

SOLITONS

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To the memory of Professor A.N. Oraevsky

Dynamics of solitons in the model of nonlinear Schrödinger equation with an external harmonic potential: I. Bright solitons

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***Abstract*. The dynamics of nonlinear solitary waves is studied by using the model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. The model allows one to analyse on the general basis a variety of nonlinear phenomena appearing both in a Bose – Einstein condensate in a magnetic trap, whose profile is described by a quadratic function of coordinates, and in nonlinear optics, physics of lasers, and biophysics. It is shown that exact solutions for a quantum-mechanical particle in a harmonic potential and solutions obtained within the framework of the adiabatic perturbation theory for bright solitons in a parabolic trap are completely identical. This fact not only proves once more that solitons behave like particles but also that they can preserve such properties in different traps for which the parabolic approximation is valid near potential energy minima. The conditions are found for formation of stable stationary states of antiphase solitons in a harmonic potential. The interaction dynamics of solitons in nonstationary potentials is studied and the possibility of the appearance of a soliton parametric resonance at which the amplitude of soliton oscillations in a trap exponentially increases with time is shown. It is shown that exact solutions of the problem found using the Miura transformation open up the possibility to control the dynamics of solitons. New effects are predicted, which are called the reversible and irreversible denaturation of solitons in a nonstationary harmonic potential.**

Keywords: *solitons, nonlinear Schrödinger and Gross – Pitaevsky equations, Bose – Einstein condensate.*

1. Model of the nonlinear Schrödinger equation with an external harmonic potential in the theory of Bose – Einstein condensation and the theory of optical solitons

The interpenetration of ideas and methods being used in various fields of science and technology becomes at present one of the decisive factors of the development of science as a whole. Among the most spectacular examples of such an interchange by ideas and theoretical methods for analysis of various physical phenomena is the problem of the dynamics of a solitary nonlinear wave described by the mathematical model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. This model is used in a variety of fields of modern science and probably will be able to play the basic role similar to that played in due time by the model of a quantum-mechanical linear harmonic oscillator in the development of modern physics.

At present among the most important applications of the NSE model with a harmonic potential are the studies of nonlinear phenomena observed upon the Bose – Einstein condensation (BEC) of atoms in vapours of alkali-earth metals. It is known that the nonlinear dynamics of a Bose – Einstein condensate in magnetic traps is described by the Gross – Pitaevsky average-field model [1, 2]:

$$i\hbar \frac{\partial}{\partial t} \Phi(\boldsymbol{r}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\rm ext}(\boldsymbol{r}) + G|\Phi(\boldsymbol{r}, t)|^2 \right) \Phi(\boldsymbol{r}, t), \quad (1)$$

$$V_{\rm ext}(\boldsymbol{r}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right), \quad G = \frac{4\pi\hbar^2 a}{m}.$$

The nonlinear term in the equation for the wave function Φ of the condensate (where G is the energy of the pair interaction between particles, a is the interaction length, $V_{\rm ext}$ is the energy of the condensate interaction with the external field of the trap, the rest of the notation is standard [1 – 3]) takes into account pair interactions between the condensate particles.

The conditions of the applicability of model (1) and of the so-called average-field approximation are discussed in detail, for example, in monograph [3] and review [4]. Because the condensate contains a macroscopically large number of particles, the wave function of the condensate becomes a classical macroscopic quantity, similarly to the strength of the electromagnetic-wave field, which becomes classical for large occupation numbers of photons in each state.

It is well known that the most complicated problem in the development of a meaningful physical model, which is not restricted only to the mathematical description of a particular phenomenon, is a passage from the *description* of the phenomenon to its *explanation*. Magnificent examples of the fine skill in the development of physically constructive ideas, the aspiration of the author to propose simple, physically clear explanations to intricate phenomena, based, in particular, on the well-known concepts of quantum

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electronics, were and remain the papers of Prof. A.N. Oraevsky (see, for example, the data base in the server of the American Institute of Physics [5]).

The analysis of the dynamic analogy between the Bose condensate of photons, an atomic condensate, and a condensate of Cooper pairs performed by Oraevsky [6–11] shows that the wave nature of matter is distinctly manifested in the condensate, and an ensemble of a sufficiently large number of particles behaves as a classical field having the amplitude and phase. The dynamics of the condensate can be treated as a substantially nonlinear process, which is completely similar to the formation of the Bose condensate of photons in a laser (see, for example, review [7] and references therein), and the study of the laser dynamics can be used as a basis for a deeper understanding and prediction of dynamic processes in Bose condensates of another type [6–11].

The method of analogies used by Oraevsky for the formulation of new problems in the BEC theory is a powerful tool for analysis of various physical phenomena. Analogies between the BEC, superfluidity, and superconductivity, the Bose condensation of photons, and lasing were studied in numerous papers, in particular, in [12–20]. The authors of paper [19] presented the review of concepts of the coherence and coherent states of the field and discussed the optical coherence, quantum-mechanical coherence, photon statistics, self-induced transparency, superconductivity and superfluidity, and Dicke superradiance. The theory of simulated emission and phase transitions was developed by Oraevsky already in papers [8, 9].

Note, however, that while the problem of localisation of the Bose condensate of photons has been already solved in pioneering papers of Basov and Prokhorov with co-workers (see, for example Nobel lectures [21, 22], pioneering papers [23–25] and references therein), the problem of localisation of a neutral atom still remains one of the complicated problems up to now. The solution of this problem was first proposed by Letokhov [26], who showed that atoms can be localised in nodes or antinodes of a standing light wave whose frequency is far from the atomic transition frequencies. At present the method of laser manipulation of an atomic condensate is generally accepted.

