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Kinematics of bosons in a two-dimensional potential well

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***Abstract*. Analysis of the kinematics of bosons (photons, atoms) in two-dimensional potential channels with the transverse eigenvalues slowly varying along the longitudinal axis of the channel reveals the existence of Bose condensation without temperature lowering, the stratification of an atomic flux over phase states, and also processes which phenomenologically reproduce phenomena in nonlinear media (shock-like waves). These phenomena can form the basis for new experimental methods.**

Keywords: *bosons, two-dimensional potential well, Bose condensation.*

1. Introduction

The kinematics of bosons in a two-dimensional potential well in the absence of external energy and dissipation sources, i.e., at the constant total energy of a particle, is determined by the energy exchange between its discrete states in a given transverse potential and a continual longitudinal motion. If the transverse potential is a slowly varying function of the longitudinal coordinate z, then such an exchange gives rise to various phenomena such as the Bose – Einstein condensation without varying temperature, the stratification of an atomic flux over phase states, etc., as well as to effects that are phenomenologically similar to effects appearing only in nonlinear media (although the potential well under study has no nonlinearity), for example, shock-like waves. These phenomena are of interest not only in themselves, but can also probably open up new experimental possibilities, in particular, for solving problems related to the manipulation by cooled fluxes of neutral atoms.

2. Electromagnetic wave in a waveguide

The simplest but instructive example of the kinematics of bosons is the propagation of electromagnetic waves in an ideal metal waveguide (two-dimensional infinitely deep potential well for photons) with the cross section variable along the longitudinal coordinate z. If the cross section changes smoothly enough by preserving its shape, so that the waveguide mode type can be assumed invariable, the critical frequency ω_{nm} proves to be a function of the longitudinal coordinate z. Thus, when the cross section decreases with increasing z, we have

$$\frac{d\omega_{nm}}{dz} > 0. \tag{1}$$

Correspondingly, the propagation constant

$$k(z) = \pm\frac{\omega}{c}\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{1/2} \tag{2}$$

and the group velocity

$$v_{gr}(z) = c\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{1/2} \tag{3}$$

of the wave also decrease during the wave propagation in the waveguide, so that

$$\frac{dv_{gr}}{dz} < 0 \tag{4}$$

up to

$$v_{gr} \to 0 \quad \text{for} \quad \omega_{nm} \to \omega. \tag{5}$$

Here, ω is the wave frequency, n and m are integer subscripts, and c is the speed of light.

This description of the wave propagation in a waveguide with the variable critical frequency (1) is, of course, only an approximation to the exact solution of the problem. The exact solution for a waveguide with a slowly varying cross section should consist of the main constant component and a small variable component and can be found by representing the integral of the corresponding wave equation as a sum of the known solution for a waveguide with a constant cross section and a small addition determined by the above-mentioned small variable component. Such an approach can be also applied to other non-electromagnetic waves, in particular, the waves of matter considered above by replacing the wave equation for the field vectors by the Schrödinger equation for ψ functions.

3. Shock-like electromagnetic wave

It follows from (1) – (5) that during the propagation of a wave in the positive direction along the z axis, the

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conditions are produced for the formation of a wave with an intensity increasing at the wave front due to a decrease in the group velocity of the electromagnetic energy flux compared to the velocity of its subsequent fragments and a decrease in the cross section of the flux at its invariable total intensity $Q = \text{const}$. This reproduces phenomenologically the known picture of a shock wave: the electromagnetic energy W per unit length of the waveguide increases during the wave propagation along the z axis as

$$W = \frac{Q}{v_{\text{gr}}} = \frac{Q}{c}\left[1 - \left(\frac{\omega_{nm}}{\omega}\right)^2\right]^{-1/2}, \tag{6}$$

and the volume energy density w increases even stronger due to the decrease in the cross section S of the narrowing waveguide:

$$w = \frac{W}{S}, \tag{7}$$

and the field strengths increase as $|\mathscr{E}| \sim w^{1/2}$ and $|\mathscr{H}| \sim w^{1/2}$.

