SOLITONS

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To the memory of Prof. A.N. Oraevsky

Dynamics of solitons in the model of nonlinear Schrödinger equation with an external harmonic potential: II. Dark solitons

C. Hernandez Tenorio, E. Villagran Vargas, V.N. Serkin,
M. Agüero Granados, T.L. Belyaeva, R. Peña Moreno, L. Morales Lara

***Abstract*. The dynamics of dark solitons is studied within the framework of the mathematical model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. A comparative analysis of the solutions of nonstationary problems is performed for a linear harmonic oscillator and the NSE model with a harmonic potential for different signs of the self-action potential. It is shown that the main specific feature of the dynamics of dark NSE solitons in a parabolic trap is the formation of solitons with dynamically changing form factors producing the periodic variation in the modulation depth (the degree of 'blackness') of dark solitons. The oscillation period of the dark soliton does not coincide with the oscillation period of a linear quantum-mechanical oscillator, which is caused by the periodic transformation of the black soliton to the grey one and vice versa. The conditions of applicability of the method of inverse scattering problem are presented, the generalised Lax pair is found, and exact soliton solutions are given for the mathematical NSE model with an external harmonic potential.**

Keywords: *solitons, nonlinear Schrödinger and Gross – Pitaevsky equations, Bose – Einstein condensate.*

1. Dark optical solitons in optical fibres and dark soliton waves of matter in a Bose – Einstein condensate

One of the most spectacular examples of using the method of mathematical analogies in modern science, which allows one to analyse various physical processes based on a unified approach, is the problem on the dynamics of a nonlinear solitary wave described by the model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. This model makes it possible to analyse a variety of nonlinear phenomena, for example, in nonlinear fibre optics, laser physics, biophysics, and the theory of Bose – Einstein condensation of atoms in a magnetic trap with profile described by a quadratic function of coordinates.

Numerous experiments have shown that the nonlinear dynamics of a Bose – Einstein condensate (BEC) in magnetic traps is well described by the mathematical model of the average Gross – Pitaevsky field for the wave function of the condensate [1 – 4]. Analysis of the dynamic analogy between the BEC of photons, the atomic condensate and the condensate of Cooper pairs in a semiconductor performed by Oraevsky [5 – 10] have shown that an ensemble of a sufficiently large number of particles in the BEC behave as a classical field having the amplitude and phase. The condensate dynamics can be considered as an essentially nonlinear process, which is completely similar to the formation of the BEC of photons in a laser (see, for example, review [6] and references therein).

Indeed, as has been pointed out in pioneering experimental papers on the generation of solitons in a BEC [11 – 14], there exists a deep mathematical analogy between the theory of soliton waves of matter and the theory of optical solitons in optical fibres (see, for example, papers [11 – 17] and monographs [18 – 20] and references therein). Note that nonlinear collective excitations were first discovered in a BEC in the so-called cigar-shaped traps with the transverse size much smaller than the longitudinal size [11 – 14]. In this case, the Gross – Pitaevsky equation is substantially simplified [1 – 4]:

$$\mathrm{i}\frac{\partial u}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}+R|u|^2u-\frac{1}{2}\Omega^2\tau^2u=0. \qquad (1)$$

Depending on the sign of the nonlinearity parameter R (corresponding to different physical scenarios of the nonlinear interaction of atoms: attraction or repulsion), either bright or dark soliton waves of matter are formed [11 – 17]. The experimental discovery of the soliton waves of matter in a BEC [11 – 14] (dark for $R < 0$ and bright for $R > 0$) was stimulated to a great degree by the fact that model (1) in the limiting case $\Omega = 0$ transforms to the well-studied NSE describing, in particular, optical solitons of the envelope of an electromagnetic field [18 – 26] predicted by Hasegawa and Tappert [23] long before the development of low-loss fibres, and also bright solitons discovered in [24] and dark solitons discovered in papers [25 – 26].

The aim of this paper is to study the dynamics of dark solitons within the framework of the mathematical NSE model with an external harmonic potential. The features of the formation and interaction dynamics of bright solitons in a harmonic trap potential were considered in the first part of our paper [27].

Let us write first of all the exact solutions of Eqn (1) for dark solitons. For $\Omega = 0$ and $R = 1$, Eqn (1) can be written in the form of the well-known NSE for the complex

C. Hernandez Tenorio, E. Villagran Vargas, V.N. Serkin, M. Agüero Granados, T.L. Belyaeva, R. Peña Moreno, L. Morales Lara Benemérita Universidad Autonoma de Puebla, Puebla, 72001 A.P. 502, Mexico; e-mail: vserkin@yahoo.com

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conjugate function u^*, which is convenient for analysing solitons in a BEC,

$$\mathrm{i}\frac{\partial u^*}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u^*}{\partial \tau^2} - |u^*|^2 u^* = 0. \qquad (2)$$

Following pioneering work [23], we represent solution of (2) in the form

$$u(\xi,\tau) = \rho^{1/2}(\tau)\exp[\mathrm{i}\sigma(\xi,\tau)], \qquad (3)$$

where the real (amplitude) and complex (phase) parts are described by the expressions

$$\rho(\tau) = \rho_0[1 - a^2\mathrm{sech}^2(\rho_0^{1/2}a\tau)], \qquad (4)$$

$$\sigma(\xi,\tau) = \rho_0\sqrt{1-a^2}\tau + \arctan\left[\frac{a}{\sqrt{1-a^2}}\tanh\left(\rho_0^{1/2}a\tau\right)\right] + \frac{\rho_0}{2}(3-a^2)\xi. \qquad (5)$$

