation, which is similar to that obtained in [7] but is simpler. The derivation is based on the scalar wave equation for the field. This relation is applicable to a broad class of systems with a cavity that can be filed with a random gain medium with a weak dispersion. In the case of a homogeneous medium, this relation coincides with (2).

Note that the orthogonality relation that we obtained was used earlier in paper [8] and its vector analogue in [9]; however, as far as we know, its derivation is reported for the first time.

2. Derivation of the orthogonality relation

Consider a cavity restricted by the planes $z = 0$ and $z = L$ and filled with an isotropic medium with the complex permittivity $\varepsilon_\omega(\boldsymbol{r})$. We assume that outside the cavity (i.e., for $z < 0$ or $z > L$), $\varepsilon_\omega(\boldsymbol{r}) = 1$.

The modes of such a cavity in the scalar approximation are the solutions of the wave equation

$$\Delta U(\boldsymbol{r}) + \frac{\omega^2}{c^2}\,\varepsilon_\omega(\boldsymbol{r})U(\boldsymbol{r}) = 0 \tag{3}$$

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containing for $z < 0$ and $z > L$ only the waves leaving the cavity. Such solutions exist only for a discrete set of values of the frequency ω. We denote these values and the corresponding solutions as ω_m and $U_m(\boldsymbol{r})$, respectively. The values ω_m are complex in the general case. The positive or negative sign of $\mathrm{Im}\omega_m$ corresponds to the increase or decrease of the field in time, respectively, which corresponds to the time dependence of the field $\sim\exp(-\mathrm{i}\omega_m t)$.

Let us write Eqn (3) for two different modes U_m and $U_{m'}$ (assuming that $\omega_m \neq \omega_{m'}$):

$$\Delta U_m + \frac{\omega_m^2}{c^2}\,\varepsilon_{\omega_m} U_m = 0,$$
$$\Delta U_{m'} + \frac{\omega_{m'}^2}{c^2}\,\varepsilon_{\omega_{m'}} U_{m'} = 0. \qquad (4)$$

By multiplying the first equation by $U_{m'}$ and the second one by U_m, taking their difference and integrating over the cavity volume, we obtain

$$\int (U_{m'}\Delta U_m - U_m \Delta U_{m'})\mathrm{d}V + \int\left(\frac{\omega_m^2}{c^2}\,\varepsilon_{\omega_m} - \frac{\omega_{m'}^2}{c^2}\,\varepsilon_{\omega_{m'}}\right) U_m U_{m'}\mathrm{d}V = 0. \qquad (5)$$

Due to the identity $U_{m'}\Delta U_m - U_m\Delta U_{m'} \equiv \nabla(U_{m'}\nabla U_m - U_m\nabla U_{m'})$, the first integral in this equality is reduced to the integral over the area of the cavity boundaries. For a typical relation that the functions U_m and $U_{m'}$ vanish at the cavity side facets, only contributions from the output facets remain in this integral:

$$\int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)_{z=L}\mathrm{d}x\mathrm{d}y - \int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)_{z=0}\mathrm{d}x\mathrm{d}y + \int\left(\frac{\omega_m^2}{c^2}\,\varepsilon_{\omega_m} - \frac{\omega_{m'}^2}{c^2}\,\varepsilon_{\omega_{m'}}\right) U_m U_{m'}\mathrm{d}V = 0. \qquad (6)$$

We will show now that the integrals over the output facets can be neglected with certain accuracy. For the piecewise continuous permittivity $\varepsilon(\boldsymbol{r})$, the expression

$$\int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)\mathrm{d}x\mathrm{d}y$$

is continuous over z, and the first two integrals in Eqn (6) can be calculated for $z = L + 0$ and $z = 0 - 0$, respectively, i.e., outside the cavity. The function U_m outside the cavity satisfies the wave equation (3) with $\varepsilon = 1$. Consider, for example, the right facet and write the Fourier expansion for the function U_m for $z = L$:

$$U_m(x, y, L) = \iint B_m(k_x, k_y)\exp[\mathrm{i}(k_x x + k_y y)]\mathrm{d}k_x\mathrm{d}k_y. \qquad (7)$$

The solution of the wave equation for $z > L$, containing only the waves coming from the laser, can be written in the form

$$U_m(x, y, z) = \iint B_m(k_x, k_y)\exp\left\{\mathrm{i}\left[k_x x + k_y y + \left(\frac{\omega_m^2}{c^2} - k_x^2 - k_y^2\right)^{1/2}(z - L)\right]\right\}\mathrm{d}k_x\mathrm{d}k_y,\ z > L. \qquad (8)$$

By using (8) and analogous expression for the function $U_{m'}$, we obtain

$$\int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)_{z=L}\mathrm{d}x\mathrm{d}y = \mathrm{i}(2\pi)^2\,\frac{\omega_m - \omega_{m'}}{c} \times \iint \frac{(\omega_m^2/c^2 - k_x^2 - k_y^2)^{1/2} - (\omega_{m'}^2/c^2 - k_x^2 - k_y^2)^{1/2}}{(\omega_m - \omega_{m'})/c} \times B_m(k_x, k_y)B_{m'}(-k_x, -k_y)\mathrm{d}k_x\mathrm{d}k_y. \qquad (9)$$