In the absence of an external potential, Eqn (1) is the NSE, which is well studied in the theory of self-focusing. Because the one-dimensional NSE belongs to the class of exactly integrated equations [27] and has many exact solutions [28], the model of a condensate in the so-called cigar-shaped trap with the transverse dimensions far smaller than the longitudinal size proved to be attractive. It is in cigar-shaped traps that nonlinear collective excitations in the BEC were first discovered, which were called bright and dark soliton waves of matter; and it is in pioneering experimental studies on the generation of solitons in the BEC [29–32] that a profound mathematical analogy between the theory of soliton waves of matter and the theory of optical solitons in optical fibres (see also monographs [33–36] and comprehensive references therein).

The passage to the one-dimensional dimensionless NSE with a harmonic potential

$$\mathrm{i}\,\frac{\partial u}{\partial \xi}+\frac{1}{2}\,\frac{\partial^2 u}{\partial \tau^2}+R|u|^2u-\frac{1}{2}\,\Omega^2\tau^2u=0 \qquad (2)$$

is described in detail, for example, in [37, 38]. Note that one-dimensional model (2) was developed in fact simultaneously in the BEC theory and the theory of optical solitons. For example, this model was considered in papers [39, 40] in the development of the concept of quasi-solitons in fibreoptic communication links with periodic variations in the group-velocity dispersion (this field of practical applications of solitons is discussed comprehensively in books [33, 34, 41]). This model also appears in the study of generation of solitons in the forbidden region of group-velocity dispersion. In [42, 43], the situation was considered, in particular, when a pair of solitons was used as the trap potential. In this case, two control soliton pulses form a nearly parabolic well for a laser pulse with a different wavelength lying, for example, in a spectral region forbidden for the generation of solitons. A soliton captured in a parabolic trap not only exists in the forbidden region of parameters but also preserves its unique properties even in the femtosecond time range [44].

The above examples of using the mathematical NSE model with an external harmonic potential in the BEC theory and problems of nonlinear fibre optics by no means do not exhaust the list of possible applications of the model under study. Thus, the NSE model with a harmonic potential opens up new possibilities in simulations of nonlinear mechanisms of energy transfer in long biological polymer molecules. The study of these mechanisms is important for the explanation of the appearance of soliton waves in DNA (see, for example, pioneering works of Davydov [45], paper of Oraevsky [46], and a recent review in this field [47]).

From the point of view of practical applications, one of the central problems of the theory is the search for new possibilities to control the dynamics of solitons. This determined the scope of problems that we considered in this paper. The investigation of the BEC dynamics includes the analysis of the role of boundaries of a cigar-shaped trap whose longitudinal size is assumed comparable with the region of variations in the order parameter of the BEC. The nonstationary problem of the dynamics of formation and interaction of solitons in the BEC is considered for bright, dark, and grey solitons. From the point of view of possible applications in high-speed soliton optical communication links, of practical interest can be the conditions of a complete compensation of forces between solitons discovered by us.

In this paper, we study the new possibilities for controlling the parameters of solitons produced in nonstationary potentials. In particular, we predicted the possibility of a soliton parametric resonance, when the amplitude of soliton oscillations in a trap increases exponentially with time. The effect of soliton denaturation that we discovered can be used to construct the simplest model for explaining the physical mechanisms of the DNA denaturation (the detailed results are presented below). The investigations of the dynamics of dark solitons are summarised in the second part of the paper. By using the mathematical apparatus developed for applications in high-speed fibreoptic communication links [48–54], we discovered a new class of mathematical NSE models, which are exactly integrated by the method of the inverse scattering problem. The corresponding solutions are also presented in the second part of this paper [55].

2. Comparative analysis of transient processes in the model of a linear harmonic oscillator and the NSE model with a harmonic potential

It is well known that the time dependences of the average values of the momentum and coordinate in the model of a linear quantum-mechanical oscillator

$$\mathrm{i}\hbar\,\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\Psi(x,t)}{\partial x^2} + \frac{m\omega^2x^2}{2}\,\Psi(x,t), \quad (3)$$

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2\mathrm{d}x = 1$$

in the state with the wave function

$$\Psi_0(t=0,x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left[-\frac{m\omega}{\hbar\pi}(x-x_0)^2 + \mathrm{i}\,\frac{p_0x}{\hbar}\right] \quad (4)$$

(where ω is the circular frequency, x is the displacement of a particle with the mass m from the equilibrium position) are determined by the well-known expressions [56]

$$\overline{x(t)} = x_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t, \quad (5)$$

$$\overline{p(t)} = p_0\cos\omega t - m\omega x_0\sin\omega t. \quad (6)$$

Let us show that by using the methods of the adiabatic perturbation theory for solitons [57–63], we can obtain analytic expressions for the main parameters of NSE solitons in a parabolic potential, which are completely mathematically equivalent to expressions (5) and (6), and thereby approximately describe the motion of solitons as the motion of material points in the Newton mechanics.