Unlike a bright soliton

$$u(\xi,\tau) = \eta\,\mathrm{sech}[\eta(\tau - q_0 + \delta\xi)]\exp\left[\mathrm{i}\left(\eta^2 - \delta^2\right)\xi/2 - \mathrm{i}\delta\tau\right], \qquad (6)$$

a dark soliton (4), (5) has the additional parameter a, which determines the modulation depth [the hole depth in the intensity (4)] and phase (5) of the soliton. An important feature of the dark soliton is the time dependence of its phase (5) – the phase modulation. When $a = 1$, the dark soliton becomes black,

$$u(\xi,\tau) = \rho_0^{1/2}\tanh\left(\rho_0^{1/2}\tau\right)\exp(\mathrm{i}\rho_0\xi) \qquad (7)$$

with the phase shift by π at the central part of the pulse. Because function (7) is two-valued by definition (taking the root), $u(\xi,\tau) = \pm\eta\tanh(\eta\tau)\exp(\mathrm{i}\eta^2\xi)$, it has two singularities of opposite signs (jumps at $\pm\eta$) at the boundary conditions $\tau \to \pm\infty$ and describes the so-called topological soliton and antisoliton with topological charges $+1$ and -1. Solitons with identical topological charges are repulsed and with opposite charges are attracted.

Note that the concept of a topological soliton has become now generally accepted and combines the whole family of solitons discovered in various fields of physics, for example, dislocations in crystals, kinks in the field theory, fluxons in Josephson junctions, waves in ferromagnetics, 2π-pulses in quantum electronics and nonlinear optics, dark solitons in optical fibres, and dark solitons of matter in a BEC.

It is interesting to follow the history of the appearance of the concept of a topological soliton. Skyrme [28, 29] was the first to attempt in 1959 – 1962 to construct the integrated nonlinear model of the field theory

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + m^2\sin\Phi = 0. \qquad (8)$$

The 'jargon' name of this model – the sine-Gordon equation, was yet absent then. It appeared later in the paper of Rubinstein [30], who nevertheless pointed out in the footnote on the second page that this name of the equation was given by Martin Kruskal. However, the first three references in [30] are the Skyrme papers. Skyrme performed in fact one of the first numerical experiments in the field theory and observed the elastic interaction of particles, for which he managed to find absolutely correct analytic expressions, which are now known as kinks and breathers. The term soliton appeared only four years later, but Skyrme observed the mutual repulsion between kinks and the mutual attraction between kinks and antikinks in his numerical experiments on the Harwell Mercury Computer already in 1958 – 1962,

$$\Phi(x,t) = 4\arctan\exp\left[\pm\frac{m}{\sqrt{1-v^2}}(x - vt)\right], \qquad (9)$$

and assigned to kinks and antikinks the topological charges $+1$ and -1, respectively, by defining them by the normalisation conditions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial\Phi(x,t)}{\partial x}\mathrm{d}x = \frac{1}{2\pi}[\Phi(\infty,t) - \Phi(-\infty,t)] = N_i. \quad (10)$$

The concept of a topological charge was introduced first of all because of the nature of forces between solitons, as we say now [31], observed in the numerical experiments of Skyrme. The second important circumstance is the different asymptotics of solutions at $\pm\infty$. A kink (this term was introduced by Finkelstein in 1966 [32]) changes from 0 to 2π and has a discontinuity in the boundary conditions. In addition, it is difficult to introduce the concept of vacuum in model (8). Indeed, the Hamiltonian of system (8)

$$H = \frac{1}{2}\left(\Phi_t^2 + \Phi_x^2 + 4m^2\sin^2\frac{1}{2}\Phi\right) \qquad (11)$$

has a minimum in the vacuum states $\Phi = 2\pi n$. Therefore, the concept of numerous vacuums appears, which are defined by integers $\Phi = 2\pi n$, and topological charges are in fact the 'jumps' [normalised by condition (10)] between different vacuum states of the field.

A similar situation also takes place for dark NSE solitons (7), where the topological charges of the soliton and antisoliton determine the jumps between the field states $\pm\eta$ for the boundary conditions at $\pm\infty$

$$\frac{1}{2\eta}\int_{-\infty}^{\infty}\frac{\partial u(\tau,\xi)}{\partial\tau}\mathrm{d}\tau = \frac{1}{2\eta}[u(\infty,\xi) - u(-\infty,\xi)] = N_i. \quad (12)$$

Therefore, solution (7) is written in the form $u(\xi,\tau) = N_i\eta\tanh(\eta\tau)\exp(\mathrm{i}\eta^2\xi)$, where $N_i = \pm1$ is the topological charge.

The theoretical idealisation requiring the presence of infinite fronts of a constant intensity for a dark soliton gives rise to competing nonlinear effects in real experiments with dark optical solitons. The main effects are SBS (for dark solitons) and SRS (for dark and bright solitons). Because of this, the so-called base-pulse method has received wide acceptance in the experimental studies of optical solitons. This method uses a long additional base pulse on which a phase jump required for the generation of a dark soliton is produced.

