Quantum Electronics **35** (10) 923 – 928 (2005)

QUANTUM MEASUREMENTS

PACS numbers: 03.67. –a
DOI: 10.1070/QE2005v035n10ABEH009911

Information properties of generalised quantum measurements*

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***Abstract*. The concept of generalised quantum measurement is introduced as a transformation establishing the correspondence between the initial state of an object and the final state of the object – instrument system with the help of a classical information index one-to-one related to a classically compatible set of the states of the object – instrument system. It is shown that this measurement includes all the types of measurements: standard projective, entangling, soft, and generalised measurements with a partial or complete destruction of the initial information contained in the object. A special class of partially destructing measurements is considered which establish the correspondence between a continual set of states of finite-dimensional quantum systems and systems with an infinite-dimensional space of states. The information sense of these measurements is discussed and some information characteristics are calculated.**

Keywords: *quantum measurement, quantum information.*

1. Introduction

Progress in the development of methods for transformation of quantum information [1] makes natural the transition from the standard quasi-classical concept of a quantum measurement [2, 3] as the establishment of a one-to-one correspondence between the eigenstates of the measurable variable of an object and readings of a quasi-classical instrument to a more general concept of a measurement as the establishment of such a correspondence between an object and a quantum instrument in a substantially quantum form including in the general case the entanglement in the object – instrument system [4] (the so-called entangling measurement).

Such a generalisation is necessary, for example, for a complete description of the measurement dynamics in experiments of the type described in [5], where the nonperturbing measurement of the number $n = 0$, 1 of photons in the resonator was performed for the first time by using a probe atom and a photodetector as a measuring instrument. In this case, a quantum entanglement exists during the response of a photodetector in the object – instrument system, while the type of representation of the measurement result during the formation of the photodetector signal changes from completely coherent (with an accuracy to minor distortions) in an atom to completely incoherent, i.e., dequantisied in the detector signal.

Within the framework of such a generalised insight, the concept of a soft measurement also naturally appears whose accuracy is restricted by the internal quantum uncertainty of the states of an indicator variable used in measurements [6]. In the case of a continually valued measured variable, a consideration of measurements of this type reveals in fact the information content of the internal quantum uncertainty of nonorthogonal quantum states of the indicator, describing it not in traditional terms of fluctuations of physical variables (for example, the Heisenberg inequality) but directly in terms of quantum states themselves, i.e., irrespective of the values of physical variables.

A natural addition to the requirement of the absolute accuracy of a classical perfect measurement is the requirement that the system is not perturbed during the measurement, whereas in the quantum case this requirement cannot be fulfilled in principle due to the specificity of a set of quantum states as a linear space. Any interaction with a quantum system inevitably changes a part of its possible states, whereas this measurement, being always substantial in terms of the Hilbert space of quantum states, is neglected in the algebra of physical variables in a classical system described by the passage to the limit $\hbar \to 0$.

In a quantum system, the requirement of the absence of perturbation can be fulfilled only for a measured variable, which distinguishes the class of nonperturbing measurements. However, because of the uniqueness – the impossibility to copy the total quantum information [7], the total nonperturbing measurement means simultaneously a complete destruction of the coherent, i.e., essentially quantum information in the initial state of the object [4], which is distributed in the case of a completely coherent measurement between the object and instrument and is not contained in the individual components of the object – instrument system.

The soft nonperturbing measurement performed instead of the total measurement preserves a part of coherent information in the object. The opposite situation – the transfer of a greater part of information to the instrument, can be realised only by using the perturbing measurement, when the object state is perturbed. Examples of a completely

* Reported at IV seminar to the memory of D.N. Klyshko, Moscow State University, 17 – 19 May, 2005

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Received 7 June 2005; revision received 8 July 2005
Kvantovaya Elektronika **35** (10) 923 – 928 (2005)
Translated by M.N. Sapozhnikov

perturbing measurement, which eliminates all initial information, are a completely coherent transfer of excitation from oscillator A to oscillator B in a set of oscillators or a purely incoherent measurement of a state of a two-level atom by detecting an emitted photon.

