*INTERACTION OF LASER RADIATION WITH MATTER. LASER PLASMA*

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# Evolution of tubular singular pulsed beams in a nonlinear dielectric medium upon ionisation

R.A. Vlasov, O.Kh. Khasanov, T.V. Smirnova

***Abstract*. The dynamics of a high-power femtosecond tubular pulsed beam in a dielectric medium is numerically analysed upon optically induced ionisation. It is found that the balance between nonlinearities of opposite sign and different magnitude in the case of multiphoton ionisation favours the establishment of a quasi-soliton regime of radiation propagation over a distance exceeding several diffraction lengths. The use of these beams enables attaining high-density light fields and generate high-density plasmas.**

***Keywords***: *femtosecond pulses, singular phase, topological charge, nonlinear medium, focusing and defocusing, multiphoton ionisation.*

## 1. Introduction

By now a new field of physical optics has come into being – singular optics [1 – 5], which is concerned with the study of generation and propagation of light beams whose wave fronts possess phase singularities (screw dislocations), forming optical vortices [3]. Structurally, these are tubular laser beams with a characteristic intensity distribution dip at the centre, which carry the orbital angular momentum (topological charge) [4]. Due to their unique properties, the singular beams are of special interest from the standpoint of practical applications. Of these applications, mention should be made of the possibility to use the specified beams as optical tweezers for manipulating micro- and nanoparticles as well as biological cells, including their cooling [6], and to produce and control Bose – Einstein condensates [7]. Investigations of the plasma produced in the field of high-power tubular beams showed that the obtained waveguide may be employed for electron acceleration [8] and high-order harmonic generation up to the X-ray region [9]. The inherence of orbital angular momenta in the photons and the possibility to produce their superposition states allow employing tubular optical beams in quantum computers and quantum teleportation [10].

**R.A. Vlasov** B.I. Stepanov Institute of Physics, National Academy of Sciences of Belarus, prosp. F Skoriny 70, 220072 Minsk, Belarus; e-mail: lvp@dragon.bas-net.by;
**O.Kh. Khasanov, T.V. Smirnova** Institute of Solid-State and Semiconductor Physics, National Academy of Sciences of Belarus, ul. P. Brovki 17, 220072 Minsk, Belarus; e-mail: khasanov@ifttp.bas-net.by

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The central problem in the theoretical and experimental investigation of beams with a singular phase is their stability in linear and nonlinear propagation regimes for a non-unit topological charge $m$. Until now this problem has been considered mainly within the framework of stationary or quasistationary interaction with media. In particular, optical vortices were shown to be unstable in the media with a self-focusing nonlinearity [11]. Furthermore, in Ref. [12] it was experimentally demonstrated that vortex beams with a topological charge $m$ split into $2m$ filaments under the conditions of absorption saturation in sodium vapour. Note for comparison that vortex solitons can be stable in the media with a defocusing cubic nonlinearity [13].

The vortex soliton stability problems were discussed in detail in several papers [14 – 20]. An analysis of two- and three-dimensional models in the media with quadratic [14] and saturable [15] nonlinearities suggests that the vortex beams with a non-unit topological charge are highly unstable due to azimuth perturbations, which was later confirmed experimentally [16]. The search for nonlinear media in which the vortex solitons would be stable showed that a combined nonlinearity type, when, for instance, the cubic nonlinearity is focusing and the fifth-order nonlinearity is defocusing, may provide the conditions for the stable propagation of such solitons [17 – 18].

The 'stability windows' for two-dimensional spatiotemporal vortex beams were determined in Ref. [19]. It was shown that, when the dimensions and amount of energy of a vortex soliton are large enough, the instability develops so slowly that the soliton may be thought of as being practically stable. More recently, the stability conditions were analysed for sufficiently broad three-dimensional vortex solitons with a unit topological charge in the above media and a conclusion was drawn that they can be stable in the media with competing quadratic focusing and cubic defocusing nonlinearities [20].

