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Nonlinear dynamics of solid-state ring lasers

N.V. Kravtsov, E.G. Lariontsev

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***Abstract*. The state of the art in the nonlinear dynamics of cw solid-state ring lasers with the homogeneously broadened gain line is systematically analysed. Diverse lasing regimes appearing upon variation of laser parameters are considered and physical mechanisms determining the conditions of their development and stability are analysed. Relaxation processes and temporal and spectral characteristics of radiation are studied.**

Keywords: *nonlinear dynamics, solid-state ring lasers, lasing regimes, theoretical model of a solid-state ring laser.*

N.V. Kravtsov, E.G. Lariontsev D.V. Skobel'tsyn Institute of Nuclear Physics, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119992 Moscow, Russia; e-mail: kravtsov@npi.phys.msu.su

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1. Introduction

Beginning with the development of the first pulsed [1 – 5] and then cw solid-state lasers [6 – 9], solid-state ring lasers (SRLs) continue to attract the attention of researchers solving a variety of scientific and applied problems. Their applications for various precision measurements in the fields of fundamental physics and metrology are based, on the one hand, on a high sensitivity of bidirectional ring lasers to the presence of weak optical nonreciprocities and, on the other, on a high stability of radiation from high-power single-frequency travelling-wave solid-state lasers.

Along with gas ring lasers, SRLs can be used to verify the postulates and predictions of the quantum electrodynamics and theory of relativity [10 – 13], to study fundamental interactions and verify the parity conservation [10], to detect nonlinear optical phenomena in vacuum [14, 15], and to develop optical frequency standards [16, 17]. Solid-state ring lasers and devices based on them are also widely used in laser instrument making (for example, in Doppler location, optical communication, and navigation systems). The development of diode-pumped single-frequency monoblock ring lasers (ring chip lasers) [18 – 25] has stimulated interest in SRLs and their dynamics.

Solid-state ring lasers use various active media (crystals and glasses doped with active ions), which differ in the type of gain-line broadening. The rate of establishment of the polarisability of the active medium in these lasers, which is determined by the homogeneous broadening of the active transition, greatly exceeds the decay rate of the field in the resonator, whereas the inverse population is established much slower than the field. Such a relation between the relaxation rates is typical for lasers of class B [26]. The interaction dynamics of counterpropagating waves and modes of different types substantially depends on the type of gain-line broadening. In most papers devoted to the dynamics of bidirectional SRLs, lasers with the homogeneously broadened gain line have been studied.

A solid-state ring laser with the homogeneously broadened gain line is a complicated nonlinear system in which the interaction between counterpropagating waves can give rise to specific regimes of bidirectional and unidirectional lasing, which are absent in linear lasers. The nonlinear radiation dynamics of such lasers is very sensitive to variations in the parameters of a ring resonator such as its frequency and amplitude nonreciprocities, the Q factor, coupling coefficients of counterpropagating waves, dynamic inverse-population gratings produced due the interference of counterpropagating waves in the active medium, etc.

The results of studying the dynamics of flashlamp-pumped SRLs of the first generation consisting of discrete elements were systematised in paper [27]. It is impossible to obtain stable parameters and eliminate the influence of technical fluctuations on the radiation dynamics of such lasers. This substantially limited and in some cases virtually excluded a detailed study of a number of problems of nonlinear dynamics. The instability of laser parameters also complicated a comparison of the theoretical and experimental results. The advent of highly stable diode-pumped monolithic ring lasers (ring chip lasers) of a new generation allowed these problems to be solved.

By using lasers of a new generation, considerable progress has been achieved in studying the nonlinear dynamics of SRLs. Among the most interesting achievements are the investigations of transient processes in stationary and nonstationary lasing regimes, parametric interactions of self-modulation and relaxation oscillations, the conditions of appearance of the dynamic chaos, and the phase radiation dynamics of counterpropagating waves.