In this paper, we consider a more general class of measurements, which unlike the class of soft measurements defined in [6], includes the possibility of perturbation of the initial state in any variables. We analyse the most interesting case (with respect to the qualitative content of mapped information) of nonselective mapping of all the states $\psi = \sum_k c_k|k\rangle$ of the Hilbert space of a quantum system with the orthogonal basis $|k\rangle$ by the classically distinguishable basis states $|\psi\rangle$ of the continually valued variable of another more complicated system with the Hilbert space of states $\Psi = \int c_\psi|\psi\rangle \mathrm{d}\psi$. Apart from the determination of potential resources of such a transformation for the development of new methods for transformation of quantum information, these measurements are of fundamental interest as the qualitative interpretation of the quantum theory. They allow one to find the most general relations between the physical content of transformations of quantum systems and classical information contained in the values of the information index establishing the one-to-one correspondence between the initial and final quantum states.

2. Determination of the indexing perturbing measurement and its correspondence to the generalised measurement

The state of the instrument before the measurement can be considered specified, so that the measurement can be characterised not by the transformation in the object–instrument system but by a simpler isometric mapping of the object states to the states of the object–instrument system. The physical sense of the isometric mapping V from the space corresponding to the object A to the space corresponding to the system A + B is that it always can be predetermined to the unitary transformation U in the system A + B, which corresponds to the mapping V for the fixed initial state $|0\rangle_\mathrm{B}$ of the instrument:

$$U|\psi)_\mathrm{A}|0\rangle_\mathrm{B} = V|\psi)_\mathrm{A}, \ \ \forall\psi \in H_\mathrm{A}.$$

Therefore, the isometry is the condition for the physical realisation of the transformation in the form of dynamically reversible evolution in the combined object–instrument system.

Consider the isometric transformation of the type

$$V = \sum_\alpha v_\alpha^{1/2}|\alpha\rangle_\mathrm{AB}(\alpha|_\mathrm{A} \tag{1}$$

from the Hilbert space H_A of the object A to the $H_\mathrm{A}\otimes H_\mathrm{B}$ space of the object–instrument system A + B. Here, vectors $|\alpha\rangle_\mathrm{AB}$ determine the orthogonal basis in the object–instrument space indexed by the values α of an indicator variable and realising a new representation of the initial quantum information, which is read out in the general case with the help of the nonorthogonal 'test' states $(\alpha|_\mathrm{A}$ (the use of the parenthesis indicates that the states of a set under study are nonorthogonal). The set of positive numbers v_α characterises the multiplicity of elementary mappings $|\alpha)_\mathrm{A}\to|\alpha\rangle_\mathrm{AB}$ from which the resulting transformation V is constructed as a coherent (depending on the phases of wave functions) superposition of the corresponding generalised projectors.

The condition of the isometry of the given transformation has the form

$$\hat{V}{+}V \equiv \sum_\alpha v_\alpha|\alpha)_\mathrm{A}(\alpha|_\mathrm{A} = \hat{I}_\mathrm{A}. \tag{2}$$

This condition can be obviously fulfilled if the set of the mapped states $|\alpha)_\mathrm{A}$ is a set of orthogonal bases arbitrarily turned with respect to each other. In particular, for N equally represented bases in the D-dimensional space, we have $v_\alpha = 1/(N)$. Transformation (1) is the generalised modification of the canonical representation of the isometric mapping $V = \sum_k |k\rangle_\mathrm{C}\langle k|_\mathrm{A}$ as the transformation of the total orthogonal basis in H_A to an orthogonal set in an arbitrary space H_C corresponding to a physical system C. This, first, specifies the structure of the mapping space as the space of the states of the object–instrument system A + B and, second, uses in the general case the overfilled set of states $|\alpha)$ for the representation of a set of the initial states.

The index α in representation (1) accumulates in the classical form the information associated with a set of the initial quantum states $|\alpha)$ of the object. The values of α are mapped in a one-to-one manner by a set of classically distinguishable stages $|\alpha\rangle_\mathrm{AB}$ in the composite system. This correspondence, i.e., the presence of the physical representation for the transferred information abstractly described by the index α is used for the representation of transformation (1) as a variety of a purely coherent measurement, which represents the output information about the object in the form of the entangled $|\alpha\rangle_\mathrm{AB}$ state. Taking into account the dequantisation effects, which are manifested in a partial loss of coherence of the measurement results without the loss of classical information, such a measurement is described by the corresponding superoperator [8]

$$\mathscr{M} = \mathscr{D}(V\odot V^+) = \sum_{\alpha\beta} R_{\alpha\beta}(v_\alpha v_\beta)^{1/2}|\alpha\rangle_\mathrm{AB}(\alpha|_\mathrm{A}\odot|\beta)_\mathrm{A}\langle\beta|_\mathrm{AB}, \tag{3}$$