By considering external NSE potential (2) as a small perturbation

$$\mathrm{i}\,\frac{\partial u}{\partial\xi} + \frac{1}{2}\,\frac{\partial^2u}{\partial\tau^2} + |u|^2u = \mathrm{i}\varepsilon(u), \quad (7)$$

we can write the solution of Eqn (2) in the form of a soliton with dynamically changing parameters (amplitude, the centre-of-mass position, phase, and velocity)

$$u(\xi,\tau) = \eta(\xi)\mathrm{sech}[\eta(\xi)(\tau - q(\xi))]\exp[\mathrm{i}\varphi(\xi) - \mathrm{i}\delta(\xi)\tau], \quad (8)$$

which are related by simple differential equations

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = -\delta, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \frac{1}{2}(\eta^2 - \delta^2). \quad (9)$$

Within the framework of the adiabatic perturbation theory for solitons, these four parameters are described by the system of equations [57–61]

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \mathrm{Re}\int_{-\infty}^{\infty}\varepsilon(u)u^*(\tau)\mathrm{d}\tau, \quad (10)$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}\xi} = -\mathrm{Im}\int_{-\infty}^{\infty}\varepsilon(u)\tanh[\eta(\tau-q)]u^*(\tau)\mathrm{d}\tau, \quad (11)$$

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = -\delta + \frac{1}{\eta^2}\mathrm{Re}\int_{-\infty}^{\infty}\varepsilon(u)(\tau-q)u^*(\tau)\mathrm{d}\tau, \quad (12)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \mathrm{Im}\int_{-\infty}^{\infty}\varepsilon(u)\left\{\frac{1}{\eta} - (\tau-q)\tanh[\eta(\tau-q)]\right\} \times u^*(\tau)\mathrm{d}\tau + \frac{1}{2}\left(\eta^2-\delta^2\right) + q\,\frac{\mathrm{d}\delta}{\mathrm{d}\xi}. \quad (13)$$

For the parabolic interaction potential, we obtain from Eqns (10)–(13)

$$\frac{\mathrm{d}\delta}{\mathrm{d}\xi} = \Omega^2q, \quad \frac{\mathrm{d}q}{\mathrm{d}\xi} = -\delta, \quad (14)$$

which leads to two equations for a harmonic oscillator

$$\frac{\mathrm{d}^2q}{\mathrm{d}\xi^2} = -\Omega^2q, \quad \frac{\mathrm{d}^2\delta}{\mathrm{d}\xi^2} = -\Omega^2\delta, \quad (15)$$

whose solutions have the form

$$q(\xi) = q_0\cos(\Omega\xi) - \frac{\delta_0}{\Omega}\sin(\Omega\xi), \quad (16)$$

$$\delta(\xi) = q_0\Omega\sin(\Omega\xi) - \delta_0\cos(\Omega\xi), \quad (17)$$

where the parameters with the subscript '0' correspond to the initial values of the velocity δ and the centre-of-mass position q.

Therefore, the main result of the approach developed in the paper is the conclusion that analytic results obtained for a nonstationary quantum-mechanical harmonic oscillator and for solitons in a harmonic trap are completely mathematically equivalent. To be certain that this is the case, it is sufficient to change the sign of the initial pulse in expressions (5), (6), and (16), (17) and consider the results of numerical experiments.

We emphasize, however, that while the results for linear model (3) are exact [expressions (5) and (6)], expressions (16) and (17) for nonlinear model (7) are valid only within the framework of the adiabatic perturbation theory for solitons. Recall that the so-called adiabaticity of perturbations, which allows the use of the perturbation theory, means that a change in the soliton shape remains small during characteristic times corresponding to the period of a harmonic oscillator and at distances corresponding to the length of the dispersion spread of a wave packet.

Our comparative numerical analysis of the dynamics of NSE solitons in a parabolic trap described by model (2) for $R \neq 0$ and the dynamics of a linear oscillator [equation (2) for $R = 0$) revealed a number of general qualitative properties.

Consider the typical results of numerical experiments presented in Figs 1–5 both for single and interacting wave packets. Figures 1–3 compare the dynamics of a linear oscillator, which is initially in the state with wave function (4), whose centre of gravity is initially displaced with respect to the equilibrium position, and the dynamics of the NSE soliton in a parabolic potential. The nonstationary problem for a linear oscillator, which has exact analytic solution (5), (6), illustrates the possibilities and stability of the numerical algorithm (calculations were performed with a double accuracy), the contour map (equal-level lines) is represented at the logarithmic scale. Because NSE (2) for $R = 0$ transforms to the usual Schrödinger equation for a linear

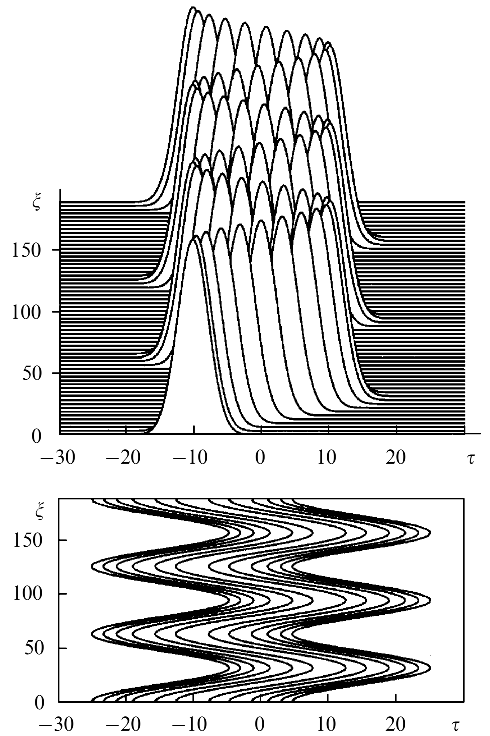

Figure 1. Spatiotemporal dynamics of the ground state of an oscillator with the centre of gravity initially displaced with respect to the equilibrium position. The contour map (equal-level lines) for the normalised wave function is presented at the logarithmic scale, beginning from the value 10^{-10} with the step 10^2. Calculations are performed with a double accuracy for Eqn (2) for $\Omega = 0.1$ in the absence of self-action ($R = 0$).