It would appear natural that the inclusion of other defocusing factors, in particular the effect of electron plasma generated by multiphoton ionisation (MPI) in the field of the pulse, would also promote the stabilisation of the pulsed beams that carry a vortex with a non-unit topological charge. To the best of our knowledge the propagation stability problem for high-power pulsed vortex beams in this formulation is posed for the first time.

At the same time it was shown [21] that the balance between the Kerr self-focusing and the defocusing plasma effect can ensure the waveguide propagation regime for vortex-free pulsed beams in dielectric media, in fused silica in particular, through distances far exceeding the diffraction length. The conditions for this propagation depend on the ratio between the input beam power $P_{in}$ and the critical power $P_{cr}$ for self-focusing [22–24].

In Ref. [22] it was shown that the beam filamentation is determined by the quasidynamic equilibrium between self-focusing and MPI: for $3 < P_{in}/P_{cr} < 10$ the temporal beam profile splits into two peaks [22], while for $P_{in}/P_{cr} \leqslant 2$ for zero group velocity dispersion the wave packet possesses a temporal profile with one peak [23]. In this case, the pulse shortens in duration and may become as short as several femtoseconds even over one diffraction length. Theoretical and experimental investigations into the propagation of a focused femtosecond pulse showed that the dimension and shape of the tracks of optical breakdown in the field of the pulse correspond to the region where the electron density produced by the MPI and the electron avalanche amounts to $\sim 10^{20}$ cm$^{-3}$ [24], which is far below the optical damage threshold.

This work is concerned with a numerical investigation of the features of the propagation of tubular singular beams with different topological charges through a medium with a nonlinearity combining third- and fifth-order components as well as the component due to MPI. We found the conditions for the stable propagation of the beams under discussion. A comparative analysis is made of free-electron generation in the field of vortex and nonvortex pulsed beams. Considering that a substantial experience has been gained in the experimental production of optical vortices [5, 17], the results of numerical simulation may be helpful in planning and carrying out experiments involving pulsed singular beams.

## 2. Mathematical model

Consider a situation close to the experimental one [25]: a light pulse with a wavelength $\lambda = 800$ nm and a duration $\tau_p = 70-100$ fs passes through a fused quartz sample less than 2.4 mm in thickness. The input pulse intensity $I_0$ varies in the 2–20 TW cm$^{-2}$ range. For these incident radiation intensities it is expedient to investigate the saturation of nonlinearity and photoionisation of the medium. We analyse the propagation of an axially symmetric polarised light beam with a spiral spatiotemporal phase structure and an annular field intensity distribution over the beam section.

The pulse propagation through the medium is described by a system represented in the dimensionless form, which consists of the generalised nonlinear Schrödinger equation for the complex envelope of the electric field $\mathscr{E} = E(r', t', z')$ and the kinetic equation for free-electron density $\rho_e$ [26]:

$$\frac{\partial u}{\partial z} = \mathrm{i}\hat{T}^{-1}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)u - \mathrm{i}\frac{L_{df}}{L_{ds}}\frac{\partial^2 u}{\partial t^2} + \mathrm{i}\frac{L_{df}}{L_{nl}} \times\hat{T}(|u^2| - \kappa|u|^4)u - \mathrm{i}\frac{L_{df}}{L_{pl}}\left(1 - \frac{\mathrm{i}}{\omega\tau_c}\right)\rho u - \left(\frac{L_{df}}{L_{mp}}|u|^{2(s-1)} + \mathrm{i}\frac{m^2}{r^2}\right)u, \quad (1)$$

$$\frac{\partial \rho}{\partial t} = |u|^{2s} - \frac{\rho\tau_p}{\tau_r}. \quad (2)$$