where $\mathscr{D} = \sum_{\alpha\beta} R_{\alpha\beta}|\alpha\rangle_\mathrm{AB}\langle\alpha|_\mathrm{AB}\odot|\beta\rangle_\mathrm{AB}\langle\beta|_\mathrm{AB}$ is the dephasing superoperator and $R_{\alpha\beta}$ is an arbitrary positively defined matrix with the unit diagonal, which describes the dephasing of states, and the symbol $\odot$ specifies the place of the substitution of he operator being transformed (density matrix). For $R_{\alpha\beta}\equiv 1$, i.e., in the absence of dephasing, this superoperator simply describes the representation of the transformation V in terms of the density matrix.

The general representation of the measurement by expression (3) and its purely coherent variant (1) include:

(i) Standard projective and entangling measurements [4] by using a complete set of classically compatible events (the orthogonal basis $|k\rangle_\mathrm{A}$) as information being mapped and by using the doubled basis $|k\rangle_\mathrm{A}|k\rangle_\mathrm{B}$ as $|\alpha\rangle_\mathrm{AB}$.

(ii) The soft measurement [6] by replacing the orthogonal basis $|k\rangle_\mathrm{B}$ of the instrument by the nonorthogonal set $|k)_\mathrm{B}$.

(iii) The generalised measurement considered in this paper with a partial destruction of the initial information by assuming that $|\alpha\rangle_\mathrm{AB} = |e_\alpha)_\mathrm{A}|\alpha\rangle_\mathrm{B}$, where the set of states $|e_\alpha)_\mathrm{A}$ is completely arbitrary, while the set $|\alpha\rangle_\mathrm{B}$ is orthogonal and uniquely maps the values of the information index α, whereas the set $(\alpha|_\mathrm{A}$ can also contain nonorthogonal states.

Transformation (1) corresponding to the generalised measurement takes the form

$$V = \sum_{\alpha} v_{\alpha}^{1/2} |e_{\alpha})_{\mathrm{A}} |\alpha\rangle_{\mathrm{B}} (\alpha|_{\mathrm{A}}, \qquad (4)$$

where in the case of the nonorthogonal set $(\alpha|_{\mathrm{A}}$, the information index α is not related uniquely with classically discernible states, and its statistics includes the internal quantum uncertainty of the states $|\alpha)_{\mathrm{A}}$ being mapped. This index can be formally interpreted as the number of the elementary coherent subchannel $|\alpha)_{\mathrm{A}} \to |e_{\alpha})_{\mathrm{A}} |\alpha\rangle_{\mathrm{B}}$ relating in the general case the classically incompatible input states $|\alpha)_{\mathrm{A}}$ of the object with the states $|e_{\alpha})_{\mathrm{A}} |\alpha\rangle_{\mathrm{B}}$ of the object–instrument system.

In the case of a soft nonperturbing measurement, the orthogonality of the set $|e_{\alpha})_{\mathrm{A}} = |k\rangle_{\mathrm{A}}$ leads to the unique correspondence with the information index $\alpha = k$ and, therefore, to the nonperturbing measurement over variables of the type $\hat{\lambda} = \sum_k \lambda_k |k\rangle_{\mathrm{A}} \langle k|_{\mathrm{A}}$, as well as to a complete absence of the coherent information of the instrument with respect to the initial state of the object [6]. The nonorthogonality of the set $|e_{\alpha})_{\mathrm{A}}$ reduces information preserved in the object, i.e., leads to the perturbing measurement and the transfer of a part of coherent quantum information to the states of the instrument, whose readings α contain the quantum uncertainty with respect to the states $|\alpha)_{\mathrm{A}}$ of the object only when the latter have the internal quantum uncertainty, and are uniquely represented by the output states $|\alpha\rangle_{\mathrm{B}}$ of the instrument. In the limiting case $|e_{\alpha})_{\mathrm{A}} \equiv |0)_{\mathrm{A}}$, transformation (4) corresponds to a complete transfer of the initial information from A to B.

When the orthogonal bases $|k\rangle_{\mathrm{A}}$ are used as the sets $|e_{\alpha})_{\mathrm{A}}$ and $|\alpha)_{\mathrm{A}}$, transformation (4) corresponds to a completely coherent entangling measurement [4] resulting in the identical redistribution of the initial information between A and B and a complete absence of the coherent information about the initial state for individual components of the A + B system. In the general case, the distribution of the initial information about the object between the object and instrument is determined by the metric $Q_{\alpha\beta} = (e_{\alpha}|e_{\beta})_{\mathrm{A}}$ of the set $|e_{\alpha})_{\mathrm{A}}$.

