PACS numbers: 42.65.Sf
DOI: 10.1070/QE2005v035n11ABEH006584

On the theory of signal filtration in a bistable moderately dissipating oscillatory system

A.N. Dombrovskii, S.A. Reshetnyak, V.A. Shcheglov

***Abstract*. The interaction of a weak periodic signal with noise is studied theoretically in bistable systems. Based on the generalised Smoluhowski equation, it is shown that, depending on the relation between the friction coefficient and the oscillation eigenfrequency, the solution of this equation has the aperiodic or oscillatory character. Within the framework of the linear response theory, the stochastic resonance effect is explained and the optimum value of the friction coefficient is found at which the output signal-to-noise ratio of a bistable system exceeds the input signal-to-noise ratio.**

Keywords: *stochastic resonance, bistable system, moderate dissipation.*

1. Introduction

Optical bistable systems, which are very convenient for studying the nonlinear interaction of a signal and noise, attract considerable recent attention [1 – 3]. In particular, a ring laser with an acousto-optic modulator [1] was the first experimental setup in which the anomalous increase in the signal-to-noise ratio caused by the stochastic resonance (SR) effect was found with increasing the noise intensity. The propagation of laser radiation either clockwise or counter-clockwise corresponds to two stable equilibrium positions in a bistable system. The interaction between the signal and noise was performed by varying determinately and randomly the parameters of an acousto-optic modulator. The review of papers on the SR effect is given in [4, 5].

However, as follows from theoretical studies [6 – 9], aside from the SR effect, another anomalous phenomenon can be observed in optically bistable systems, which lies in the fact that the output signal-to-noise ratio of a bistable system exceeds the signal-to-noise ratio at its input only due to energy transfer from the noise to signal. According to [10], we call this phenomenon the stochastic filtration effect. This paper continues the theoretical study performed in [9]. While in [9] the first-order system was analysed, here we study these effects in the second-order system, which has been studied mainly numerically so far [11 – 13].

Without referring to any specific optically bistable system, we consider the behaviour of an abstract physical quantity η obeying the system of Langevin equations

$$\frac{dv}{dt} + \gamma v + W'(\eta) = h\cos(\omega_s t) + \xi(t), \qquad (1)$$

$$v = \frac{d\eta}{dt}, \quad \langle\xi(t)\rangle = 0, \quad \langle\xi(t)\xi(t_1)\rangle = 2\varepsilon\delta(t - t_1),$$

where $W(\eta) = -\alpha\eta^2/2 + \beta\eta^4/4$ is the bistable potential and γ is the friction coefficient. One can see that (1) is the system of equations of nonlinear oscillations caused by the action of the signal $h\cos(\omega_s t)$ and the white noise $\xi(t)$ of the intensity ε at the input of the bistable system. The angle brackets in (1) denote averaging over time or an ensemble of realisations of the random force $\xi(t)$. The potential function $W(\eta)$ has two minima or stable equilibrium positions at the points $\pm\eta_0$, where $\eta_0 = (\alpha/\beta)^{1/2}$. Between the equilibrium positions the potential barrier of width $W_0 = \alpha^2/(4\beta)$ is located.

It is well known [14] that the behaviour of systems described by the second-order Langevin equations depends on the value of the friction (or dissipation) coefficient. In the limiting case of large friction coefficients, relaxation to the equilibrium state is aperiodic and described by the Smoluhowski equation. In the limiting case of small friction coefficients, relaxation has the oscillatory behaviour and is described by the energy diffusion equation. In this paper, we show that in the case of moderate dissipations, the description involving both aperiodic and oscillatory features can be used. Within the framework of such an approach and the linear response theory, we explained the effects of stochastic resonance and stochastic filtration and also found the optimal friction coefficients at which the conditions are improved for detecting weak signals against the background noise.