Here, $u = E(r', t', z')/\bar{E}_0$ is the ratio between the field amplitude in the medium and the initial amplitude averaged over the beam aperture; $\omega$ is the optical frequency; $\rho = \rho_e/\rho_0$ is the electron density normalised to the initial density $\rho_0 = 10^9$ cm$^{-3}$; $L_{df} = kw_0^2/2$ is the diffraction length; $w_0$ is the initial pulse radius; $L_{ds} = \tau_p^2/\beta_2$ is the dispersion length; $\beta_2 = 361$ fs$^2$ cm$^{-1}$ is the group velocity dispersion; $L_{nl} = c/(\omega n_2 I_0)$ is the nonlinear length; $I_0 = n_0 c|E_0|^2/2\pi$ is the initial peak intensity; $n_0$ is the linear refractive index; $n_2 = 3 \times 10^{-16}$ cm$^2$ W$^{-1}$ is the nonlinear refractive index; $L_{pl} = 2\rho_0/(\sigma\omega\tau_c)$ is the plasma absorption length; $\sigma = 1.55 \times 10^{-18}$ cm$^2$ is the cross section for inverse bremsstrahlung; $\tau_c = 23.3$ fs is the electron–electron collision relaxation time; $L_{mp} = (\beta_s \times I_0^{s-1})^{-1}$ is the length of multiphoton absorption (MPA) with an absorption coefficient $\beta_s$; $s$ is the number of photons in the MPA process (for fused silica, $s = 6$); $\kappa = n_4\bar{I}_0/n_2$ is the saturation parameter related to the fifth-order nonlinearity; $n_4 = 1.25 \times 10^{-29}$ cm$^4$ W$^{-2}$ [27]; $m = \pm 1, \pm 2...$ is the topological beam charge; $\tau_r = 300$ fs is the electron recombination time resulting from electron–phonon interaction; $z = z'/L_{df}$ and $r = r'/w_0$ are the dimensionless longitudinal and transverse variables; and $t = (t' - z'/v_{gr})/\tau_p$ is the dimensionless time in the frame of reference moving with a velocity $v_{gr}$. All coefficients that enter the equations are real and are expressed in terms of the physical parameters of the problem.

In the system under study, the terms in Eqn (1) describe the mechanisms of diffraction, group velocity dispersion, cubic nonlinearity and fifth-order nonlinearity, plasma defocusing, and MPA for the dielectric. The operator

$$\hat{T} \equiv 1 + \mathrm{i}\frac{\partial}{\partial t'}\frac{1}{\omega\tau_p},$$

which enters Eqn (1) and characterises the measure of departure from the approximation of a slowly varying envelope, introduces a correction arising from the effects of spatiotemporal focusing and increase in envelope steepness.

As regards the kinetic equation for the free-electron density, the first term in the right-hand side of Eqn (2) describes MPI and the second accounts for electron recombination. Provided that the duration of the propagating pulse is far shorter than the electron–phonon relaxation time, the effect of recombination can be neglected in the subsequent discussion.

The boundary conditions are of the following form:

$$u(z=0) = u_0\frac{r^{|m|}}{m^2}\exp\left(-\frac{r^2}{2m^2} - \frac{t^2}{2}\right), \; \frac{\partial u}{\partial r}\bigg|_{r=0} = u\bigg|_{r=R} = 0. \quad (3)$$

Here, the parameter $u_0 = E_0/\bar{E}_0$; $E_0$ is the peak field amplitude at the input; $R$ is the transverse beam dimension, which is taken to be long enough to satisfy the condition $u|_{r>R} = 0$ for all $z \in [0, L]$, where $L$ is the dimensionless length of the sample.

We solved numerically the obtained system in the domain $D = [0, R] \times [0, L] \times [-T, T]$ (where $T$ is the characteristic time by introducing a mesh uniform in $t$ and nonuniform in $r$, $z$, which became finer in the region of strong gradients. On this mesh, the initial problem was approximated by a system of difference equations realised on the basis of the iterative alternating direction method [28].

The system of equations with three-diagonal matrices was solved by the sweep technique combined with iterations, the electron density equation was integrated by the Runge–Kutta method. In the region of strong spatial gradients, use was made of steps variable in the $z'$-coordinate.