The typical picture of the generation dynamics of a black soliton in this case is presented in Fig. 1. Note that the use of a base pulse gives rise to the interesting effect of a soliton 'rolling down' from the base (Fig. 2). This effect still remains not studied; however, as shown below, it plays a substantial role in analysis of the interaction dynamics of dark solitons in a BEC. Because the influence of boundaries of the base pulse cannot be accurately described by the method of inverse scattering problem [33–37], we consider the results of direct numerical simulation.

Consider the process of formation of a dark soliton on a base pulse in more detail. A finite width of the base pulse leads, first, to a substantial broadening of the dark soliton (this fact was revealed already in earlier numerical experiments [38]) and, second, the combined action of dispersion and nonlinearity causes the flattening of the top of the base pulse, so that analytic expressions (3)–(5) describing the hole in the intensity against the background of an infinitely wide constant base pulse more and more correspond to the exact solution, while the fronts of the base pulse no longer affect substantially the dynamics of the dark soliton (see Figs 1 and 2).

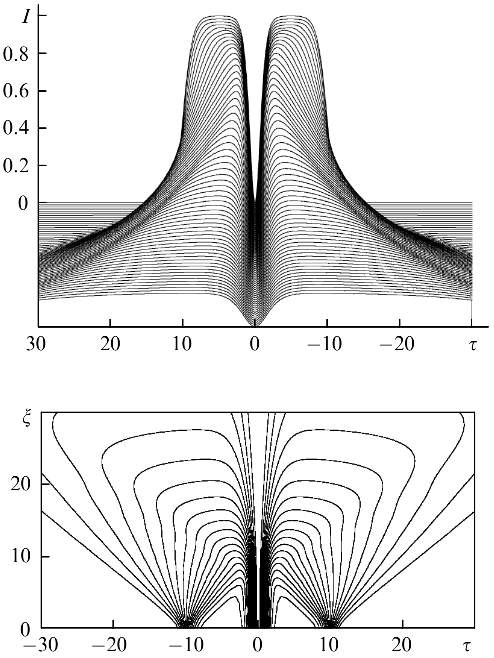

Figure 1. Dynamics of a dark (black) soliton formed on a super-Gaussian base pulse. The presented projection is turned by 180° (the pulse propagated to the reader). The contour map of equiscalar lines is represented in the standard projection.

When the centre of the black pulse is shifted with respect to the centre of gravity of the base pulse, a new effect appears – the soliton pulse is accelerated and, what is the most interesting, transforms to a grey soliton because its modulation depth changes (Fig. 2). Indeed, according to expression (5), a change in the dark-soliton velocity is uniquely related to a change in its modulation depth, because according to exact topological solutions (4), (5) for topological solitons, only the grey soliton moves in the coordinate system connected with the base, while the dark soliton, according to the exact solution (7), is at rest in the coordinate system connected with the group velocity of the

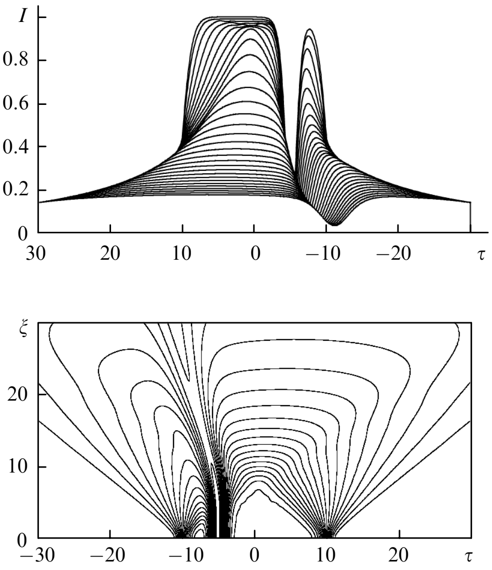

Figure 2. Effect of the base-pulse boundaries on the dynamics of a black soliton resulting in its rolling off from the base pulse and transformation to a grey soliton.

pulse. Figure 2 demonstrates the transformation of the black soliton to the grey one in the corresponding projection.

This effect is important for the interpretation of the results of various experiments with optical solitons and solitons in a BEC. The 'sliding off' of two dark solitons from the edges of the base pulse shown in Fig. 3 can be erroneously treated in a particular experiment as the repulsion interaction between dark solitons. The general case of the formation and interaction of black and gey solitons is shown in Fig. 4.

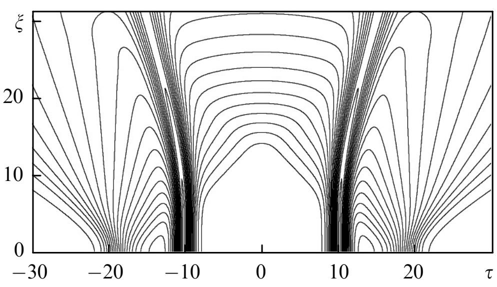

Figure 3. Effect of the base-pulse boundaries on the dynamics of two black solitons resulting in their rolling off from the super-Gaussian base pulse and transformation to grey solitons.