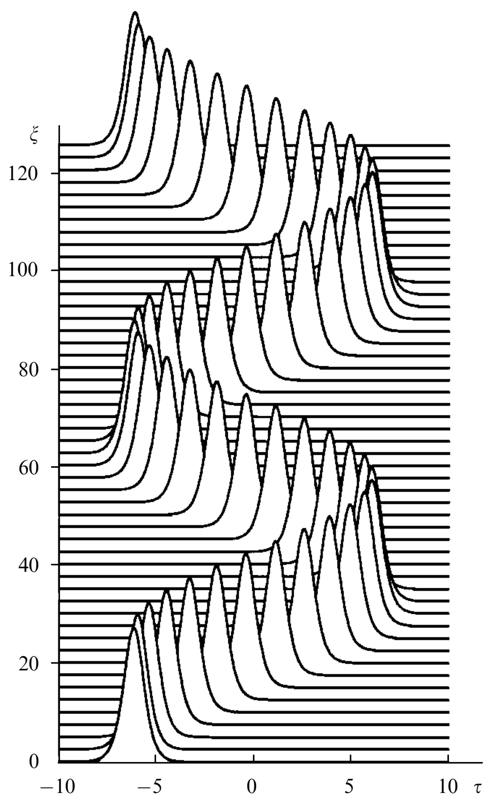

Figure 2. Nonlinear dynamics of the NSE soliton in a harmonic potential calculated within the framework of model (2) for $\Omega = 0.1$ and $R = 1.0$.

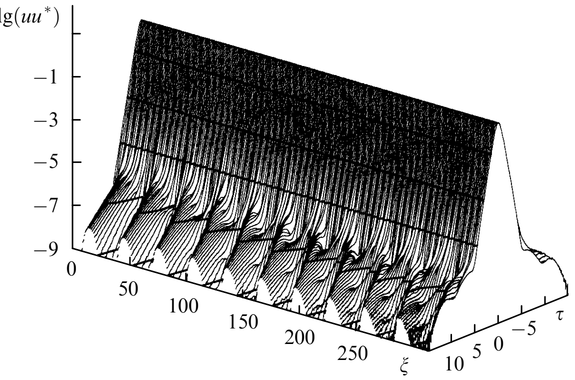

Figure 3. Soliton dynamics presented at the logarithmic scale, beginning from the value $\lg(uu^*) = -9$ with the step 2.

harmonic oscillator, its ground-state wave function in the dimensionless form is described by the expression

$$u_0(\xi, \tau) = \left(\frac{\Omega}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{\Omega}{2}\,\tau^2 - \mathrm{i}\,\frac{\Omega}{2}\,\xi\right).$$

Although the calculated dynamics of the NSE soliton in a harmonic potential presented in Fig. 2 completely agrees with analytic estimates (16), (17) obtained using the perturbation theory (7)–(13), nevertheless it does not allow us to make a certain conclusion about the real dynamics of the soliton envelope. Indeed, if we consider the distortions of the soliton shape at a 'deeper' logarithmic scale (Fig. 3), we will see how the distortions of the soliton shape gradually appear, which tend to accumulate, whereas the central part of the pulse does not change substantially.

The dynamics of a linear harmonic oscillator found initially in the state with the wave function representing a linear superposition

$$u_{1+2}(\xi = 0, \tau) = u_0(\tau - q_0) + u_0(\tau + q_0)\exp(\mathrm{i}\varphi)$$

of the two wave functions separated is space, where the parameter φ describes their relative phases, is presented in Fig. 4. The contour maps of equal-level lines at the logarithmic scale clearly show that the initial state can be considered as two virtually nonoverlapping Gaussians with parameters corresponding to the wave function of the ground state of a harmonic oscillator. The qualitative picture of their interaction, which is similar due to the optical-quantum-mechanical analogy, for example, to the interaction of two Gaussian beams in a gradient waveguide, is determined by the initial phase difference. In the case of in-phase initial states ($\varphi = 0$), their interaction at the trap centre ($\xi = 0$) corresponds to attraction, while in the case of out-of-phase states ($\varphi = \pi$), their interaction corresponds to repulsion, which is clearly seen in the map in Fig. 4. Thus, the equal-level lines in Fig. 4a intersect at the trap centre, while in Fig. 4b they do not intersect at the trap centre, by forming a gap.

It is well known that the dynamics of NSE solitons is also determined by phase relations between pulses. The attraction of in-phase solitons and repulsion of out-of-phase NSE solitons are described by analytic expressions obtained, in particular, by the methods of the adiabatic perturbation theory (all the priority papers on the interaction of solitons in the NSE model are cited, for example, in review [63]).

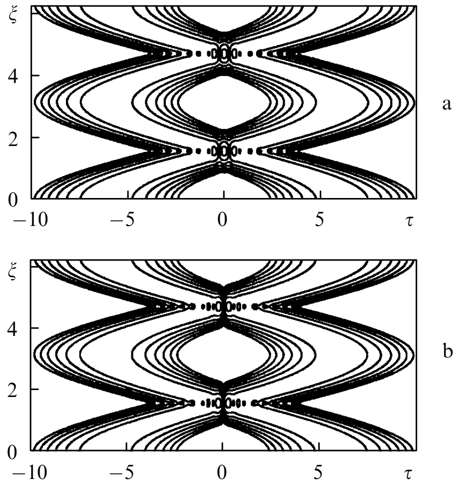

Figure 4. Contour maps (at the logarithmic scale) illustrating the dynamics of a linear harmonic oscillator initially in the states corresponding to the in-phase (a) and out-of-phase (b) wave functions. Calculations are performed for $\Omega = 1.0$ and $R = 0$ for Eqn (2). The contour lines begin with the value $\lg(uu^*) = -6$ and are drawn with the step 1.