Note that nonuniform approximative mesh construction in the investigation of pulse filamentation in the air was used to advantage in Ref. [29], which allowed the authors of that work to substantially optimise the simulation procedure in the attainment of high precision. The construction algorithm we employ implies the use of a step in the coordinate $r'$, which varies by the geometric progression law {the denominator $q = [(1+\varepsilon/N_r)^{N_r} - 1]/\varepsilon$, where $N_r$ is the number of steps, $\varepsilon \ll N_r$} with mesh refinement in the paraxial region. To provide the conservatism of the constructed difference scheme, the balance method was used.

## 3. Results of numerical analysis

Note that a stationary problem of the interaction between a radiation beam with a structure close to that under our study and the medium combining third- and fifth-order nonlinearities was considered in Ref. [30]. It was therein emphasised that the existence of saturation is one of the necessary conditions for the quasisoliton propagation regime for a beam with a spiral wave front. Another condition consists in the observation of the required dynamic equilibrium between the energy localised in the beam and its certain value depending on the topological charge $m$. Considered in the present work, unlike Ref. [30], is a fundamentally transient propagation regime of a vortex-carrying beam.

It is well known that conservation laws play an important part in the investigation of in-medium radiation waveguide propagation regimes. It is easy to show that Eqn (1) in the absence of absorption-related dissipative terms and the operator $\hat{T}$ is characterised by the integral of motion

$$I_1 = \iint |u|^2 r\mathrm{d}r\mathrm{d}t, \qquad (4)$$

which has the significance of the energy conservation law. An equation close to Eqn (1) (in the absence of nonlinearity saturation) was investigated in Ref. [21], which analysed the propagation of high-power femtosecond pulses of Gaussian shape in argon and air. Invoking the variational approach with the employment of the conservation law (4) and the Hamiltonian of the problem allowed the authors of Ref. [21] to draw a conclusion that MPI may prevent the collapse of the beam over long propagation lengths with a simultaneous distortion of its temporal profile. In this case they emphasise that, when the input pulse power exceeds $P_{\rm cr}$ by an order of magnitude, the effect of Raman response inertia facilitates the stable pulse propagation over distances far exceeding the diffraction length.

To compare our results with the available data, we estimated the input radiation parameters and the critical power, which permit revealing the role of the processes that essentially govern the evolution of the pulsed radiation. For the samples under study, the critical power $P_{\rm cr} = 0.159 \times \lambda_0^2/(n_0 n_2) \approx 2.2$ MW, the ratio $P_{\rm in}/P_{\rm cr} = 1.5 - 20$. The initial beam radius $w_0$ was varied in the 10–25 μm range. The diffraction length of the beam with $w_0 = 10$ μm is equal to $\sim 0.57$ mm, which is several times shorter than the sample length. For the initial intensities employed, the nonlinear length $L_{\rm nl} \ll L$. The dispersion length $L_{\rm ds}$ exceeds $L_{\rm df}$ by three order of magnitude for $\tau_{\rm p} = 100$ fs; the plasma length $L_{\rm pl} \approx 10^6$ m, and $L_{\rm mp} \approx 10^2$ m. The effect of the operator $\hat{T}$ can be neglected, because its action can manifest itself over lengths which exceed the lengths under consideration by an order of magnitude. It is noteworthy that the analysis of the problem reduces to the quasistationary case when the plasma density in the field of the travelling pulsed beam varies only slightly relative to the initial density. Fulfilment of the discrete analogue of the conservation law in the stationary case allowed us to estimate the accuracy of the computational scheme constructed, to make preliminary estimates, and compare the resultant data with the available data.

As to the saturation coefficient, its magnitude, which is related to the aperture-averaged initial intensity (see the definition of $\kappa$ given above), depends on the topological charge. For vortex beams the peak-to-averaged ratio is the greater, the higher is the topological charge, and vice versa. For intensities ranging into the terawatts, the saturation parameter $\kappa$ for $m = 2, 3$ varied in the $2.5 \times 10^{-4} - 5 \times 10^{-3}$ range.