2. Basic properties of the dynamics of dark solitons in a harmonic potential

In the limit $R = 0$, Eqn (1) is transformed to the usual Schrödinger equation for a harmonic oscillator, whose general solution for the stationary states of the oscillator is determined by the Hermitean functions [39]

$$u_n(\xi, \tau) = \frac{\sqrt{\Omega}}{\sqrt{2^n n! \sqrt{\pi}}} \exp(-\mathrm{i}\lambda\xi) H_n(\tau) \exp\left(-\frac{\Omega\tau^2}{2}\right), \quad (13)$$

where

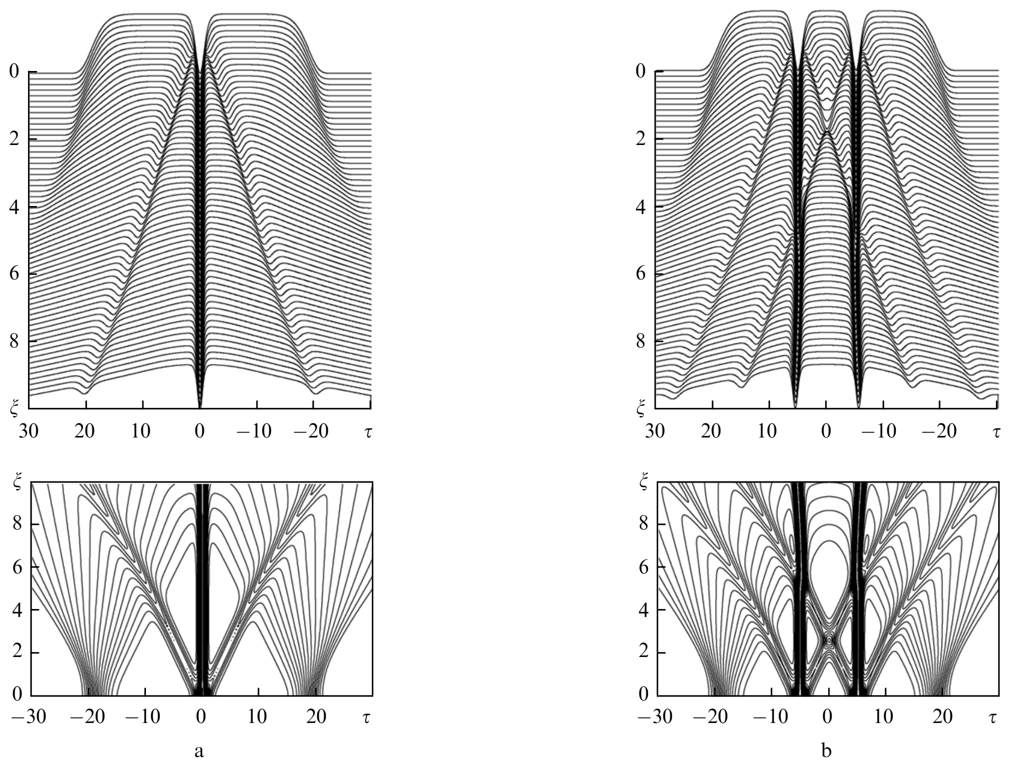

Figure 4. Formation of a black and two grey solitons (1) under the initial conditions $u = 2\tanh\tau$, $R = -1$, and $\Omega = 0$ (a) and the interaction dynamics of black and grey solitons for the same parameters of the problem (b).

$$H_n(\tau) = \exp(-\tau^2)(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\exp(-\tau^2); \quad \lambda = \Omega\left(n + \frac{1}{2}\right).$$

We are interested in the solution for $n = 1$

$$u_1(\tau) = 2\tau\frac{\sqrt{\Omega}}{\sqrt{2\sqrt{\pi}}}\exp\left(-\frac{\Omega\tau^2}{2}\right)\exp\left(-\mathrm{i}\frac{3}{2}\Omega\xi\right), \qquad (14)$$

which, as one can easily see, is similar to the form of the initial condition in the study of a dark (black) soliton on a Gaussian base pulse

$$u(\tau) = \eta\tanh(\eta\tau)\exp\bigl(-\beta\tau^2\bigr) \approx \tau\eta^2\exp\bigl(-\beta\tau^2\bigr). \qquad (15)$$

The parameters η and β in (15) determine the duration of the dark soliton (the phase jump duration) and the width of the Gaussian base pulse, respectively. State (14), which is initially shifted from the equilibrium position, oscillates in the parabolic potential (Fig. 5) similarly to the oscillations of the bright soliton [27], while the interaction between two spatially separated wave functions (14) is totally elastic (Fig. 6), as in the case $n = 0$ [27].

Our calculations show that the self-action effects do not prevent the formation of quasi-soliton states both for the positive and negative types of self-action (Fig. 7). The latter statement requires a more detailed explanation.

Numerical experiments allow one to study the oscillator dynamics in different approximations. In the absence of the trap potential, the dynamics of the self-action process depends on the nonlinearity sign. The initial condition (12) in a self-focusing medium is reduced to the appearance of two diverging soliton-like pulses, whereas in a self-defocusing medium these pulses are rapidly spreading (see Figs 7a, b). The switching on of a harmonic potential drastically changes the dynamics of the linear oscillator (Figs 7a, b). For $R < 0$ and $R > 0$, one can observe in

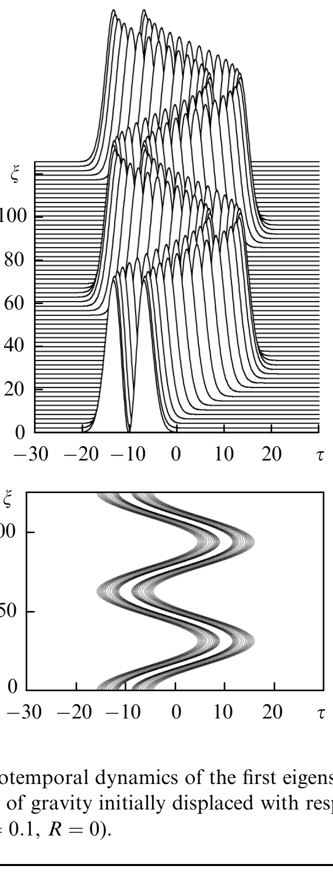

Figure 5. Spatiotemporal dynamics of the first eigenstate of an oscillator with the centre of gravity initially displaced with respect to the potential minimum ($\Omega = 0.1$, $R = 0$).