Let us compare the interaction dynamics of in-phase and out-of-phase NSE solitons in a harmonic potential (Fig. 5) with that of a harmonic oscillator presented in Fig. 4. The initial state is also taken in the form of two almost nonoverlapping functions, which are now represented not by two Gaussians but by two hyperbolic secants, each of them being the exact solution of the NSE in the absence of a harmonic potential. One can see at the logarithmic scale how the distortions of the soliton shape appear during the interaction of solitons, out-of-phase-phase solitons never being overlapped (by repelling). It follows from our numerical experiments performed in a broad range of variations of the parameters of the problem that the interaction of out-of-phase solitons drastically differs from the dynamics of in-phase solitons, being more stable (Fig. 5).

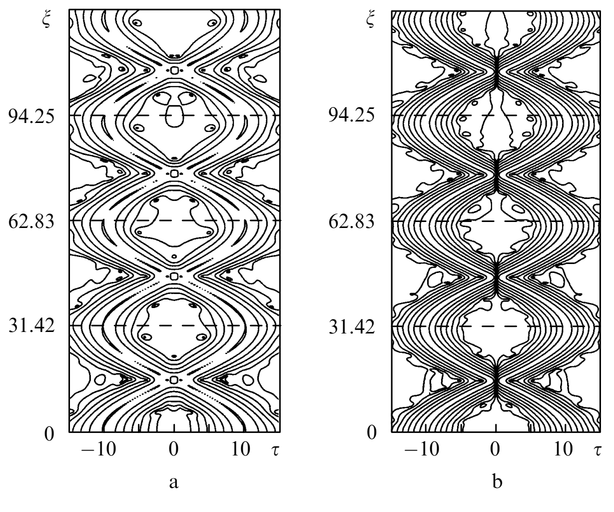

Figure 5. Contour maps (at the logarithmic scale) illustrating the interaction dynamics of in-phase (a) and out-of-phase (b) NSE solitons in a harmonic potential. Calculations are performed for $\Omega = 0.1$ and $R = 1.0$ for Eqn (2). The contour lines begin with the value $\lg(uu^*) = -6$ and are drawn with the step 1.

Note by analogy that the multisoliton solutions of the Gross–Pitaevsky equation (1) are not the multiboson wave functions. For a harmonic oscillator (3), Ψ is the one-particle wave function, while the function Φ in the Gross–Pitaevsky equation is a collective variable – the order parameter reflecting the evolution of the spatial density of the condensate, its spatial argument rather than the coordinates of bosons in the condensate.

The mathematical formulation of the problem proves to be completely similar to (7)–(13); however, now the soliton pair

$$u_{1,2}(\xi,\tau) = \eta_{1,2}(\xi)\operatorname{sech}[\eta_{1,2}(\xi)(\tau - q_{1,2}(\xi))] \times \exp[i\varphi_{1,2}(\xi) - i\delta_{1,2}(\xi)\tau] \tag{18}$$

is used as the initial condition. By substituting (18) into (7), we obtain the perturbed NSE model for two solitons

$$i\,\frac{\partial u_1}{\partial \xi} + \frac{1}{2}\,\frac{\partial^2 u_1}{\partial \tau^2} + |u_1|^2 u_1 = \frac{1}{2}\,\Omega^2\tau^2 u_1 - 2|u_1|^2 u_2 - u_1^2 u_2^*, \tag{19}$$

$$i\,\frac{\partial u_2}{\partial \xi} + \frac{1}{2}\,\frac{\partial^2 u_2}{\partial \tau^2} + |u_2|^2 u_2 = \frac{1}{2}\,\Omega^2\tau^2 u_2 - 2|u_2|^2 u_1 - u_2^2 u_1^*. \tag{20}$$

The three terms in the right sides of (19) and (20) describe the confinement (capture) of solitons by a parabolic trap and nonlinear interaction between overlapping solitons. Note that depending on the sign of the parameter Ω^2, Eqns (19) and (20) describe either the confining (well) or repulsive (barrier) potentials. Let us also emphasize that in the absence of an external potential, the problem of the interaction of NSE solitons within the framework of the adiabatic perturbation theory has been already solved in classical papers [57–61], while the model considered here is in fact the generalisation of the previous results to the case of an external harmonic potential. After transformations similar to (7)–(17), we obtain finally the equation describing the interaction dynamics for a pair of NSE solitons in a harmonic potential

$$\frac{d^2 q}{d\xi^2} = -\Omega^2 q - 4\eta^3 \exp(-2\eta q)\cos\varphi, \tag{21}$$

where $q = (q_1 - q_2)/2$ is the distance and $\varphi = (\varphi_1 - \varphi_2)/2$ is the phase between solitons.

The most interesting results following from the analysis of our analytic model are:

(i) The oscillation period of a pair of solitons in a parabolic trap is determined by the combined action of two forces. The first force increases linearly with distance and dominates at large distances between solitons. The second force is a nonlinear short-range force (exponentially decreasing with distance) and depends on the phases of interacting solitons. It begins to play the role only when the wave functions are well overlapped and solitons closely approach each other.

(ii) The phase dependence of forces and the sign of the potential (attraction or repulsive external potential) allow the efficient control of the dynamics of Schrödinger solitons. When these two forces are exactly compensated, for example, for out-of-phase solitons in the attraction potential or for in-phase solitons in the repulsive potential, a stationary state can be formed. The study of the stability of the stationary regime by the usual method of linearisation of equations with respect to stationary values proves to be quite simple and shows that a stable state forms only out-of-phase solitons in the attraction potential, while the bound states of solitons in the repulsive potential are always unstable.