Numerical experiment demonstrated the existence of radial compression and spreading stages in the pulse evolution in the medium. As the beam propagates, at the first stage there occurs beam compression, which is due to the dominating contribution of focusing cubic nonlinearity to the field intensity. Figure 1 shows the radial distribution of the peak intensity of the pulse with a topological charge $m = 2$ at the stage of compression, when the beam radius shrinks to 1/3 of its initial value and its intensity increases. As the intensity increases, there also increases the effect of defocusing related to the fifth-order nonlinearity. The spreading stage is primarily attributable to the effect of defocusing.

It was found that these structural changes of the pulsed beam exhibit a clearly defined periodicity as the beam propagates along the $z'$ axis (Fig. 2). The maximum amplitude increases and the pulse narrows down until

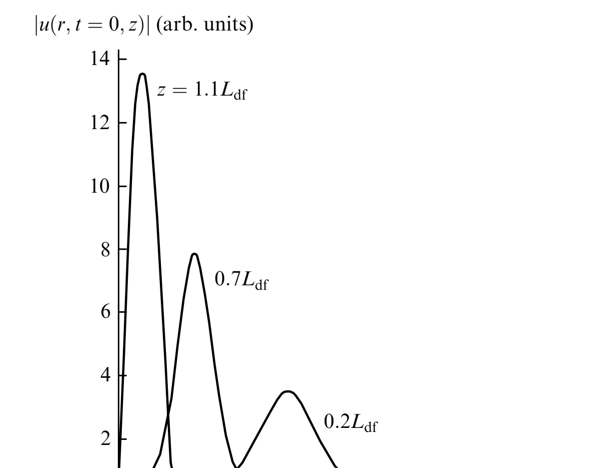


**Figure 1.** Radial amplitude distribution for a pulsed beam with a singular phase and a duration $\tau_{\rm p} = 100$ fs for $I_0 = 2$ TW cm$^{-2}$, $w_0 = 10$ μm, $\lambda = 800$ nm, and $m = 2$ shown at different propagation distances.

the effect of defocusing nonlinearities, which becomes stronger with an increase in the intensity, terminates its further increase. In this case, the beam aperture becomes smaller. In other words, a multiple self-focusing of the beam is observed. With increasing the saturation parameter (with an increase in $n_4$ or $\bar{I}_0$), the oscillation periods of the beam intensity and its aperture become shorter. When the generated plasma is not the dominant defocusing factor [estimates of the contributions of the nonlinearity-related terms in Eqn (1) show that this takes place for $P/P_{\rm cr} \simeq 1.5$] and exerts only an additional defocusing effect, it limits the growth of the intensity and shortens the period of its oscillations. The energy dissipation due to multiphoton and plasma absorption has the effect that the beam intensity lowers and the period of its oscillations increases as the beam propagates. We note that the effect of group delay dispersion is insignificant in the case under consideration, which is also confirmed by the numerical analysis.

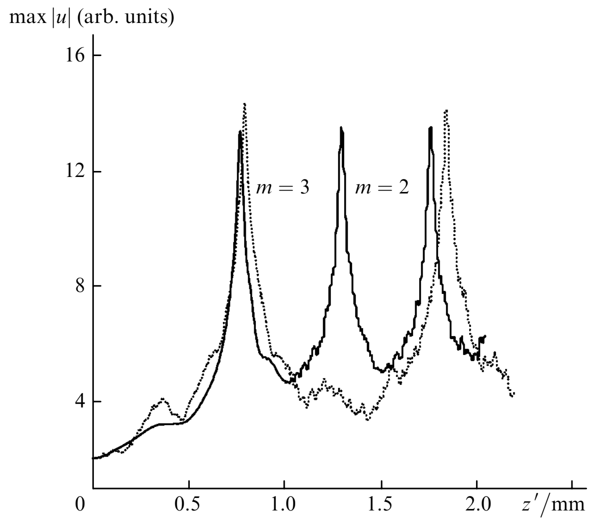


**Figure 2.** Space-periodic dependence of the peak amplitude value of the pulsed beam with $I_0 = 2.25$ TW cm$^{-2}$, $w_0 = 20$ μm, and $\tau_{\rm p} = 100$ fs in its propagation through the sample for $m = 2$ and 3.