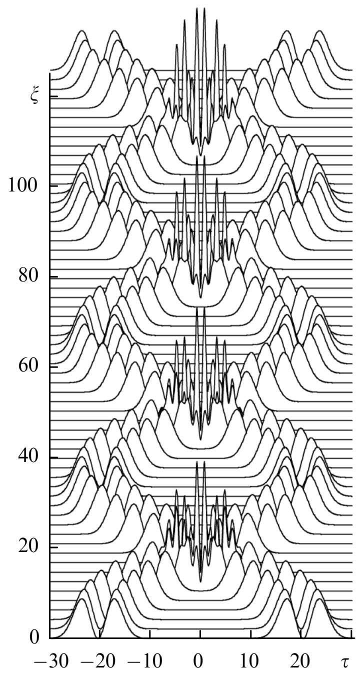

Figure 6. Interaction dynamics of two spatially separated oscillator states ($\Omega = 0.1,\ R = 0$).

numerical experiments the formation of quasi-soliton configurations of the field, which are characterised by an almost elastic interaction (Fig. 8). At the usual scale of the representation of graphic results, the pulse dynamics in this figure does not differ from the typical picture for real solitons. The inelastic type of the interaction is revealed at the logarithmic scale.

Therefore, numerical experiments performed in a broad range of variation of basic parameters of the problem reveal the following main properties of the dynamics of dark solitons in a harmonic potential:

(i) The oscillation period of dark (black) solitons depends on the specific experiment. If a dark soliton is formed on a base pulse oscillating in a harmonic trap, the oscillation period virtually coincides with the oscillation period of a linear oscillator, while the dynamics of the dark soliton is similar to that of a classical particle obeying the Newton mechanics laws (Fig. 9).

(ii) In the case of an immobile base pulse with the shape periodically oscillating due to self-action effects in a harmonic potential, the oscillation period of the black soliton considerably increases because of the periodic transformation of the black soliton to the grey one and vice versa. The soliton becomes black at the turning points in the harmonic potential and grey at the instant of its passage through the trap centre, where its velocity is maximal. In this case, its oscillation period is only approximately determined by relations obtained in previous papers [15–17, 20]. Typical pictures of the soliton evolution and changes in its modulation depth in this case are shown in Figs 10 and 11.

(iii) In the case of both positive and negative sign of nonlinearity, stable quasi-soliton configurations of the field can be formed, which correspond to the first stationary state of the harmonic oscillator and are characterised by virtually an elastic interaction (see Fig. 8).

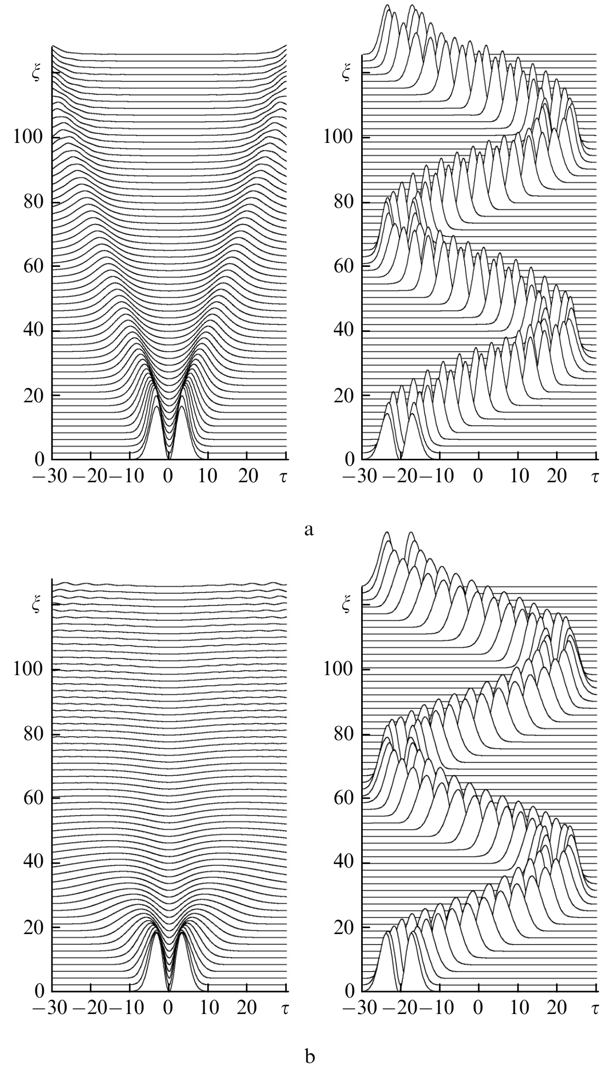

Figure 7. Formation of a quasi-stationary state in a nonlinear Schrödinger oscillator in the case of the positive (a) and negative (b) sign of the self-action potential. Both figures show at the left the soliton dynamics in the absence of a harmonic potential.