Before proceeding to the analysis of system (1), we find the conditions under which it transforms to the first-order equation for η studied earlier in [9]. By representing the potential near one of the equilibrium positions in the form

$$W(\eta) = -W_0 + \frac{1}{2}\omega_0^2(\eta - \eta_0)^2 + \ldots, \qquad (2)$$

A.N. Dombrovskii Moscow State Institute of Radio Engineering, Electronics, and Automatics (Technical University), prosp. Vernadskogo 78, 119454 Moscow, Russia; e-mail: dmbr@rambler.ru;
S.A. Reshetnyak Moscow State Institute of Radio Engineering, Electronics, and Automatics (Technical University), prosp. Vernadskogo 78, 119454 Moscow, Russia; P.N. Lebedev Physics Institute, Russian Academy of Sciences, Leninskii prosp. 53, 119991 Moscow, Russia;
V.A. Shcheglov P.N. Lebedev Physics Institute, Russian Academy of Sciences, Leninskii prosp. 53, 119991 Moscow, Russia

Received 18 March 2005; revision received 30 August 2005
Kvantovaya Elektronika **35** (11) 1033 – 1038 (2005)
Translated by M.N. Sapozhnikov

where $\omega_0 = (2\alpha)^{1/2}$ is the eigenfrequency of oscillations near the equilibrium position, we find the established in time response of the system to a signal in the absence of noise, which is described by the expression

$$\eta = \eta_0 + \frac{h\cos(\omega_s t - \varphi)}{\left[\left(\omega_0^2 - \omega_s^2\right)^2 + (\gamma\omega_s)^2\right]^{1/2}}, \qquad \tan\varphi = \frac{\gamma\omega_s}{\omega_0^2 - \omega_s^2}. \tag{3}$$

It can be easily verified that expression (3) for signal frequencies $\omega_s \ll \omega_0$ and eigenfrequencies $\omega_0 \ll \gamma$ coincides with the solution [9] of the first-order Langevin equation

$$\frac{d\eta}{dt} + u'(\eta) = A\cos(\omega_s t), \tag{4}$$

where $u(\eta) = W(\eta)/\gamma$; $A = h/\gamma$.

It follows from (4) that in he limit of low signal frequencies ω_s, the theory based on the system of equations (1) should contain the results obtained earlier [9]. However, the questions related to the behaviour of a bistable system in the frequency region $\omega_s \sim \omega_0$ and the dependence of the signal-to-noise ratio on the friction coefficient γ remain open. It is these questions that we considered in this paper. We have managed to obtain analytic results only for small amplitudes h of a signal.

2. Kinetic equation for small friction coefficients

Note that the system of equations (1) is universal and is used in many fields of physics. For $h = 0$, it forms the basis for the theory of motion of Brownian particles with the unit mass in the potential field $W(\eta)$. Let us pass from system (1) to the kinetic equation for the probability density distribution F [10]:

$$\frac{\partial F}{\partial t} + \upsilon\frac{\partial F}{\partial \eta} - W'\frac{\partial F}{\partial \upsilon} + h\cos(\omega_s t)\frac{\partial F}{\partial \upsilon} = \frac{\partial}{\partial \upsilon}\left(\gamma\upsilon F + \varepsilon\frac{\partial F}{\partial \upsilon}\right), \tag{5}$$

$$F|_{t=0} = \delta(\eta - \eta_*)\delta(\upsilon - \upsilon_*), \quad \int F d\eta\, d\upsilon = 1,$$

where η_* and υ_* are the initial values of η and its derivative, respectively. We also assume that a signal begins to act on the system at the instant $t = 0$.

By considering the equations for the first two moments following from Eqn (5),

$$\frac{\partial \mathscr{M}_0}{\partial t} + \frac{\partial \mathscr{M}_1}{\partial \eta} = 0,$$

$$\frac{\partial \mathscr{M}_1}{\partial t} + \frac{\partial \mathscr{M}_2}{\partial \eta} + \left[W' - h\cos(\omega_s t)\right]\mathscr{M}_0 = -\gamma\mathscr{M}_1,$$

where $\mathscr{M}_n = \int \upsilon^n F d\upsilon$, we obtain the exact equation

$$\frac{\partial^2 \mathscr{M}_0}{\partial t^2} + \gamma\frac{\partial \mathscr{M}_0}{\partial t} + h\cos(\omega_s t)\frac{\partial \mathscr{M}_0}{\partial \eta} = \frac{\partial}{\partial \eta}\left(W'\mathscr{M}_0 + \frac{\partial \mathscr{M}_2}{\partial \eta}\right). \tag{6}$$