3. Relation between the generalised measurement transformation and its representation in the form of the positive operator measure

Consider superoperator (3) mapping generalised transformation (4) taking into account dephasing

$$\mathscr{M} = \sum_{\alpha\beta} R_{\alpha\beta} (v_{\alpha} v_{\beta})^{1/2} |e_{\alpha})_{\mathrm{A}} |\alpha\rangle_{\mathrm{B}} (\alpha|_{\mathrm{A}} \odot |\beta)_{\mathrm{A}} \langle\beta|_{\mathrm{B}} (e_{\beta}|_{\mathrm{A}}. \qquad (5)$$

The probability distribution $P(\alpha) = \langle\alpha|_{\mathrm{B}} \hat{\rho}_{\mathrm{B}} |\alpha\rangle_{\mathrm{B}}$ corresponds to this superoperator, where $\hat{\rho}_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}} \mathscr{M} \hat{\rho}_{\mathrm{A}}$ and $\hat{\rho}_{\mathrm{A}}$ are the density matrices of the instrument and object, respectively, for the results of measurement α physically realised as the quantum states of the instrument. This distribution has the form

$$P(\alpha) = \mathrm{Tr}_{\mathrm{A}} \hat{E}_{\alpha} \hat{\rho}_{\mathrm{A}} \qquad (6)$$

where $\hat{E}_{\alpha} = v_{\alpha} |\alpha)_{\mathrm{A}} (\alpha|_{\mathrm{A}}$ is the positive operator measure (POM). This expression depends neither on the degree of the transformation coherence nor the form of its representation in the output state of the object and the corresponding entanglement of the object–instrument system after measurement because it describes only classical (internally compatible) information of the instrument about the initial state. It describes, however, not the quantum result of measurement by the resulting nonselective information preserved in the quasi-classical form in the object.

The resulting information, although mapped in a classically discernible form, is described by the reduced object–instrument superoperator

$$\mathscr{M}_{\mathrm{B}} = \sum_{\alpha\beta} R_{\alpha\beta} (v_{\alpha} v_{\beta})^{1/2} (e_{\beta}|e_{\alpha})_{\mathrm{A}} |\alpha\rangle_{\mathrm{B}} (\beta|_{\mathrm{A}} \odot |\beta)_{\mathrm{A}} \langle\beta|_{\mathrm{B}}, \qquad (7)$$

which takes into account quantum correlations with the initial state. Even for a completely coherent measurement, expression (7) contains the decoherence factor $R_{\alpha\beta} = (e_{\beta}|e_{\alpha})_{\mathrm{A}}$ caused by the neglect of the coherent information represented as the entanglement of the object–instrument system. This dequantisation is not complete. The consideration of a complete dequantisation of the output information for $R_{\alpha\beta} = \delta_{\alpha\beta}$ leads, as one can easily see, to the transformation

$$\mathscr{M}_{\mathrm{B}} = \sum_{\alpha} v_{\alpha} |\alpha\rangle_{\mathrm{B}} (\alpha|_{\mathrm{A}} \odot |\alpha)_{\mathrm{A}} \langle\alpha|_{\mathrm{B}},$$

to which the probability distribution (6) in the algebra of classical events described by a set of compatible states $|\alpha\rangle_{\mathrm{B}}$ and its subsets still corresponds.

Generalised measurements in terms of the POM were widely discussed earlier, in particular, in the problem of the optimal measurement of continual quantum variables of the type of coordinates and momenta [9–11]. The consideration presented above is qualitatively different due to a finite dimension of the space H_{A} allowing the discussion of information about all the quantum states of the Hilbert space to which the analysis of measurements in infinite-dimensional quantum systems performed earlier was not obviously applicable.