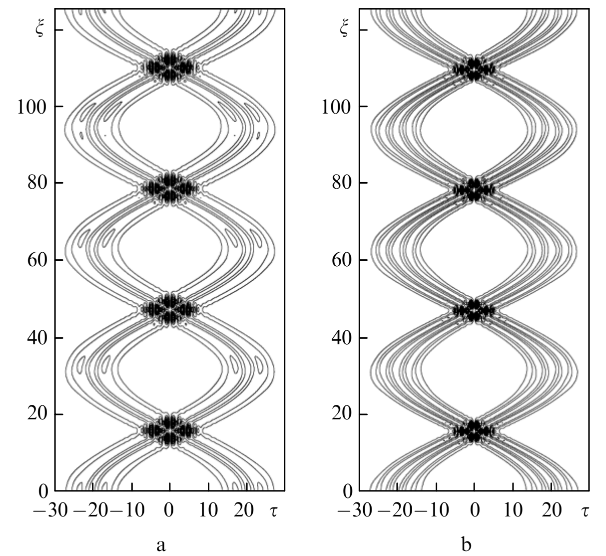

Figure 8. Interaction of quasi-soliton states for $\Omega = 0.1$ and different signs of the self-action potentials: $R = 1$ (a) and -1 (b).

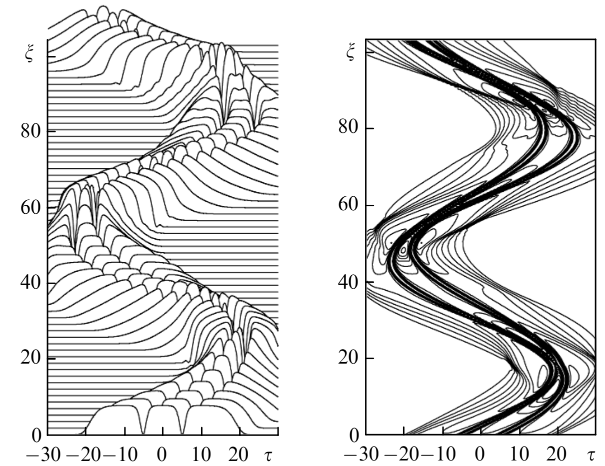

Figure 9. Dynamic of dark solitons in a parabolic trap (for a base pulse oscillating in the trap).

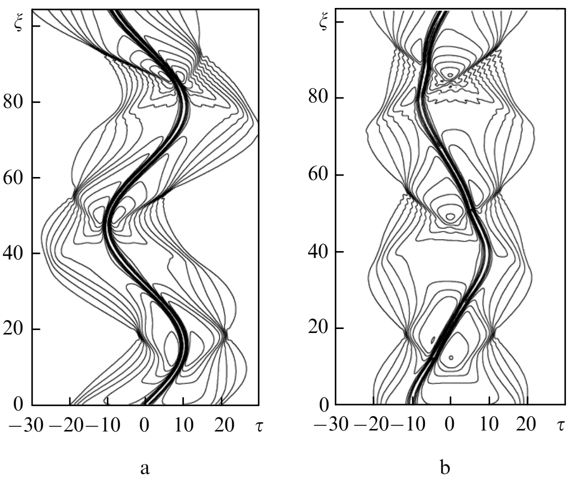

Figure 10. Change in the oscillation period of dark solitons in the parabolic potential of a trap depending on the experimental conditions [the base pulse oscillating in the trap (a) is compared with the base pulse at rest (b)].

3. Lax pair and exact analytic soliton solutions for the model of nonlinear Schrödinger equation with an external harmonic potential

One of the most significant achievements in the theory of nonlinear waves was the discovery of the exact method for integrating the Korteweg-de Vries equation in 1967 [33], which resulted in the development of the method for solving nonlinear evolution equations – the so-called method of the inverse scattering problem (ISP) [33–37]. It was proved that, if the so-called Lax representation is found [34], then this equation can be solved by the ISP method and the existence of the Lax pair proves a complete integrability of the model. The problem of the search for evolution equations possessing the property of complete integrability has a rather long history (see priority papers [33–37] and monographs [40–46]).

In the last years, methods are being extensively developed which generalise the inverse spectral problem of Zakharov–Shabat (ZS) [35] and Ablowitz–Kaup–Newell–Segur (AKNS) [37]. The aim of these studies is to

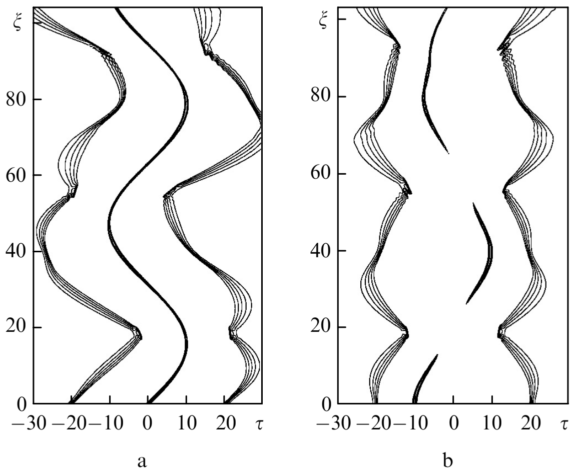

Figure 11. Effect of the periodic transformation of a black soliton to a grey one and vice versa. For the proof, contour lines do not achieve the maxima of pulses but begin with the value 0.05 and come to 0.25. One can see that in the case of the oscillating base pulse (a), the soliton remains black; in the case of the base pulse at rest (b), the sopped soliton becomes black, and breaks in the contour lines at maximum velocities show that the soliton became grey because its minimum intensity exceeds 0.25. The excess over this level makes it possible to observe the nontrivial dynamics of the periodic transformation of the black soliton to the grey one.