When the plasma plays the dominant part in defocusing (the radiation power localised in the transmitted beam exceeds the incident radiation power by an order of magnitude), the periodicity exhibits features arising from the inertial character of plasma formation. One might expect that the periodical beam behaviour would also change. This is borne out by the periodic behaviour (in the absence of dissipation) of the radius of a vortex-free beam obtained in Ref. [31] on the basis of an analysis of conservation laws for a ratio $P_{\rm in}/P_{\rm cr} \leqslant 10$ as well as by the multi-focusing character of the focused femtosecond pulse with $m = 0$ in fused silica, which was experimentally observed in Ref. [32]. Our simulations ($m \neq 0$) showed that the periodicity is violated when the defocusing effects of plasma and fifth-order nonlinearity are comparable in magnitude (for $P_{\rm in}/P_{\rm cr} \sim 5$).

It turned out that the use of pulsed tubular beams enables a substantial increase in free-electron plasma density in comparison with the case of Gaussian pulses due to a rise in laser radiation intensity arising from the very structure of the singular beam. We investigated the dynamics of free-electron density distribution in the radiation propagation through the sample as a function of the topological charge.

Our numerical calculations showed that when the input intensity $I_0$ was varied in the 2–9 TW cm$^{-2}$ range, the free-electron density increased by a factor of 3–5 compared to the case of ordinary (nonsingular) Gaussian beams [33] (Fig. 3 and the inset). The density distribution is determined by the transverse beam profile, specifically by the increase in its steepness with an increase in the topological charge $m$. In this case, the dip in intensity distribution broadens at the centre of the tubular beam and its 'walls' become thinner. The higher is the topological charge, the smaller is the volume occupied by the field. This leads to an increase in the field density and the density of free electrons, other parameters being equal. Further density increase is possible with the use of beams with a higher topological charge (Fig. 4). In this case, the increase in plasma density quickens with increasing the input radiation intensity, the free-electron density peak is observed to shift towards shorter distances in the sample. In other words, in the field of high-power singular pulsed beams the plasma density may reach critical values. The use of such beams may therefore turn

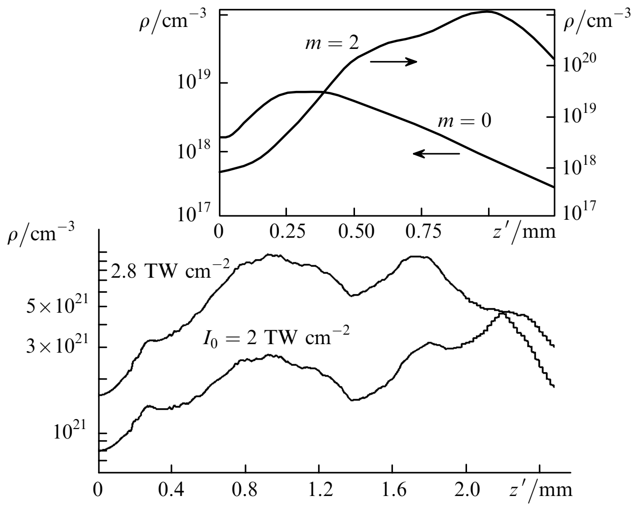


**Figure 3.** Time-averaged free-electron density distribution for singular beams with $\tau_{\rm p} = 100$ fs, $w_0 = 10$ μm, $m = 2$, $I_0 = 2$ and 2.8 TW cm$^{-2}$. The inset: the same for $I_0 = 1.2$ TW cm$^{-2}$, $m = 0$ and 2.