4. Selective generalised measurement

A special case is the selective measurement which differs from the entangling measurement by the generalised form of the output states of the instrument: $|k\rangle_{\mathrm{A}} \to |e_{\alpha})_{\mathrm{A}} = |k)_{\mathrm{A}}$, $k = 1, \ldots, D$, which now differ from the basis states $|k\rangle_{\mathrm{A}}$ of the measured variable and, being represented in the general case by a nonorthogonal set, do not allow the object to preserve the initial state for the measured variable. In the case of a purely coherent measurement, the instrument acquires as a compensation the nonzero coherent information about the initial state of the object, which is complete in the limiting case $|k)_{\mathrm{A}} \equiv |0\rangle_{\mathrm{A}}$, i.e., quantitatively equal to the initial entropy of the object. In this case, information relations for the object–instrument mapping obviously reproduce the same relations for the object–object mapping for a soft measurement, for which the transformation differs by the inversion $H_{\mathrm{A}} \rightleftharpoons H_{\mathrm{B}}$ of the resulting states of the object and instrument. Because of this, the corresponding dependences for the coherent object–object information in a two-level system presented in [6] are also valid in the case under study. The quasi-classical information acquired by the instrument is always complete due to the absolute accuracy of the measurement, i.e., it quantitatively coincides with the entropy of the measured variable.

5. Nonselective generalised measurement

Another special case is the nonselective measurement for which a set of mapped states $|\alpha)_{\rm A}$ includes all the quantum states of an object. In this case, the information index α uniquely maps all the physically discernible elements of the Hilbert space $H_{\rm A}$, and the set of its values is naturally represented by the unit $(2D-2)$-dimensional sphere of the real Euclidean space. The corresponding generalised measurement is the $H_{\rm A} \to H_{\rm A} \otimes H_{\rm B}$ mapping with the state space $H_{\rm B} = L_2(H_{\rm A})$ of the instrument and the wave functions $\psi_{\rm B}(\alpha)$ of the continual argument. The multiplicity of the states $\mathrm{d}v = \sum_{\mathrm{d}\upsilon} v_\alpha$ corresponding to the element of the set $\alpha \in \mathrm{d}\upsilon$ has in this case the form $\mathrm{d}v = D\mathrm{d}\upsilon/\upsilon$, where υ is the total volume of a hypersphere of physical states. Figure 1 shows the set of states of a two-level quantum system. It corresponds, in particular, to the measurement of the polarisation vector of a single photon or the wave function of a single two-level atom.

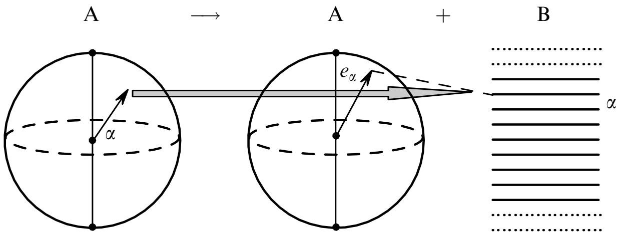

Figure 1. Mapping of the elementary states during the nonselective generalised measurement. The new states $|e_\alpha)_{\rm A}$ of the object differ from the states $|\alpha)_{\rm A}$ in the general case.

Distribution of information between the object and instrument. The amount of information preserved in the object is determined by the information content of the overfilled basis $|e_\alpha)_{\rm A}$, which partially duplicates information represented by the states $|\alpha\rangle_{\rm B}$ of the instrument, but corresponds in the general case to a partial or complete loss of the initial information $|\alpha)_{\rm A}$ of the object. In the case of a completely coherent measurement, i.e., for $R_{\alpha\beta} \equiv 1$, the information content of this basis for a pure input state $\hat\rho_{\rm A} = |\psi\rangle_{\rm A}\langle\psi|_{\rm A}$ is characterised by the entanglement $E(|\psi\rangle_{\rm AB})$ of the resulting object–instrument state $|\psi\rangle_{\rm AB} = V|\psi\rangle_{\rm A}$. The instrument in this case contains all available information [12] about the total Hilbert space of the object states, which is represented, however, in the quantum form including entanglement with the object. This form passes to the classical form either after the additional projective measurement or after the totally dephasing transformation $\mathscr{D}$ in (3) for $R_{\alpha\beta} = \delta_{\alpha\beta}$, which is equivalent informatively to this measurement.