To break the chain of equations for the moments, we find the relation between $\mathscr{M}_2$ and $\mathscr{M}_0$. In the absence of a signal, Eqn (5) describes the establishment of the Maxwell–Boltzmann distribution. The establishment time of the velocity-equilibrium Maxwell distribution is of the order of γ^{-1} [10], which is substantially shorter than the time of establishment of the Boltzmann distribution over η, which is determined by the quantity inverse to the Kramers frequency [14] or by the average time of transition of a Brownian particle through a potential barrier. Taking into account the fast transition of the velocity distribution function to the equilibrium function, we consider relaxation for the time $t > \gamma^{-1}$. Then, the relation between $\mathscr{M}_2$ and $\mathscr{M}_0$ can be considered the same as in the case of the established distribution function of Eqn (5). We assume that in this case the theory will give correct quantitative results for signal frequencies $\omega_s < \gamma$ and correct qualitative results for $\omega_s > \gamma$. For small signal amplitudes established in time, the solution of Eqn (5), taking (2) into account, has the form

$$F_0 \sim \exp\left[-\frac{(\upsilon - \bar{\upsilon})^2 + \omega_0^2(x - \bar{x})^2}{2\theta}\right], \tag{7}$$

where $x = \eta - \eta_0$; $\theta = \varepsilon/\gamma$ is the analogue of the motion temperature of Brownian particles; and $\bar{x}$ and $\bar{\upsilon}$ are the average values of x and υ satisfying the equations

$$\frac{d\bar{\upsilon}}{dt} + \gamma\bar{\upsilon} + \omega_0^2\bar{x} = h\cos(\omega_s t), \quad \frac{d\bar{x}}{dt} = \bar{\upsilon}. \tag{8}$$

Note that the solution of Eqn (8) for average values exactly coincides with the solution (3) of linearised Eqn (1) describing forced oscillations in the absence of noise.

By using (7) and considering signals with amplitudes $h \ll (\varepsilon\gamma)^{1/2}$, we find

$$\mathscr{M}_2 = (\theta + \bar{\upsilon}^2)\mathscr{M}_0 \simeq \theta\mathscr{M}_0.$$

In this case, Eqn (6) takes the form

$$\frac{\partial^2 f}{\partial t^2} + \gamma\frac{\partial f}{\partial t} + h\cos(\omega_s t)\frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta}\left(W'f + \theta\frac{\partial f}{\partial \eta}\right), \tag{9}$$

where $f = \mathscr{M}_0$.

In the case of low signal frequencies ($\omega_s \ll \omega_0$), the second derivative with respect to time from the distribution function can be neglected and Eqn (9) transforms to the known Smoluhowski equation, which is equivalent to the first-order Langevin equation (4). For $\omega_s \sim \omega_0$, the second derivative should be taken into account. Note that Eqn (9) was obtained for the parameters of a bistable system satisfying inequalities

$$\omega_s < \gamma, \quad h \ll (\varepsilon\gamma)^{1/2}. \tag{10}$$

3. Analysis of the kinetic equation for small-amplitude signals

We assume that the signal amplitude satisfies the condition $h \ll \omega_0\sqrt{\theta}$, which is more strict than condition (10). Then, the solution of Eqn (9) can be found in the first order of the nonstationary perturbation theory in a small parameter h.

Because this solution is constructed similarly to that in [9], we present here the main results of calculations.

We seek the distribution function in the form of a series

$$f = \rho + \tilde{\rho} + \tilde{\tilde{\rho}} + \ldots, \tag{11}$$

where ρ is the solution of unperturbed Eqn (9), which takes the form

$$\frac{\partial^2 \rho}{\partial t^2} + \gamma \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \eta}\left(W' \rho + \theta \frac{\partial \rho}{\partial \eta}\right), \tag{12}$$

$$\rho|_{t=0} = \delta(\eta - \eta_*), \quad \int \rho \, \mathrm{d}\eta = 1;$$

where $\tilde{\rho}$ is the perturbed part of the distribution function proportional to h and defined by the equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} + \gamma \frac{\partial \tilde{\rho}}{\partial t} = \frac{\partial}{\partial \eta}\left(W' \tilde{\rho} + \theta \frac{\partial \tilde{\rho}}{\partial \eta}\right) - h \cos(\omega_s t) \frac{\partial \rho}{\partial \eta}, \tag{13}$$

$$\tilde{\rho}|_{t=0} = 0, \quad \int \tilde{\rho} \, \mathrm{d}\eta = 0;$$

$\tilde{\tilde{\rho}}$ is the perturbed part of the distribution function proportional to h^2, etc. For small signal amplitudes h considered here, it is sufficient to find the solution of (9) in the form of a sum of the two first terms of series (11). From the physical point of view, the terms ρ and $\tilde{\rho}$ in (11) describe the response of the system to noise and signal, respectively.