For convenience, we factored out the term $(\omega_m - \omega_{m'})/c$ from the integral. Let us now estimate the obtained integral by replacing the fraction on the integrand by unity. Such an approximation will give a correct order of the integral magnitude if at least one of the functions $B_m(k_x, k_y)$ and $B_{m'}(k_x, k_y)$ is small for values $k_\perp \equiv (k_x^2 + k_y^2)^{1/2}$ close to $k_{0m,m'} \equiv \omega_{m,m'}/c$ (and also for $k_\perp > k_{0m,m'}$) or, in other words, if the output radiation from the cavity does not contain at least for one of the modes any noticeable components propagating at angles close to 90° to the cavity axis (and also components experiencing total internal reflection from cavity mirrors). Thus, in this approximation, we can write

$$\int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)_{z=L}\mathrm{d}x\mathrm{d}y \sim \mathrm{i}(2\pi)^2\,\frac{\omega_m - \omega_{m'}}{c} \times \iint B_m(k_x, k_y)B_{m'}(-k_x, -k_y)\mathrm{d}k_x\mathrm{d}k_y. \qquad (10)$$

By using the transformation inverse to (7), we obtain

$$\int\left(U_{m'}\,\frac{\partial U_m}{\partial z} - U_m\,\frac{\partial U_{m'}}{\partial z}\right)_{z=L}\mathrm{d}x\mathrm{d}y \sim \mathrm{i}\,\frac{\omega_m - \omega_{m'}}{c} \times \iint U_m(x, y, L)U_{m'}(x, y, L)\mathrm{d}x\mathrm{d}y. \qquad (11)$$

For the integral over the left facet in equality (6), we obtain similarly

$$\int \frac{\omega_m^2\varepsilon_{\omega_m} - \omega_{m'}^2\varepsilon_{\omega_{m'}}}{\omega_m^2 - \omega_{m'}^2}\,U_m U_{m'}\mathrm{d}V \sim \mathrm{i}\,\frac{c}{\omega_m + \omega_{m'}} \times \left[\iint U_m(x, y, L)U_{m'}(x, y, L)\mathrm{d}x\mathrm{d}y + \iint U_m(x, y, 0)U_{m'}(x, y, 0)\mathrm{d}x\mathrm{d}y\right]. \qquad (12)$$

One can see from this that with an accuracy of the order of λ/L (where $\lambda = 2\pi c/\omega$ is the radiation wavelength in vacuum) for $m \neq m'$, the orthogonality relation

$$\int \frac{\omega_m^2\varepsilon_{\omega_m} - \omega_{m'}^2\varepsilon_{\omega_{m'}}}{\omega_m^2 - \omega_{m'}^2}\,U_m U_{m'}\mathrm{d}V = 0 \qquad (13)$$

is fulfilled, i.e, for the functions U_m and $U_{m'}$ normalised by the relation $\int nn_{\rm g}U^2{\rm d}V = 1$, integral (13) is of the order of λ/L ($nn_{\rm g} = \lim_{\omega_m\to\omega_{m'}}[(\omega_m^2\varepsilon_{\omega_m} - \omega_{m'}^2\varepsilon_{\omega_{m'}})/(\omega_m^2 - \omega_{m'}^2)]$, where $n \equiv \sqrt{\varepsilon}$ and $n_{\rm g} \equiv n + \omega\,\partial n/\partial\omega$ are the phase and group refractive indices, respectively).

For a medium with a weak dispersion in the case $|\omega_m - \omega_{m'}| \ll \omega_{m,m'}$, relation (13) for $m \neq m'$ is reduced to

$$\int nn_{\rm g}U_mU_{m'}{\rm d}V = 0. \tag{14}$$

For a medium without dispersion, it follows immediately from (13) for $m \neq m'$ that

$$\int n^2U_mU_{m'}{\rm d}V \equiv \int \varepsilon U_mU_{m'}{\rm d}V = 0. \tag{15}$$

If a medium is homogeneous, relation (13) is reduced to (2).

The use of the traditional (1) or refined (14) orthogonality relation in the calculation of the contribution from spontaneous emission leads to the corresponding form of the normalising factor for the field amplitude of a laser mode. In the case of relation (1), the amplitude is proportional to $[\int|U_m|^2{\rm d}V]^{-1}$, and in the case (14) – to $[\int U_m^2{\rm dV}]^{-1}$ [if the relative variations in the phase and group refractions in the cavity volume in expression (14) can be neglected]. In most cases, the final result of the calculation of radiative characteristics of the laser is the radiation intensity or spectral density, i.e., quantities quadratic (in modulus) in the field amplitude. The use of refined relation (14) leads to the result which differs by the factor $|\int|U_m|^2{\rm d}V|^2\times|\int U_m^2{\rm d}V|^{-2}$ from that obtained by using expression (1). This factor can be reduced in particular cases to the Petermann [10] and Henry [4] factors; if the dependence of the field on the coordinate z and transverse coordinates x, y is factorised, this factor is a product of the above-mentioned factors. As pointed out in Introduction, this factor for typical semiconductor lasers can be several times greater than unity, which requires the use of the orthogonality relation namely in the form (13), (14).

3. Conclusions

We have obtained approximate orthogonality relations (13), (14) for the modes of an open cavity. These relations are quite simple and at the same time take into account the spatial inhomogeneity of the active medium in the most general form. Thus, relation (14) was used [11] for calculating the 'natural' linewidth and intensity fluctuations for a distributed feedback laser with a tilted grating, whose cavity was a complicated optical system. It is shown that the accuracy of the orthogonality relation has been determined by the ratio λ/L and, therefore, it should be quite acceptable for a broad class of semiconductor lasers.

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