The calculation can be illustrated by the example of a two-level system ($D = 2$) by using as $|e_\alpha)_{\rm A}$ all the states of the part of the Bloch sphere with the angular variables ϑ and φ obtained by mapping $\vartheta \to q\vartheta$, where $0 \leqslant q \leqslant 1$ is the compression coefficient of the initial Bloch sphere mapped into its part corresponding to $0 \leqslant \vartheta \leqslant \pi q$. In the case of such mapping ($q \leqslant 1$), there exists symmetry with respect to the value of the polar angle s of the initial state $\alpha_0 = (s, \varphi_0)$, which is maximal for $q = 0$ and is absent for $q = 1$. The entanglement of the object–instrument system appearing after the measurement can be represented as the entropy $S(\hat\rho'_{\rm A}) = -\mathrm{Tr}\,\hat\rho'_{\rm A}\log_2\hat\rho'_{\rm A}$ of the partial density matrix $\hat\rho'_{\rm A} = \mathrm{Tr}_{\rm B}|\psi\rangle_{\rm AB}\langle\psi|_{\rm AB}$ of the transformed state of the object*. The corresponding dependence $E(s,q)$ is shown in Fig. 2.

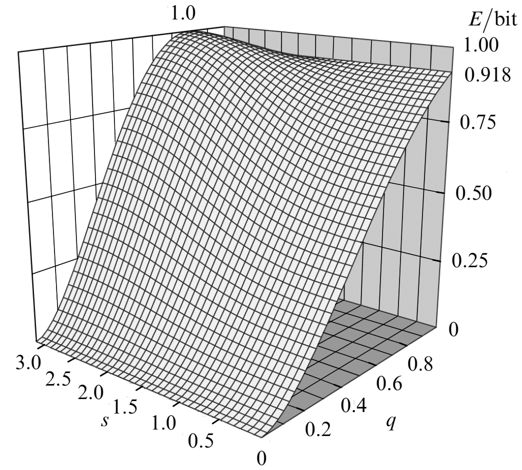

Figure 2. Dependence of the entanglement E on the compression coefficient q of the Bloch sphere and the angle $\vartheta = s$ corresponding to the initial state. The maximum value 1 bit is achieved for $s = \pi$ and $q = 0.7978$, while for $q = 1$ the entanglement is independent of s and corresponds to the entropy $S = E_0 = 0.918$ bit of the pure qubit state after its complete depolarisation.

The result of calculations for a completely nonselective representation of the final state of the object for $q = 1$ and, therefore, $|e_\alpha)_{\rm A} = |\alpha)_{\rm A}$ is obvious without complete calculations because in this case $\hat\rho'_{\rm A}$ corresponds to the totally depolarised initial state {see, for example, expression (3.115) for $p = 1$ in [13]}:

$$\hat\rho'_{\rm A} = (2/3)|\alpha_0\rangle\langle\alpha_0| + (1/3)|\bar\alpha_0\rangle\langle\bar\alpha_0|,$$

where $|\bar\alpha_0\rangle$ is orthogonal to $|\alpha_0\rangle$. Therefore, irrespective of α_0, we obtain $E = E_0 = (2/3)\log_2(3/2) + (1/3)\log_2(3/1)$.

The result $E = 1$ bit, i.e., the complete entanglement achieved in the initial state characterised by the angle $s = \pi$, which is represented on the Bloch sphere by the point opposite to the compression point $\vartheta = 0$, and the intermediate value of the compression coefficient seems to be somewhat unexpected and requires qualitative explanation. This is alleviated because for the given orientation the symmetry with respect to the Bloch sphere axis takes place; as a result, the density matrix is diagonal in the corresponding basis and has the form $\hat\rho'_{\rm A} = p_1|1\rangle_{\rm A}\langle 1|_{\rm A} + p_2|2\rangle_{\rm A}\langle 2|_{\rm A}$. In this case, because the direction of the state vector with $\vartheta = 0$ is opposite to that of the initial state vector with $s = \pi$, according to the above expression for $q = 1$, the probability is $p_1 = 1/3$. When the compression parameter changes to the value $q = 0$ corresponding to the collapse of the Bloch

*The calculation of the trace over B in the partial matrix $\hat\rho'_{\rm A}$ in the continual limit under study is reduced to the corresponding integral over the Bloch sphere with the differential $\mathrm{d}v = \sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi/2\pi$.

sphere to the point $\vartheta = 0$, the probability p_2 of the opposite state decreases to zero and, hence, the probability p_1 changes to unity, according to the analytic expression

$$p_1 = 1 - p_2 = \frac{3 - 2q^2 + \cos \pi q}{4(1 - q^2)} - \frac{1 - \cos \pi q}{4(4 - q^2)}.$$

In this case, because of continuity, the probability p_1 passes through the value 1/2 corresponding to the maximum entanglement of the two systems, one of which is a qubit.