Solutions for ρ and $\tilde{\rho}$ are represented in the form of series in the basis functions φ_n:

$$\rho = \rho_0 \sum_{n=0}^{\infty} C_n(t) \varphi_n(\eta), \quad \tilde{\rho} = \rho_0 \sum_{n=0}^{\infty} \tilde{C}_n(t) \varphi_n(\eta), \tag{14}$$

where $\rho_0 = N_0^{-1} \exp(-W/\theta)$ is the equilibrium distribution function of Eqn (12) normalised to unity and N_0 is the normalisation constant. The basis functions φ_n are the eigenfunctions of the boundary problem

$$\theta \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\rho_0 \frac{\mathrm{d}\varphi_n}{\mathrm{d}\eta}\right) = -v_n^2 \rho_0 \varphi_n, \tag{15}$$

where the positive eigenvalues have the dimensionality of the square of frequency and are denoted by v_n^2. They can be conveniently numbered in the order of increasing as $v_0 = 0$, $v_1 < v_2 < v_3 \ldots$.

The eigenfunctions φ_n have the orthogonality and completeness properties [9]. For the symmetric potential W considered here, all φ_{2n} are even and φ_{2n+1} are odd functions of the variable η. There exists the method [15] for successive calculations of all the eigenfunctions and eigenvalues, beginning from φ_1 and v_1^2. The first term of a series for ρ in (14) is the equilibrium probability density distribution in the absence of a signal and it corresponds to $v_0 = 0$ and $\varphi_0 \equiv 1$. As follows from calculations [15], unlike the higher eigenvalues, the first nonzero eigenvalue v_1^2 exponentially depends on the noise intensity,

$$v_1^2 \simeq \frac{\omega_0^2}{\pi\sqrt{2}} \exp\left(-\frac{W_0}{\theta}\right). \tag{16}$$

The quantity gives the Kramers frequency for the transitions of a Brownian particle from one potential well to another caused by the action of noise on a bistable system. The quantity inverse to the Kramers frequency determines also the characteristic time of establishment of the Boltzmann distribution over the variable η. The frequencies v_2 and v_3 coincide by the order of magnitude with the frequency ω_0 of the natural oscillations of the bistable system near one of the equilibrium positions and weakly depend on temperature θ. The higher eigenvalues v_n^2 describe the details of the behaviour of the bistable system during its relaxation to the stationary state and are of minor importance.

By substituting (14) into (12) and taking into account the orthogonality of the eigenfunctions φ_n, we obtain equations for the coefficients C_n of expansion (14)

$$\frac{\mathrm{d}^2 C_n}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}C_n}{\mathrm{d}t} + v_n^2 C_n = 0,$$

$$C_n|_{t=0} = \varphi_n(\eta_*), \quad \left.\frac{\mathrm{d}C_n}{\mathrm{d}t}\right|_{t=0} = 0. \tag{17}$$

One can easily see that, taking into account the completeness of the eigenfunctions, the initial conditions in (17) correspond to the initial condition for ρ

$$\rho|_{t=0} = \rho_0 \sum_{n=0}^{\infty} C_n(0) \varphi_n(\eta)$$

$$= \rho_0 \sum_{n=0}^{\infty} \varphi_n(\eta_*) \varphi_n(\eta) = \delta(\eta - \eta_*).$$

We find from Eqn (17) that

$$C_n(t) = \varphi_n(\eta_*) \frac{k_{n1} \exp(k_{n2} t) - k_{n2} \exp(k_{n1} t)}{k_{n1} - k_{n2}}, \tag{18}$$

where k_{n1} and k_{n2} are the roots of the characteristic equation $k_n^2 + \gamma k_n + v_n^2 = 0$. It follows from (18) that $C_0 = 1$, while the rest of C_n decrease exponentially with time because here the roots k_{n1} and k_{n2} are either real and negative or complex with the negative real parts.