develop the mathematical algorithm for the search for new integrable NSE models and to elaborate new methods for controlling the soliton dynamics [47–57]. The generalisation of the Lax pair in the ISP method to the case of the time-dependent spectral parameter and the use of the concept of moving foci in the self-focusing theory resulted in a substantial progress in this direction.

Note that the idea of a variable spectral parameter is not new and was first used already in pioneering paper [58] in 1976 for solving by the ISP method the nonlinear Schrödinger equation with an external potential linear in the spatial coordinate. The use of the concept of moving foci in the ISP [49–50] provided a considerable progress in the search for new integrable NSE models. It is important to note that the apparatus developed in papers [47–57] makes it possible to use new methods to control the parameters of solitons at which, however, the NSE solitons accelerated or amplified in external potentials do not lose their individuality, by preserving their functional form and interacting elastically with each other [47–57].

In our study of the dynamics of solitons in a harmonic potential (summarised in the two parts of this paper), we used the methods of the adiabatic perturbation theory [27] and direct numerical experiment. However, these studies would be incomplete if we had not considered the possibility of constructing completely integrable models for NSE solitons in parabolic traps by the ISP methods proposed in [45–57].

The general algebraic formulation of the problem is as follows. First we will write the NSE with a varied harmonic potential as the compatibility condition

$$\hat{F}_{\xi} - \hat{G}_{\tau} + \left[\hat{F}, \hat{G}\right] = 0 \qquad (16)$$

for a pair of linear equations

$$\psi_\tau = \hat{F}\psi(\xi,\tau), \qquad \psi_\xi = \hat{G}\psi(\xi,\tau), \tag{17}$$

where the matrix elements F and G depend on the unknown variables of the evolution equation and determine the scattering potential $u(\xi,\tau)$.

Equation (16) is known as the generalised Lax pair, and the spectral ZS–AKNS problem is described by the system of matrix equations (17). By representing the matrices in the form

$$\hat{F} = -\mathrm{i}\Lambda(T)\hat{\sigma}_3 + \hat{L}\hat{\phi}, \tag{18}$$

$$\hat{G} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{19}$$

where

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{20}$$

$$\hat{\phi} = \begin{pmatrix} \exp[-\mathrm{i}\Theta(\xi,\tau)/2] & 0 \\ 0 & \exp[\mathrm{i}\Theta(\xi,\tau)/2] \end{pmatrix}; \tag{21}$$

$$\hat{L} = \sqrt{R(\xi)}\begin{pmatrix} 0 & u(\xi,\tau) \\ -u^*(\xi,\tau) & 0 \end{pmatrix}, \tag{22}$$

and using the concept of the moving focus [50]

$$\Theta(\xi,\tau) = P(\xi)\tau^2, \tag{23}$$

we, omitting the details of calculations, will write finally the elements of the matrix G (the AKNS elements in the mathematical literature) and the required nonlinear evolution equation in the form

$$A = \frac{\mathrm{i}}{2}R(\xi)|u|^2 - \mathrm{i}\Lambda\tau P(\xi) - \mathrm{i}\Lambda^2, \tag{24}$$

$$B = \sqrt{R(\xi)}\left[\frac{1}{2}\tau u R(\xi) + \frac{\mathrm{i}}{2}u_\tau + \Lambda u\right]\exp(\mathrm{i}\Theta/2), \tag{25}$$

$$\mathrm{i}\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} \pm R(\xi)|u|^2 u + \frac{1}{2}\left[R(\xi)\frac{\partial^2}{\partial \xi^2}\frac{1}{R(\xi)}\right]\tau^2 u = 0. \tag{26}$$

Thus, the NSE model with a harmonic potential (1) will be completely integrable then and only then when its basic parameters are interdependent. Therefore, the nonstationary harmonic potential is related to nonlinearity by the expression

$$\Omega^2(\xi) = R(\xi)\frac{\partial^2}{\partial \xi^2}\frac{1}{R(\xi)}, \tag{27}$$

where $R(\xi)$ is an arbitrary doubly differentiable function.