Note also that the maximum entanglement $E_0 = 0.918$ bit, achieved upon a complete reproduction of all the states of the Hilbert space by the object after the measurement, is very close to the complete entanglement achieved in the completely coherent nonperturbing entangling measurement. However, in the latter case this is realised only for the optimal choice of the initial wave function of the object. In the case of the completely nonselective measurement considered here, the value of E_0 is invariant with respect to the initial state $|\alpha_0)_{\mathrm{A}}$ because all the states are on equal terms.

6. Competition between the object and instrument in the choice of nonselective quantum information

A distinct property of the non-destructive quantum measurement is the absence of competition between the object and instrument because of the possibility of unlimited multiplication of classically compatible information obtained in such a measurement. However, during the choice of nonselective information related to the nonorthogonal overfilled set $|\alpha_0)_{\mathrm{A}}$, which is typical, in particular, for protocols of the quantum transfer of a key [14], the competition appears which is caused by the same reason – the impossibility of nonperturbing duplication of information about nonorthogonal quantum states. The competition of this type can be considered by describing information quantitatively with the help of the information Holevo measure, which directly takes into account its quantum specific.

Information for a semiclassical quantum channel characterised by the density matrix $\hat{\rho}(\alpha)$ depending on classical messages α on the channel input is described by the expression

$$I_{\mathrm{H}} = S(\bar{\hat{\rho}}) - \int P(\mathrm{d}\alpha) S(\hat{\rho}(\alpha)), \tag{8}$$

where $\bar{\hat{\rho}} = \int \hat{\rho}(\alpha) P(\mathrm{d}\alpha)$ and $P(\mathrm{d}\alpha)$ is the distribution of probability (or appearance frequency) of classical messages α. One can easily see that the classical parameter α in the considered transformation of the quantum measurement corresponds to the information index of the initial object states $|\alpha)_{\mathrm{A}}$, while the two channels of the problem (object–object and object–instrument) are described by averaging the state corresponding to the wave function $V|\alpha)_{\mathrm{A}}$ of the object–instrument system [or in the general case, the density matrix appearing after incoherent transformation (3)] over the competing system.

The density matrices of the corresponding channels for the uniform distribution $P(d\alpha)$ have the form

$$\hat{\rho}(\alpha) = \frac{D}{v} \int \mathrm{d}v_\beta |(\beta|\alpha)_{\mathrm{A}}|^2 |e_\beta)_{\mathrm{A}}(e_\beta|_{\mathrm{A}}, \tag{9}$$

$$\bar{\hat{\rho}}_{\mathrm{A}} = \frac{1}{v} \int \mathrm{d}v_\beta |e_\beta)(e_\varepsilon \beta|, \tag{10}$$

$$\hat{\rho}_{\mathrm{B}}(\alpha) = \sum_{\beta\beta'} (v_\beta v_{\beta'})^{1/2} (e_\beta|e_{\beta'})_{\mathrm{A}} (\beta'|\alpha)_{\mathrm{A}} (\alpha|\beta)_{\mathrm{A}} |\beta'\rangle_{\mathrm{B}} \langle\beta|_{\mathrm{B}}, \tag{11}$$

$$\bar{\hat{\rho}}_{\mathrm{B}} = \frac{1}{D} \sum_{\beta\beta'} (v_\beta v_{\beta'})^{1/2} (e_\beta|e_{\beta'})_{\mathrm{A}} (\beta'|\beta)_{\mathrm{A}} |\beta'\rangle_{\mathrm{B}} \langle\beta|_{\mathrm{B}} \to \hat{\bar{\hat{\rho}}}_{\mathrm{B}}$$

$$= \frac{1}{v} \int \mathrm{d}v_\beta |\beta)_{\mathrm{A}} |e_\beta^*)(e_\beta^*|(\beta|_{\mathrm{A}}. \tag{12}$$

The second expression in (12) is the isometric mapping of the continual density matrix of the instrument to the discrete space $H_{\mathrm{A}} \otimes H_{\mathrm{A}}$, which realises the active subspace of states and is used for quantitative calculations. The entropies of density matrices $\hat{\rho}_{\mathrm{A}}(\alpha)$ and $\hat{\rho}_{\mathrm{B}}(\alpha)$ coincide, so that there is no need to use the continual representation. Figure 3 presents the dependences of the Holevo information for the instrument and object calculated by general expression (8) by using (9)–(12).