In this paper, we made an attempt to analyse systematically the state of the art in the nonlinear dynamics of SRLs with the homogeneously broadened gain line. We considered the physical reasons and mechanisms of the development of various lasing regimes in cw SRLs. The dependence of radiation characteristics on the laser parameters was analysed in detail. The methods for stabilising bidirectional lasing regimes were also considered.

2. Control of parameters of solid-state ring lasers

2.1 Modern solid-state ring lasers

In modern SRLs with the homogeneously broadened gain line, as a rule, neodymium-doped yttrium – aluminium garnet single crystals $Nd^{3+}:Y_3Al_5O_{12}$ (Nd : YAG) are used as active elements. This is explained by a fortunate combination of their spectral and luminescence characteristics, rather strong absorption bands in the spectral range 0.808 – 0.812 μm convenient for pumping, a high optical homogeneity and excellent operation characteristics (high thermal conductivity and high optical homogeneity, low-thermal expansion coefficient, high hardness, etc.). It is also important that $Y_3Al_5O_{12}$ crystals have the cubic symmetry, which allows monoblocks of any configurations to be used because there is no need to take into account the birefringence of the active medium.

The SRL parameters and their control are determined to a great extent by the ring resonator design. Modern SRLs can be classified into three groups according to their design: conventional ring lasers consisting of discrete elements, semimonolithic, and monolithic ring chip lasers.

In lasers of the first group, all the laser elements (resonator mirrors, active element, control elements, etc.) are spatially separated and are mounted on a hard platform. An obvious advantage of such a design is a simple control of the laser parameters and the possibility of introducing various additional control elements into the laser resonator. A disadvantage is a low stability of the output parameters caused by an insufficient rigidness of the laser construction consisting of many discrete elements.

Semimonolithic ring chip lasers with a composite resonator [25] have a somewhat better stability. The principal schemes of such lasers are shown in Fig. 1. Lasers of this group consist usually of two-three rigidly coupled elements. Semimonolithic lasers allow a broader tuning by means of piezoelectric elements [28, 29] and also a simple optimisation of the loss ratio for counterpropagating waves, which is required for obtaining high-power unidirectional single-frequency lasing.

Monolithic (monoblock) SRLs (ring chip lasers) are of most scientific and practical interest. They are characterised by a high temporal, frequency, and polarisation stability of radiation, a low sensitivity to external disturbances, and a high efficiency (the efficiency of such lasers is almost an order of magnitude higher than that of conventional flashlamp-pumped lasers). The geometrical perimeter of the resonator of ring single-frequency chip lasers is

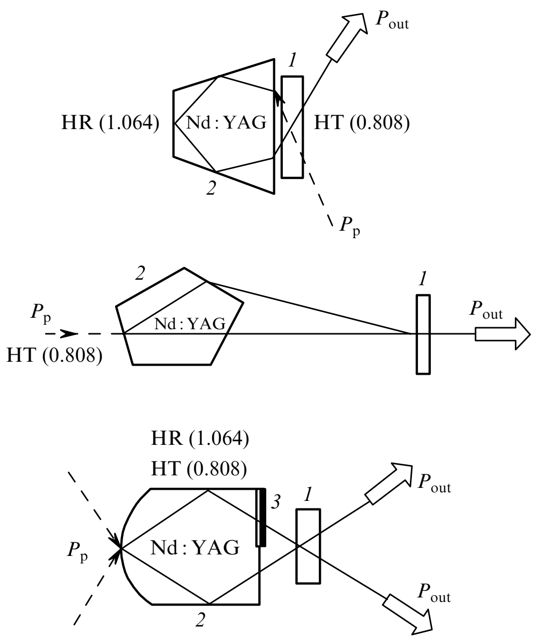

Figure 1. Principal schemes of semimonolithic ring lasers: (*1*) output mirror; (*2*) active element; (*3*) partial polariser; (P_p) pump radiation; (P_{out}) output radiation; HR and HT are high reflection and high transmission coefficients at the wavelength indicated in μm.