By using the solution found for ρ, we obtain the correlation function $k(\tau)$ of the stationary random process $\eta(\tau)$ at the bistable system output:

$$k(\tau) = \langle \eta(t+\tau) \eta(t) \rangle = \int_{-\infty}^{\infty} \eta_1 \rho(t+\tau, \eta_1, \eta)$$

$$\times \eta \rho(t, \eta, \eta_*) \rho_0(\eta_*) \mathrm{d}\eta_* \mathrm{d}\eta \mathrm{d}\eta_1, \tag{19}$$

where $\rho(t+\tau, \eta_1, \eta)$ is the probability that a random quantity has the value η at the instant t and takes the value η_1 at the instant $t+\tau$; and $\rho_0(\eta_*)$ is the equilibrium distribution function at the initial instant.

By substituting (14) into (19), we obtain

$$k(\tau) = \int_{-\infty}^{\infty} \eta \eta_1 \rho(t+\tau, \eta_1, \eta) \rho_0(\eta) \mathrm{d}\eta \, \mathrm{d}\eta_1.$$

Note now that the transition probability $\rho(t+\tau, \eta_1, \eta)$ is determined by Eqn (12) in which t should be replaced by τ,

η by η_1, and η_* by η. Therefore, taking these substitutions into account, this probability is specified by expressions (14) and (18). This gives the expression for the correlation function

$$k(\tau) = \langle \eta(\tau)\eta(0) \rangle = \sum_{k=0}^{\infty} C_{2k+1}(\tau) M_{2k+1}^2, \tag{20}$$

where $M_n = \int_{-\infty}^{\infty} \eta \rho_0 \varphi_n \mathrm{d}\eta$. Because the integrand is odd, we have $M_{2n} = 0$. In the first-order approximation [15], φ_1 is determined by the error function integral, so that $M_1 = \eta_0$.

By calculating the Fourier transform of the correlation function, we find the spectral noise density

$$N(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} k(\tau) \exp(-\mathrm{i}\omega\tau) \mathrm{d}\tau$$

$$= \frac{2\gamma}{\pi} \sum_{n=0}^{\infty} \frac{v_{2n+1}^2 M_{2n+1}^2}{(v_{2n+1}^2 - \omega^2)^2 + (\gamma\omega)^2}, \tag{21}$$

where the obvious equalities $k_{n1} + k_{n2} = -\gamma$ and $k_{n1}k_{n2} = v_n^2$ were taken into account.

Let us calculate now the response of a bistable system to a signal, which is determined by the perturbed part $\tilde{\rho}$ of the distribution function

$$\langle \tilde{\eta} \rangle = \int_{-\infty}^{\infty} \eta \tilde{\rho}(t, \eta) \mathrm{d}\eta = \sum_{n=0}^{\infty} \tilde{C}_n(t) M_n. \tag{22}$$

The equation for $\tilde{C}_n$ is obtained by substituting (14) into (13):

$$\frac{\mathrm{d}^2 \tilde{C}_n}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}\tilde{C}_n}{\mathrm{d}t} + v_n^2 \tilde{C}_n = h \cos(\omega_\mathrm{s} t) \int_{-\infty}^{\infty} \varphi_n' \rho \mathrm{d}\eta, \tag{23}$$

$$\tilde{C}_n|_{t=0} = 0, \quad \left.\frac{\mathrm{d}\tilde{C}_n}{\mathrm{d}t}\right|_{t=0} = 0.$$

Note that $\tilde{C}_{2n} = 0$ because the right-hand side of Eqn (23) vanishes for odd subscripts.

To determine the stationary response of the system to a signal, it is sufficient to find only the solution of homogeneous Eqn (23), which has the form

$$\tilde{C}_{2n+1}(t) = \frac{h \int_{-\infty}^{\infty} \rho_0 \varphi_{2n+1}' \mathrm{d}\eta}{\left[(v_{2n+1}^2 - \omega_\mathrm{s}^2)^2 + (\gamma\omega_\mathrm{s})^2 \right]^{1/2}}$$

$$\times \cos(\omega_\mathrm{s} t - \psi_{2n+1}), \tag{24}$$

$$\tan \psi_{2n+1} = \frac{\gamma\omega_\mathrm{s}}{v_{2n+1}^2 - \omega_\mathrm{s}^2}, \quad \int_{-\infty}^{\infty} \rho_0 \varphi_{2n+1}' \mathrm{d}\eta = \frac{v_{2n+1}^2}{\theta} M_{2n+1}.$$