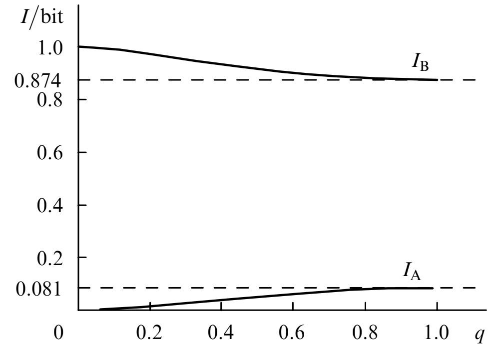

Figure 3. Dependences of the Holevo information, obtained in the instrument (I_{B}) and preserved in the object (I_{A}), about a uniformly distributed ensemble of the initial states $|\alpha)_{\mathrm{A}}$ of the object–qubit on the degree q of preservation of information in the object. The maximum value of the object information $I_{\mathrm{A}} = 0.081$ bit corresponds to the minimum value of the measured information $I_{\mathrm{B}} = 0.874$ bit.

The dependences demonstrate a relatively weak competition in the object–instrument system upon obtaining semiclassical information compared, for example, to competition during obtaining coherent information in the selective measurement, when the preservation of all information in the object leads to its complete absence in the instrument [4].

7. Conclusions

We have shown that the transformation of a partially destructing quantum measurement generalises in the unified form all the basic types of quantum measurements discussed earlier: standard projective and entangling, soft and destructing, coherent and dequantising.

The nonselective variant of such a measurement, as shown by the example of a two-level system, has an interesting feature of the possibility to achieve the maximum entanglement in the object–instrument system in the case of the intermediate degree of preservation by the object of the information initially contained in it. When the object preserves the maximum of the initial information, the entanglement achieves almost the maximum value for any initial pure states.

The nonselective measurement, realising equal measurements of all the dynamic variables in the quantum system, is characterised by a substantially weaker competition in the distribution of quantum information between the object and instruments compared to the completely selective measurement. In the experimental scheme [5], the passage to the nonselective measurement corresponds to the passage from measuring the number of photons $n = 0$, 1 to measuring the superposition state $|\alpha\rangle_A = c_0|0\rangle + c_1|1\rangle$ of the photon field itself. In this case, the most obvious difficulty involved in such measurements, as follows from the relations obtained in this paper, is related to the necessity to establish the equal correlation between each state $|\alpha\rangle_A$ and the corresponding states of the continual set of the orthogonal, i.e., classically discernible states of the instrument.

References

1. Bouwmeester D., Ekert A., Zeilinger A. *Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Calculations* (London: Springer-Verlag, 2000; Moscow: Postmarket, 2002).
2. Von Neumann J. *Mathematical Foundations of Quantum Mechanics* (Princeton, N.J.: University Press, 1955; Moscow: Nauka, 1964).
3. Sudbery A. *Quantum Mechanics and the Particles of Nature: An Outline for Mathematicians* (Cambridge: Cambridge Univer. Press, 1986; Moscow: Mir, 1989).
4. Grishanin B.A., Zadkov V.N. *Phys. Rev. A*, **68**, 022309 (2003).
5. Nogues G., Rauschenbeutel A., Osnaghi S., Brune M., Raimond J.M., Haroche S. *Nature*, **400**, 239 (1999).
6. Grishanin B.A., Zadkov V.N. quant-ph/0506045 (2005).
7. Grishanin B.A., Zadkov V.N. *Radiotekh. Radioelektron.*, **47**, 1029 (2002).
8. Grishanin B.A., Zadkov V.N. *Laser Phys. Lett.*, **2** (2), 106 (2005).
9. Helstrom C.W. *Quantum Detection and Estimation Theory* (New York: Acad. Press, 1976; Moscow: Mir, 1979).
10. Grishanion B.A. *Izv. Akad. Nauk SSSR, Ser. Tekh. Kibern.*, **11**, 127 (1973).
11. Holevo A.S. *Probabilistic and Satistical Aspects of Quantum Theory* (Amsterdam: North-Holland, 1982; Moscow: Nauka, 1980).
12. Caves C.M., Fuchs C.A. quant-ph/9601025 (1996).
13. Preskill J. *Lecture Notes on Physics 229: Quantum Information and Computation*; http://www.theory.caltech.edu/people/preskill/ph229/.
14. Bennett Ch.H., Brassard G. *Proc. IEEE Int. Conf. on Computer, System and Signal Processing* (Bangalore, India; New York: IEEE, 1984) p. 175.