In the idealised case, this should result in an infinite increase in the volume energy density up to $w \to \infty$ in the waveguide cross section with the critical coordinate $z = z_{nm}$ at which $\omega_{nm}(z_{nm}) \to \omega$ irrespective of the initial intensity, and in the wave reflection [the change in the sign of the propagation constant (2)] from the cross-section plane at $z = z_{nm}$. In a multimode waveguide in this case, the separation of critical coordinates z_{nm} for different modes occurs.

By passing to the concept of photons in a waveguide [1], we see that, according to the dispersion relation $(\hbar\omega)^2 = (\hbar\omega_{nm})^2 + (c\hbar k)^2$, these phenomena correspond to the conversion of the total energy $\hbar\omega = \text{const}$ of a photon to its so-called equivalent observable rest mass $M_{nm} = \hbar\omega_{nm}/c^2$ when the photon 'stops' completely and its kinetic energy vanishes ($c\hbar k = 0$).

Of course, shock-like waves propagate without divergence for a number of reasons described above. Thus, because the wave is non-monochromatic, having the linewidth $\Delta\omega$, the critical coordinate z_{nm} is blurred up to a finite value

$$\Delta z_{nm} = \left(\frac{\mathrm{d}\omega_{nm}}{\mathrm{d}z}\right)^{-1}\Delta\omega, \tag{8}$$

the smoother is the waveguide narrowing, the greater is this value.

The decay of the wave caused by losses in the waveguide walls prevents the infinite growth of w ($w \to \infty$). The usual description of this process with the help of the decay coefficient χ [2] with the reciprocal length dimensionality gives, due to the stop of the wave at $\omega_{nm}(z_{nm}) \to \omega$, the erroneous impression that losses increase infinitely upon approaching the critical regime. Therefore, losses near $\omega_{nm}(z_{nm}) \to \omega$ would be more properly characterised by the decay time $\tau_\chi = (\chi v_{\text{gr}})^{-1}$, which for waves with TM and TE polarisations is determined by the expressions

$$(\tau_\chi)^{-1}_{\text{TM}} = \frac{R_{\text{s}}}{2\mu_0}\left(\frac{c}{\omega_{nm}}\right)^2 \oint \left(\frac{\mathrm{d}\mathscr{E}_z}{\mathrm{d}n}\right)^2 \mathrm{d}l \Big/ \int \mathscr{E}_z^2 \mathrm{d}s \tag{9}$$

and

$$(\tau_\chi)^{-1}_{\text{TE}} = \frac{R_{\text{s}}}{2\mu_0}\oint\left[\left(\frac{\omega_{nm}}{\omega}\right)^2 \mathscr{H}_z^2 + \left(\frac{c}{\omega_{nm}}\right)^2 \frac{v_{\text{gr}}}{c} \times \left(\frac{\mathrm{d}\mathscr{H}_z}{\mathrm{d}l}\right)^2\right]\mathrm{d}l \Big/ \int \mathscr{H}_z^2 \mathrm{d}s, \tag{10}$$

respectively.

Here, R_{s} is the resistance of the unit surface of metal walls of the waveguide taking the skin effect into account; μ_0 is the permeability of vacuum; $\mathscr{E}_z$ and $\mathscr{H}_z$ are the longitudinal components of the electric and magnetic vectors of the wave; the integrals in numerators are calculated over the contour and in denominators – over the waveguide cross section; and the derivatives in integrands are calculated along the normal $\boldsymbol{n}$ and tangent $\boldsymbol{l}$ to the wall surface.

One can see that the decay times of the TM and TE waves remain finite for $\omega_{nm}(z_{nm})/\omega \to 1$ and $v_{\text{gr}} \to 0$, which points to the absence of infinite increase in losses. And finally, the condition $w \to \infty$ cannot be achieved due to the electric discharge inside the waveguide, which is probably the main restriction of the process.