By substituting (24) into (22), we obtain

$$\langle \tilde{\eta} \rangle = \frac{h}{\theta} [a_\mathrm{s} \cos(\omega_\mathrm{s} t) + b_\mathrm{s} \sin(\omega_\mathrm{s} t)], \tag{25}$$

$$a_\mathrm{s} = \sum_{n=0}^{\infty} \frac{v_{2n+1}^2 (v_{2n+1}^2 - \omega_\mathrm{s}^2)}{(v_{2n+1}^2 - \omega_\mathrm{s}^2)^2 + (\gamma\omega_\mathrm{s})^2} M_{2n+1}^2, \tag{26}$$

$$b_\mathrm{s} = \gamma\omega_\mathrm{s} \sum_{n=0}^{\infty} \frac{v_{2n+1}^2}{(v_{2n+1}^2 - \omega_\mathrm{s}^2)^2 + (\gamma\omega_\mathrm{s})^2} M_{2n+1}^2. \tag{27}$$

To calculate sums contained in expressions (21), (26), and (27), note [9] that the weight factors M_{2n+1}^2 decrease with increasing subscript n faster than $(2n+1)^{-3}$, and hence the terms with the two first eigenvalues make the main contribution to these sums. Therefore, we separate the first term of a series corresponding to the eigenvalues v_1^2, while the residual sums of series corresponding to the higher eigenvalues are replaced by the expressions describing the effective resonance curves

$$\sum_{n=1}^{\infty} \frac{v_{2n+1}^2 M_{2n+1}^2}{(v_{2n+1}^2 - \omega^2)^2 + (\gamma\omega)^2} \simeq \frac{v_\mathrm{eff}^2}{(v_\mathrm{eff}^2 - \omega^2)^2 + (\gamma\omega)^2} \frac{\theta}{2\alpha}, \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{v_{2n+1}^2 (v_{2n+1}^2 - \omega^2) M_{2n+1}^2}{(v_{2n+1}^2 - \omega^2)^2 + (\gamma\omega)^2} \simeq \frac{v_\mathrm{eff}^2 (v_\mathrm{eff}^2 - \omega^2)}{(v_\mathrm{eff}^2 - \omega^2)^2 + (\gamma\omega)^2} \frac{\theta}{2\alpha}, \tag{29}$$

where v_eff^2 is the effective higher eigenvalue, and the estimate [9]

$$\sum_{n=1}^{\infty} M_{2n+1}^2 \simeq \frac{\theta}{2\alpha}$$

for temperatures $\theta \leqslant W_0$ is used.

Let us find the effective eigenvalue v_eff^2 from the condition of the equality of areas under the resonance curves described by the left- and right-hand sides of equalities (28) and (29). By considering temperatures that are low compared to W_0, after integration with respect to the frequency and calculations (similarly to [9]) of the sums appearing in this case, we obtain $v_\mathrm{eff} \simeq \omega_0$. Note that asymptotics of the effective resonance curves for high and low frequencies ω coincide with the asymptotics of partial sums (28) and (29). In addition, the estimate of v_2 in the first-order approximation by the method [15] gives the value v_eff with accuracy to the numerical factor of the order of unity. Therefore, the approximation of these sums by the effective resonance curves leads to a small error due to a rapid decrease of terms in the sums with increasing the summation index. The consideration of the main and efficient higher eigenvalue can be called the two-relaxation time approximation, because γ/v_1^2 and γ/ω_0^2 determine the kinetic and dynamic relaxation times of the bistable system. Finally, note that for $\varepsilon \to 0$, the obtained solution (25)–(29) coincides with the response (3) of the system to a weak signal in the absence of noise, and in the limit of low-frequency signals, with the results of analysis [9] of the Langevin equation (4). Note that a more accurate estimate of the quantity v_eff will result in its weak dependence on temperature θ.

4. Signal-to-noise ratio and transfer coefficient

We define the signal-to-noise ratio S/N for weak signals as the ratio of the square of the signal amplitude to the spectral noise density at the signal frequency. The signal-to-noise ratios at the input and output of a bistable system have the form

$$(\mathrm{S/N})_\mathrm{in} = \frac{\pi h^2}{2\varepsilon}, \quad (\mathrm{S/N})_\mathrm{out} = \frac{h^2 (a_\mathrm{s}^2 + b_\mathrm{s}^2)}{\theta^2 N(\omega_\mathrm{s})}. \tag{30}$$