Let us confirm the above conclusions by particular calculations. Consider a pair of in-phase solitons with the centres of mass well separated by varying in numerical calculations only one parameter of the problem – the distance q, by decreasing it between initially stationary NSE solitons. Figure 6a shows that the role of short-range forces increases with decreasing q. These forces substantially change the interaction dynamics of in-phase solitons in a parabolic trap under the condition

$$\Omega^2 \leqslant \frac{4}{q}\,\eta^3 \exp(-2\eta q). \tag{22}$$

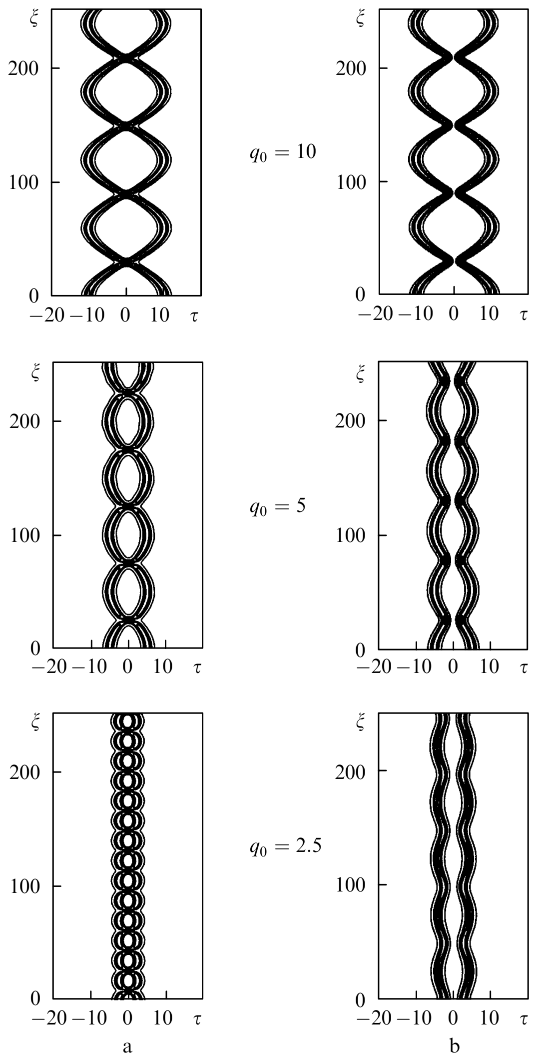

Figure 6. Dependence of the oscillation period for in-phase (a) and out-of-phase (b) NSE solitons in a harmonic potential on the distance between them. The interaction dynamics of in-phase NSE solitons is calculated within the framework of model (2) for $\Omega = 0.05$ and $R = 1.0$. Scenarios of the interaction between solitons with decreasing the initial distance between them by half ($q_0 = 10$, 5, 2.5) are presented from top to bottom.

Because in the canonical case (without potential), the oscillation period of the soliton pair with parameters $q = q_0$ and $\eta = 1$ is determined by the relation

$$T_{\rm sol} = \frac{\pi}{2}\exp(q_0) \tag{23}$$

inequality (22) relates the main parameters of the system

$$\frac{T_{\rm sol}}{T_0} \geqslant 2\sqrt{q_0}\,, \tag{24}$$

where T_0 is the oscillation period of a harmonic oscillator. The dynamics of out-of-phase solitons (which are repelled in the absence of the trap potential) is shown in Fig. 6b.

The interaction forces between two solitons are exactly compensated if

$$\Omega_0^2 = -\frac{4}{q}\,\eta^3 \exp(-2\eta q)\cos\varphi. \tag{25}$$

This gives, in particular, the condition for formation of a stable stationary state for out-of-phase solitons (Fig. 7) with parameters $q = q_0$ and $\eta = 1$,

$$\frac{T_{\rm sol}}{T_0} = 2\sqrt{q_0}\,, \tag{26}$$

and the critical frequency of the harmonic potential

$$\Omega_0^2 = \frac{4}{q}\,\eta^3 \exp(-2\eta q). \tag{27}$$

3. Nonstationary potential. Parametric resonance for solitons in a harmonic potential

The assumed adiabaticity of the NSE soliton dynamics in a parabolic trap allows one to consider more complicated processes appearing in nonstationary harmonic potentials, when the parameter Ω depends on time. By using (7)–(17), we can easily obtain the equation for the coordinate of the centre of gravity of a soliton:

$$\frac{{\rm d}^2 q}{{\rm d}\xi^2} + \Omega^2(\xi) q = 0. \tag{28}$$

It is well known that Eqn (28) appears in the theory of unclosed oscillatory systems in which the external action is reduced to temporal variations in the parameters of the system [64]. A simple example of such a system is a mathematical pendulum with the point of support experiencing a specified periodic motion in the vertical direction. When the function Ω is periodic, the so-called parametric resonance can appear in the system described by Eqn (28). This means that the state of rest of the oscillatory system in the equilibrium position becomes unstable – an arbitrarily small deviation from this state rapidly increases with time. The conditions of the appearance of the parametric resonance, when the function

$$\Omega^2(\xi) = \omega_0^2(1 + h\cos\gamma\xi) \tag{29}$$

weakly differs from the constant ω_0^2, were studied in detail, for example, in [64]. It was shown that the parametric resonance is most intense when the perturbation frequency is close to the double frequency: $\gamma = 2\omega_0 + \varepsilon$.