Consider a simple example of the TE_{10} wave in a rectangular waveguide with the cross-section sides a and b (the critical frequency is $\omega_{10} = \pi c/a$): if a linearly decreases with increasing longitudinal coordinate z as $a(z) = (\pi c/\omega)[1 + \beta(z_{nm} - z)]$, the decay time is $\tau_{10} = (\mu_0 b/R_{\text{s}})[1 + 2(b/a)]^{-1}$ in the cross section $z = z_{nm}$, and the longitudinal spread of the position of the critical cross section is $\Delta z_{nm} = \Delta\omega/\beta\omega$ (where $\Delta\omega$ is the emission linewidth). If $\Delta\omega/\omega = 10^{-5}$ and $\beta = (100\lambda)^{-1}$, then $\Delta z_{nm} = 10^{-3}\lambda$ (where λ is the wavelength).

4. Waves of matter in a potential channel

Another and probably more interesting example is the waves of matter in extended channels with the transverse potential $U(x, y)$, which are filled with a longitudinally propagating flux of deeply cooled atoms. Such an atomic ensemble is described by solutions in the form of a travelling wave

$$\Psi_{nm}(x, y, z, t) = \psi_{nm}(x, y)\exp\left[\mathrm{i}\left(\frac{p_{nm}}{\hbar}z - \frac{E}{\hbar}t\right)\right] \tag{11}$$

with the transverse eigenfunctions $\psi_{nm}(x, y)$ of the Schrödinger equation

$$\nabla_{xy}\psi(x, y) + \frac{2M}{\hbar^2}\left[E - \frac{p^2}{2M} - U(x, y)\right]\psi(x, y) = 0, \tag{12}$$

where $E = \text{const}$ is the total energy of an atom of mass M and $p_{mn}/\hbar$ is the longitudinal component of the wave vector satisfying the relation

$$p_{nm}^2 = 2M(E - E_{nm}); \tag{13}$$

where E_{nm} is an eigenvalue of Eqn (12).

Such waveguide-like channels (traps) are widely used in the laser cooling of neutral atoms [3] with the typical parabolic dependence of the potential on the transverse coordinates x and y:

$$U(x, y) = \alpha(x^2 + y^2). \tag{14}$$

If the potential U remains invariable ($\alpha = \text{const}$) along the coordinate z, then solutions (12), as in the case of a harmonic oscillator, are expressed in terms of the Hermitean polynomials $H_n(qx)$ and $H_m(qy)$ [4], so that

$$\psi_{nm}(x,y) = q(2^{m+n-1}\pi n!m!)^{-1/2} H_n(qx) H_m(qy) \times \exp\left(-q^2 \frac{x^2+y^2}{2}\right), \quad (15)$$

where

$$q^2 = \frac{(2M\alpha)^{1/2}}{\hbar} \quad (16)$$

for the energy eigenvalues

$$E_{nm} = \hbar\left(\frac{2\alpha}{M}\right)^{1/2}(n+m+1). \quad (17)$$

5. Shock-like wave of matter

If the coefficient α in (14) is no longer constant but increases slowly with increasing longitudinal coordinate z, then the solution of type (11) with invariable integer subscripts n and m can be (as in the previous electromagnetic problem) considered approximately valid also for the case of a slowly varying parabolic potential, but with the gradually increasing energy eigenvalue $E_{nm}(z)$ (17). In this case [see (13)], the longitudinal coordinate of the wave vector also slowly decreases:

$$p_{nm} = \pm[2M(E - E_{nm})]^{1/2} = \pm\left\{2M\left[E - \hbar\left(\frac{2\alpha}{M}\right)^{1/2}(n+m+1)\right]\right\}^{1/2}, \quad (18)$$

i.e., the longitudinal movement of atoms is decelerated till their complete stop ($p_{nm} = 0$) at the point $z = z_{nm}$, when $\alpha(z) = \alpha_{nm}$ and

$$E_{nm}(z_{nm}) = \hbar\left(\frac{2\alpha_{nm}}{M}\right)^{1/2}(n+m+1) \to E. \quad (19)$$