The filtering properties of the system under study can be described by the transfer coefficient

$$q = \frac{(\mathrm{S/N})_{\mathrm{out}}}{(\mathrm{S/N})_{\mathrm{in}}},$$

which, after simple transformations taking (28) and (29) into account, can be written in the form

$$q = \frac{(p\Delta_2 + \Delta_1)^2 + (\gamma\omega_s)^2(p+1)^2}{p\Delta_2^2 + \Delta_1^2 + (\gamma\omega_s)^2(p+1)}, \quad (31)$$

where $p = 8W_0\nu_1^2/(\theta\omega_0^2)$; $\Delta_1 = \nu_1^2 - \omega_s^2$; and $\Delta_2 = \omega_0^2 - \omega_s^2$.

We used the obtained expressions to calculate $(\mathrm{S/N})_{\mathrm{out}}$ and q depending on the parameters of the problem. In the region of low signal frequencies ($\omega_s \ll \omega_0$), the SR takes place both in the second-order and first-order systems, which is manifested as a local maximum of the dependence of the signal-to-noise ratio on the noise intensity shown in Fig. 1. This maximum corresponds to the condition $\omega_s \simeq \nu_1^2/\gamma$. From the point of view of filtration of weak signals, this effect is of no interest, because the transfer coefficients for parameters of the system at which it is observed is always smaller than unity.

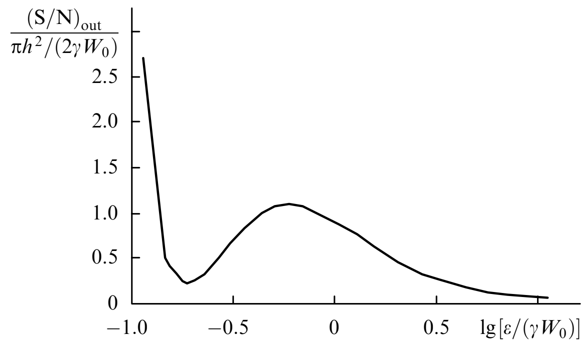

Figure 1. Dependence of the normalised signal-to-noise ratio on the noise intensity for $\gamma/\nu_{\mathrm{eff}} = 1$ and $\omega_s/\nu_{\mathrm{eff}} = 0.017$.

The frequency dependence of the output signal intensity is presented in Fig. 2. For the friction coefficient that is small compared to ω_0, this dependence has two characteristic maxima and a plateau in the low-frequency region. The first maximum (in the low-frequency region) is observed only in the presence of noise and corresponds to the signal frequency $\omega_s = \nu_1^2/\gamma$. The second maximum (at a higher frequency) corresponds to natural oscillations at the frequency ω_0 near the bottom of the potential well. As the noise intensity ε decreases, the first maximum shifts to the red, the plateau intensity increases, and the second maximum decreases and finally disappears. As ε further decreases, the second maximum appears again and increases, while the first maximum decreases. In the absence of noise, only the second maximum at the frequency ω_0 remains.

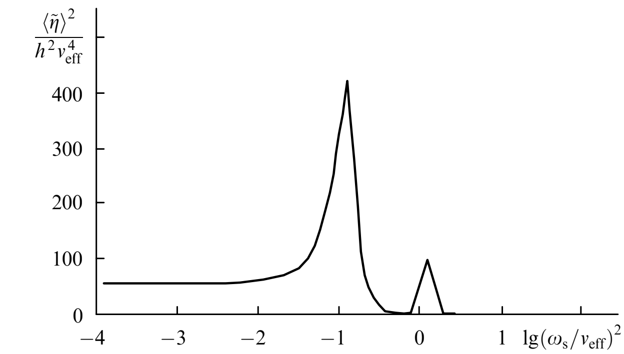

Figure 2. Dependence of the normalised intensity of the output signal on its frequency for $\gamma/\nu_{\mathrm{eff}} = 0.1$ and $W_0/\theta = 0.8$.

The calculations of the frequency dependence of the spectral noise density showed that it behaves as the curve in Fig. 2. As a result the signal-to-noise ratio and the transfer coefficient increase with increasing signal frequency and are saturated for $\omega_s \geqslant \omega_0$, i.e., the region of high-frequency signals is most convenient for filtration. Indeed, as follows from (31), for $\omega_s \geqslant \omega_0$ and $\gamma \gg \omega_s$, the transfer coefficient is $q = 1 + p$. In practice, the noise intensity ε is specified, so that it is more convenient to vary ε rather than γ. Figure 3 shows the dependence of q on γ for $\omega_s = \omega_0$. One can see that this dependence for the system parameters used in calculations has a local maximum and, therefore, the optimum friction coefficient γ exists, which is determined from the condition $\theta = W_0$. In this case, the transfer coefficient q takes the maximum value equal to 1.66. Because the value of ε is specified, the output signal-to-noise ratio similarly depends on γ.