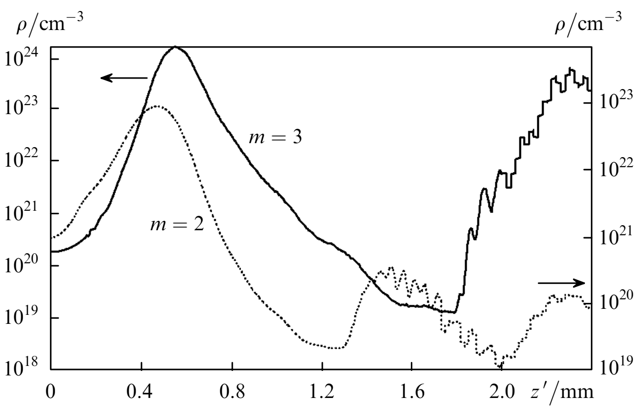


**Figure 4.** Same as in Fig. 3 for $\tau_{\rm p} = 100$ fs, $w_0 = 10$ μm, $I_0 = 1.3$ TW cm$^{-2}$, $m = 2$ and 3.

out to be preferable for the generation of higher-order harmonics, electrons, and X-ray radiation in the pulsed regime.

We emphasize that the fifth-order nonlinearity is nothing more than the first approximation in the inclusion of nonlinearity saturation in the field of a high-power pulse and its role in the stabilisation of a vortex soliton has not been studied. At the same time, the generated plasma can do more than play a significant defocusing part. The very plasma production process is actually responsible for a nonlocal character of the light – matter interaction, and this may be the main factor for the attainment of vortex beam stability condition, like other types of nonlocal nonlinearities: thermal nonlinearity [34], spatial dispersion [35].

## 4. Conclusions

For the first time an investigation was made of the evolution of a high-power pulsed femtosecond vortex beam with different values of the topological charge $m$ in a dielectric medium by the example of quartz glass with third- and fifth-order nonlinearities under optically induced ionisation conditions. For a model, we employed the system of equations consisting of the modified $(2+1)$-dimensional nonlinear Schrödinger equation and the kinetic equation for the free-electron density. To numerically analyse the problem under consideration, advantage was taken of an algorithm which has the virtue of coordinate-varied steps in the approximation of the initial problem on a mesh. The aim of numerical analysis was to elucidate the existence conditions for the quasisoliton propagation regime of pulsed beams with a singular phase. The balance between competing factors of opposite sign and different magnitude sets in with participation of the MPI-related nonlinear component. The main simulation results and conclusions can be formulated as follows.

When a vortex pulsed beam with a near-critical input power propagates through a nonlinear dielectric medium (quartz glass) for weak dissipation and dispersion effects, the competition between the third- and fifth-order nonlinearities has the effect that the beam structure behaves in a periodic manner along the propagation axis. In this case, the effect of plasma is not the controlling factor. With an increase in the input beam power, the defocusing factors related to the plasma and the fifth-order nonlinearity become comparable in magnitude and bring about a violation of the above periodicity.

The results of our numerical analysis testify to a quasisoliton character of vortex beam propagation with the prevailing effect of the fifth-order nonlinearity, because the peak radiation intensity at the output of the sample lowered by no more than 0.5 % – 3 %. In the case when the effects of defocusing factors are comparable, it is also valid to say that the propagation is quasisoliton in nature: the beam retains its shape and the value of the amplitude lowers only slightly (by less than 6 %) over the sample lengths under consideration, although the propagation characteristics are different.

The transient interaction of a singular pulsed beam with a nonlinear dielectric ionisable medium under study permits attaining higher light field densities and generating higher plasma densities in comparison with ordinary (nonsingular) pulsed beams. The plasma density increases with increasing the topological charge of the vortex beam.

An analysis of the nonlinear dynamics of a beam possessing an orbital angular momentum with the inclusion of the contribution of competing processes calls for a more detailed investigation. It is of interest to elucidate the role of group velocity dispersion (normal or anomalous) as well as of the increase in beam profile steepness and the spatiotemporal beam focusing. We intend to analyse these questions in the course of further investigations, which will be reported elsewhere.

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