Because of the continuity of an atomic flux with the constant absolute value $Q_a = \text{const}$, as the longitudinal movement of atoms is decelerated and their velocity

$$v = \frac{p_{nm}}{M} \quad (20)$$

decreases, the concentration of atoms $N \sim |\Psi\Psi^*|$ increases as

$$N = \frac{Q_a}{vS}, \quad (21)$$

where $S = S(z)$ is the effective area of the flux cross section, which decreases with increasing the coefficient $\alpha(z)$.

Thus, in the idealised case of noninteracting atoms, the gas concentration would increase infinitely upon approaching the critical cross section with the coordinate $z \to z_{nm}$, by reproducing the phenomenology of a shock wave, which, of course, does not occur in fact. The obvious limit of the concentration growth is the transition of a gas to the condensed phase.

6. Bose condensation, stratification of the matter wave and other phenomena near the critical cross section

However, the concentration (21) of a cooled gas considerably increases already before the atomic flux approaches the critical cross section and before the transition of atoms to the condensed phase, whereas when the atomic flux propagates in the negative direction along the z axis, i.e., moves away from the critical cross section, the longitudinal acceleration of atoms occurs due to the decrease in their potential energy E_{nm}.

Under certain conditions, namely, if at a constant temperature T of a gas of boson atoms, their approach to the point $z = z_{nm}$ accompanied by the increase in the concentration N leads to the increase in the degeneracy temperature

$$T_0 = 3.3\frac{\hbar^2 N^{2/3}}{(2J+1)^{2/3} k_B M} \quad (22)$$

(where J is the angular momentum of the atom and k_B is the Boltzmann constant) up to $T_0 > T$, then the gas can form a Bose condensate [5] before the formation of a usual condensed phase, and stratification along the coordinate z can appear in the sequence: gas – Bose condensate – usual condensed phase.

Apart from phenomena related to phase transitions, other factors exist which disturb the idealised picture. In particular, because the atomic ensemble is not monokinetic and has the energy dispersion ΔE, the coordinate z_{nm} has the dispersion

$$\Delta z_{nm} = \alpha_{nm}\left(\frac{d\alpha}{dz}\right)^{-1}\frac{\Delta E}{E}. \quad (23)$$

The results presented in sections 5 and 6 were obtained by assuming that solutions for the waves in channels with transverse parameters slowly varying along the coordinate z differ from solutions for channels with constant parameters only in that in the former case the corresponding eigenvalues of the transverse problem also change slowly. As in the electromagnetic problem, the exact solutions can be found by using the solutions for channels with constant transverse parameters as the first approximation, which is supplemented with corrections introduced by small variable additions to the constant transverse parameters [for example, by replacing the constant potential $U(x,y)$ by $U_0(x,y)(1+\beta z)$, where $U_0 = \text{const}$ and β is a small parameter with the reciprocal length dimensionality].

7. Conclusions

Thus, we can see that even the approximate description of the kinematic transformation of electromagnetic waves and waves of matter (deeply cooled atomic fluxes) in potential channels gives the same pictures in general terms. It seems that similar phenomena can be also observed in the wave structures of different types, in particular, in fluxes of ultracold neutrons with the de Broglie wavelength of the order of the wavelength of light. The effects considered above can decelerate the longitudinal movement of waves till their complete stop, producing extremely high wave densities. Upon backward movement, the energy of the transverse quantum sate (in the electromagnetic case, the energy equivalent to the observed rest mass of a photon) can transform to the kinetic energy of the accelerated wave. The phenomena studied in the paper can find various experimental applications, in particular, for manipulating deeply cooled neutral atoms.

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