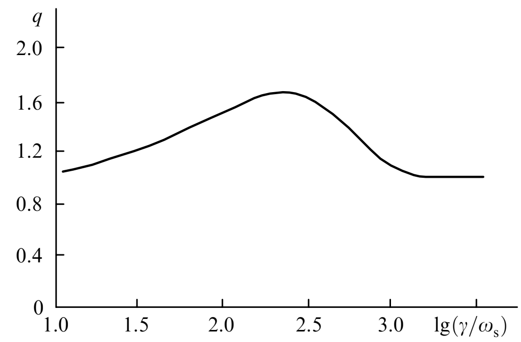

Figure 3. Dependence of the transfer coefficient on the friction coefficient for $\omega_s/\nu_{\mathrm{eff}} = 1$ and $\omega_s W_0/\varepsilon = 0.005$.

5. Conclusions

The theory developed in the paper is applicable for small signal amplitudes and low friction coefficients satisfying inequalities (10). The final expressions have been found in the two-relaxation time approximation, when sums (28) and (29) can be approximated by the effective resonance curves. We assume that because of a fast decrease of terms in these series with increasing their summation index, the error of the approximation does not exceed 20 %. The obtained results give correct solution (3) of Langevin equation (1) when the noise intensity tends to zero. In the limit of low-frequency signals, the expressions contain the results [9] obtained by solving first-order Langevin equation (4).

Our analysis has shown that the SR effect is realised in the second-order nonlinear system, as in the first-order system, for $\omega_s \ll \omega_0$. In the high-frequency region ($\omega_s \gg \omega_0$), the stochastic filtration effect is realised, i.e., the transfer coefficient exceeds unity. For the specified noise intensity ε in the high-frequency region, the optimal friction coefficient $\gamma_{\mathrm{opt}} = \varepsilon/W_0$ exists at which the transfer coefficient q takes the maximum value equal to 1.66.

Thus, optimal conditions for the weak-signal detection exist in the second-order system, and we are sure that the predicted stochastic filtration effect, as the most interesting from the practical point of view, can be found in optically bistable systems both of the first and second order.

References

1. McNamara B., Wiesenfeld K., Roy R. *Phys. Rev. Lett.*, **60**, 2626 (1988).
2. Bartussek R., Hanggi P., Jung P. *Phys. Rev. E*, **49**, 3930 (1994).
3. Barbay S. et al. *Phys. Rev. Lett.*, **85**, 4652 (2000).
4. Gammaitoni L. et al. *Rev. Mod. Phys.*, **70**, 223 (1998).
5. Anishchenko V.S., et al. *Usp. Fiz. Nauk*, **169**, 7 (1999).
6. Hanggi P. et al. *Phys. Rev. E*, **62**, 6155 (2000).
7. Kartashev V.M. et al. *Kratk. Soobshch. Fiz. FIAN.*, (9), 19 (2000).
8. Kartashev V.M. et al. *Kratk. Soobshch. Fiz. FIAN.*, (9), 12 (2000).
9. Reshetnyak S.A., Shcheglov V.A. *Kvantovaya Elektron.*, **33**, 142 (2003) [*Quantum Electron.*, **33**, 142 (2003)].
10. Klimontovich Yu.L. *Usp. Fiz. Nauk*, **169**, 39 (1999).
11. Alfonsi L. et al. *Phys. Rev. E*, **62**, 299 (2000).
12. Zhang Xue-Juan. *J. Phys. A*, **34**, 10859 (2001).
13. Kang Y.M., Xu J.X., Xie Y. *Phys. Rev. E*, **68**, 6123 (2003).
14. Kramers H.A. *Physika (The Hague)*, (7), 284 (1940).
15. Reshetnyak S.A., Shelepin L.A. *Kvazistatsionarnye raspredeleniya v kinetike* (Quasi-Stationary Distributions in Kinetics) (Moscow: IPO 'Avtor', 1996).