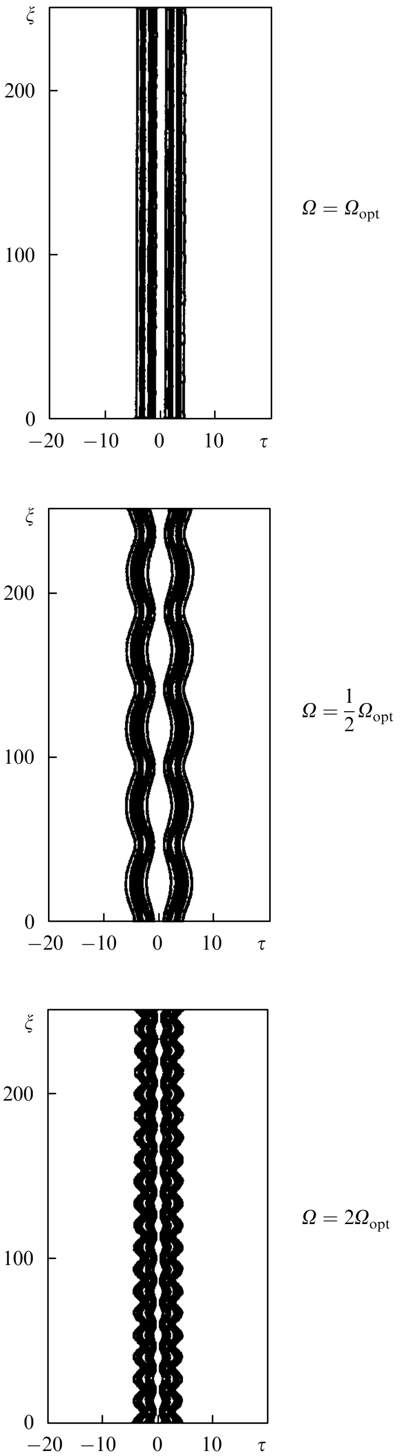

Figure 7. Formation of the stationary state of the noninteracting pair of out-of-phase solitons in a harmonic trap potential ($\Omega = \Omega_{\rm opt}$). The initial distance between solitons is $q = 3.5$, the rest of the parameters in the program were specified from the condition of compensation of forces according to expression (27). When the condition of optimum $\Omega = \Omega_{\rm opt}$ is violated, the oscillating bound states are formed, which are shown for $\Omega = \frac{1}{2}\Omega_{\rm opt}$ and $\Omega = 2\Omega_{\rm opt}$.

The solution of the Mathieu equation of motion

$$\frac{\partial^2 q}{\partial \xi^2} + \omega_0^2[1 + h\cos(2\omega_0 + \varepsilon)\xi]q = 0 \qquad (30)$$

gives the conditions for the appearance of the parametric resonance in the frequency interval $-\frac{1}{2}h\omega_0 < \varepsilon < \frac{1}{2}h\omega_0$ with the parameter $s^2 = \frac{1}{4}[(h\omega_0/2)^2 - \varepsilon^2]$ of exponential amplification of oscillations (we follow here paper [64]). It is known that the parametric resonance also takes place at the frequencies $2\omega_0/n$, where n is an integer. However, the widths of resonance instability regions rapidly decrease with increasing n. The amplification parameter s also decreases [64].

Therefore, Eqn (28) and the mathematical analogy with the parametric resonance suggest the possibility of excitation of parametric resonances also in the NSE model with a nonstationary harmonic potential (29). To verify this conclusion, we performed numerical calculations in a broad range of variations of parameters within the framework of the NSE model with potential (29). Typical results of these calculations are presented in Fig. 8.

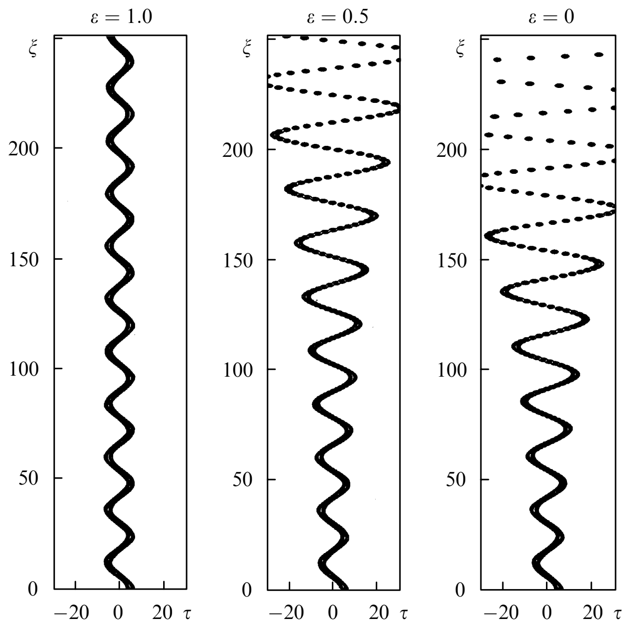

Figure 8. Parametric instability of a soliton in the time-dependent harmonic potential (30). The soliton dynamics is shown for the parameter of detuning from resonance $\varepsilon = 1.0$, 0.5, and 0.

4. Nonstationary potential. Denaturation of bound soliton states

It is well known that Eqn (28) with the time-dependent potential of the form

$$\Omega^2(\xi) = a\exp(\lambda\xi) \qquad (31)$$

has a set of exact analytic solutions expressed in terms of the Bessel functions of the first $[J_\nu(z)]$ and second $[Y_\nu(z)]$ kinds

$$q(\xi) = C_1 J_0\left[\frac{2\sqrt{a}}{\lambda}\exp\left(\frac{\lambda\xi}{2}\right)\right] + C_2 Y_0\left[\frac{2\sqrt{a}}{\lambda}\exp\left(\frac{\lambda\xi}{2}\right)\right].$$

The function of type (31) can be used to simulate the switching on and off of the harmonic potential of a trap depending on the sign of the parameter λ. Typical results of numerical calculations of the soliton dynamics in this case are presented in Fig. 9. As expected, the period of oscillations of the NSE soliton in a harmonic trap decreases with increasing the effective frequency of the oscillator in time and, on the contrary, the amplitude and period of soliton oscillations increase when the potential is switched off.