In conclusion, we will write exact soliton solutions for Eqn (26). Bright solitons are determined by the expressions

$$u(\xi,\tau) = 2f_0\sqrt{R(\xi)}\,\mathrm{sech}[\theta(\xi,\tau)]\exp\left[-\frac{\mathrm{i}}{2}P(\xi)\tau^2 - \mathrm{i}\chi(\xi,\tau)\right], \tag{28}$$

$$\theta(\xi,\tau) = 2f_0R(\xi)\tau + 4\upsilon_0 f_0\int_{\tau_0}^{\tau} R^2(z)\,\mathrm{d}z, \tag{29}$$

$$\chi(\xi,\tau) = 2\upsilon_0 R(\xi)\tau + 2(\upsilon_0^2 - f_0^2)\int_{\tau_0}^{\tau} R^2(z)\,\mathrm{d}z. \tag{30}$$

Dark solitons are represented by the expressions

$$u(\xi,\tau) = f_0\sqrt{\rho_0 R(\xi)(1 - a^2\mathrm{sech}^2\theta)} \times \exp\left[\frac{\mathrm{i}}{2}P(\xi)\tau^2 + \mathrm{i}\chi(\xi,\tau)\right], \tag{31}$$

$$\theta(\xi,\tau) = f_0 R(\xi)\tau a\sqrt{\rho_0}, \tag{32}$$

$$\chi(\xi,\tau) = f_0\tau\sqrt{\rho_0 R(\xi)(1 - a^2)} + \arctan\left\{\frac{a}{\sqrt{1-a^2}}\tanh\left[f_0\tau a R(\xi)\sqrt{\rho_0}\right]\right\} + \frac{1}{2}\rho_0(3 - a^2)f_0^2\int_{\tau_0}^{\tau} R^2(z)\,\mathrm{d}z. \tag{33}$$

By direct substitution, we can verify, for example, that solutions (28)–(33) transform (26) to identity; in addition, by integrating and differentiating the elementary functions R (algebraic, trigonometric, etc.), we can easily construct the infinite set of new NSE equations and their solutions.

Consider, for example, the solutions of Eqn (26) for

$$R = R_0\exp(\beta\xi),$$

when it is written in the form

$$\mathrm{i}\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + R_0\exp(\beta\xi)|u|^2 u + \frac{1}{2}\beta^2\tau^2 u = 0. \tag{34}$$

For bright solitons, we obtain

$$u(\xi,\tau) = 2\frac{f_0}{\sqrt{R_0}}\exp\left(\frac{\beta\xi}{2}\,\mathrm{sech}\left[2f_0\tau\exp(\beta\xi) + 4f_0\upsilon_0\frac{\exp(2\beta\xi)-1}{2\beta}\right]\exp\left\{-\mathrm{i}\frac{\beta\tau^2}{2} - \mathrm{i}\left[2\upsilon_0\tau\exp(\beta\xi) + 2(\upsilon_0^2 - f_0^2)\frac{\exp(2\beta\xi)-1}{2\beta}\right]\right\}. \tag{35}$$

The solutions for dark solitons are

$$u(\xi,\tau) = \frac{f_0}{\sqrt{R_0}} \times \exp\left(\frac{\beta\xi}{2}\right)\sqrt{\rho_0\left\{1 - a^2\mathrm{sech}^2\left[f_0\tau\exp(\beta\xi)a\sqrt{\rho_0}\right]\right\}} \times \exp\left[\mathrm{i}\frac{\beta\tau^2}{2} + \mathrm{i}\chi(\xi,\tau)\right], \tag{36}$$

$$\chi(\xi,\tau)=f_0\tau\exp(\beta\xi)\sqrt{\rho_0(1-a^2)}$$

$$+\arctan\left\{\frac{a}{\sqrt{(1-a^2)}}\tanh\left[f_0a\tau\exp(\beta\xi)\sqrt{\rho_0}\right]\right\}$$

$$+\frac{1}{2}\rho_0(3-a^2)f_0^2\left[\frac{\exp(2\beta\xi)-1}{2\beta}\right]. \quad (37)$$

These solutions are of interest because they describe the fallout of particles from the condensate for $\beta<0$ and the amplification of a soliton for $\beta>0$. To verify this, we make the change of variables

$$Q=\frac{u}{\sqrt{R_0}}\exp\left(-\frac{\beta\xi}{2}\right), \quad (38)$$

which transforms Eqn (34) to the equation with the additional complex term (amplification of absorption)

$$\mathrm{i}\frac{\partial Q}{\partial\xi}+\frac{1}{2}\frac{\partial^2 Q}{\partial\tau^2}+|Q|^2Q+\frac{1}{2}\beta^2\tau^2Q=\mathrm{i}\frac{\beta}{2}Q. \quad (39)$$

Therefore, the use of the concept of moving foci (23) in the AKNS algorithm of the inverse scattering problem (16)–(22) allows us to reveal a new class of nonlinear equations of the Schrödinger type with a harmonic potential completely integrable by the ISP method and to find their solutions (26)–(33).

Note in conclusion that the use of mutually complementary analytic methods – the adiabatic perturbation theory and the methods of the inverse scattering problem, as well as direct numerical experiments performed in a broad range of variation of the basic parameters of the problem allowed us to describe and explain the dynamics of solitons in the NSE model with a harmonic potential with the help of rather simple analogies and concepts of quantum electronics. The use of analogies often makes it possible to solve one of the most difficult problems in the construction of a physically meaningful theory that is not restricted only to the mathematical description of one or another phenomenon. This problem is the difficulty of going from the *description* of a phenomenon to its *explanation*. An excellent example of the refined mastery of the development of physically constructive ideas, *of the author's striving to propose simple and physically clear explanations based, in particular, on the well-known concepts of quantum electronics* was and remain the works of Prof. A.N. Oraevsky, to whose memory this paper is devoted. The interested reader can find information about the contribution of Prof. Oraevsky to the development of quantum electronics in the server of the American Institute of Physics [59].

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