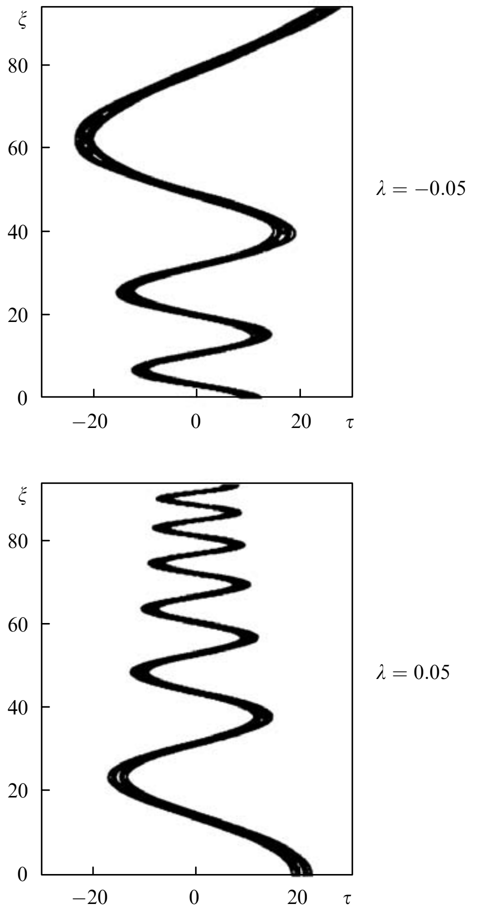

Figure 9. Soliton dynamics upon the adiabatic 'switching off' and 'switching on' of the harmonic potential according the exponential law $\Omega^2(\xi) = a\exp(\lambda\xi)$ for $\Omega = 0.1$, $R = 1.0$, $\lambda = -0.05$ and 0.05.

Let us show that the so-called Miura transformation in the soliton theory (see, for example, [28])

$$\mu(\xi) = \frac{\partial v(\xi)}{\partial \xi} - v^2(\xi) = \Omega^2(\xi) \tag{32}$$

allows one to obtain simple analytic solutions for Eqn (28) in the form

$$q(\xi) = \exp\left[-\int_0^{\xi} v(z)\mathrm{d}z\right], \tag{33}$$

where an arbitrary control function $v(\xi)$ should satisfy the conditions of integrability and differentiability. Indeed, it is easy verify that the substitution of (33) into (28) gives the identity under the condition (32).

Exact solutions (32), (33) allow us to propose the method to control the dynamics of breathers – the bound states of NSE solitons in the time-dependent harmonic potential of the trap (32). Let us emphasise that the parameters of the system should satisfy the condition reverse to inequality (22).

The adiabatic variation of the trap potential with time permits the realisation of the effect of reversible and irreversible denaturation of solitons, in which the period of NSE soliton oscillations changes in a controllable way in the nonstationary potential up to the complete decomposition of the bound state (Fig. 10). We used in calculations the functions $\Omega^2(\xi)$ in the form $\exp(-\beta\xi)$, $1-\exp[-\beta(\xi-\xi_0)^n]$, and $\tanh^2(\xi-\xi_0)$. They simulated the switching on and off of the harmonic trap and also the time-limited action on the attraction potential. We will consider the possible application of the effect of soliton denaturation for the construction of the simplest nonlinear model of the DNA denaturation in the next paper. Similarly to the destruction of the secondary and tertiary structures of protein upon its denaturation with the preservation of the primary structure, the process that we investigated preserves the primary properties of the model – soliton properties.

Note in conclusion that in practically interesting cases, as a rule, different potentials can be expanded near the minima of the potential energy in a series corresponding to the harmonic approximation, so that the dynamics of solitons near the minimum of the potential energy will obey the laws considered above.

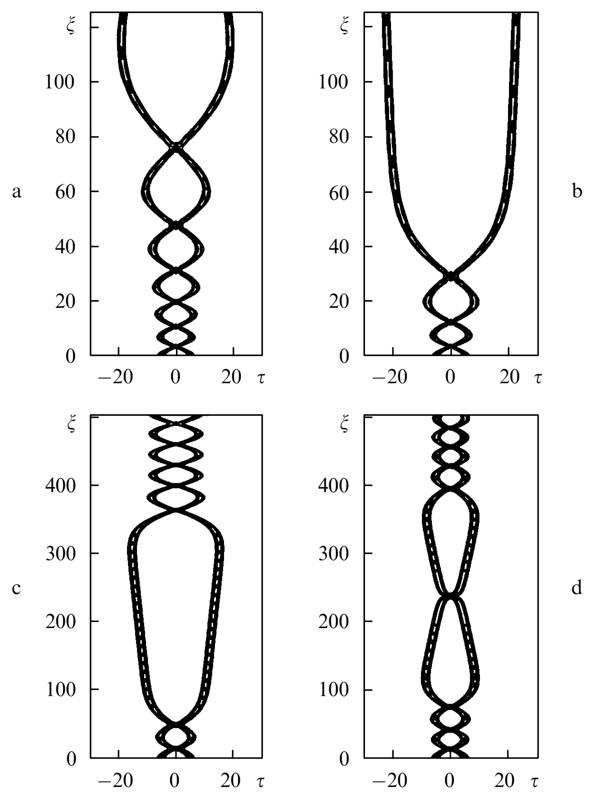

Figure 10. Effect of soliton denaturation: (a, b) the decay dynamics of the bound state of two in-phase NSE solitons in the harmonic potential calculated for $\Omega = 0.5$, $R = 1.0$, $q = 5$, $\beta = -0.5$ (a) and -0.1 (b); (c, d) the dynamics of reversible denaturation of a soliton pair in the case of temporal switching off of the harmonic potential for $\Omega = 0.1$ and $R = 1.0$.

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