20 – 50 mm and their output power can achieve tens of milliwatts.

A monolithic ring chip laser [25] is a complex polyhedral prism (Fig. 2) cut from an optically homogeneous single crystal. A doped single crystal serves as the active medium and optical resonator of such a laser. The prism configuration allows one to create a planar or nonplanar ring resonator inside the monoblock due to total internal reflections from prism faces and from a partially transmitting selective mirror deposited on one of the faces. The only disadvantage of the monolithic laser is virtually the difficulty to control some of its parameters because it is impossible to introduce control elements into the resonator.

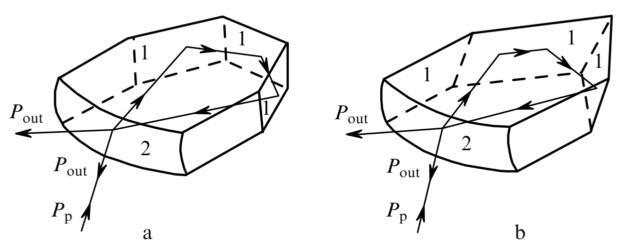

Figure 2. Principal schemes of a monolithic ring laser with planar (a) and nonplanar (b) resonators; (1) total internal reflection faces; (2) spherical faces.

An exotic monolithic ring laser is a laser in which a doped single crystal in the shape of a sphere of small diameter (from a few fractions of millimetre to a few millimetres) is used as the active medium and resonator [30]. In this case, lasing occurs on the so-called whispering-gallery modes. Such lasers are characterised by a high Q factor achieving $10^8 - 10^9$ and an extremely low lasing threshold, which can be as low as a few nanowatts [31]. This allows one, by using liquid helium temperatures, to create a microlaser containing only several (!) active ions. Such lasers attract considerable interest because they can be used in fundamental experiments in the field of quantum electrodynamics.

Detailed studies of the nonlinear dynamics of SRLs require highly stable characteristics of the output radiation, a low level of external technical disturbances, and good reproducibility of the laser parameters. Also, the possibility of a flexible control of the laser parameters determining lasing regimes is required. These parameters are the frequency and amplitude nonreciprocities of a ring resonator, the laser frequency detuning from the gain (luminescence) line centre, moduli and phases of the coupling coefficients of counterpropagating waves, and the pump power and polarisation.

Consider some methods for controlling these parameters by giving main attention to a diode-pumped monolithic ring chip laser.

2.2 Ring resonator nonreciprocity

Optical nonreciprocity is a specific property inherent only in ring lasers. In the absence of nonreciprocity, the eigenfrequencies of a ring resonator and losses during a round trip of light in the resonator are the same for counterpropagating waves. Nonreciprocity means that such invariance is violated, i.e., the eigenfrequencies and intracavity losses become different for identical electromagnetic waves propagated in the ring resonator by the same path in opposite directions (corresponding to the frequency and amplitude nonreciprocity, respectively). There also exist the polarisation [32, 33] and spectral [34] nonreciprocities, which we will not consider here because their role in the nonlinear dynamics of SRLs is not studied at present.

Consider the possibilities to control the amplitude and frequency nonreciprocities of a ring resonator. The effects responsible for their appearance and widely used to control the parameters of the ring resonator can be divided into several groups according to their physical nature [32]. The first group includes effects based on the use of the magneto-optical Faraday effect, the second one – nonreciprocal effects appearing during acousto-optic interactions, and the third one includes effects produced during the rotation of a ring laser.

Nonreciprocal magnetooptical effects appearing when a magnetic field is applied to a medium through which electromagnetic radiation propagates are used most often. The optical anisotropy induced by the magnetic field in the absence of absorption is manifested as a difference in the refractive indices for light waves with different polarisations (the Faraday effect). As a result, these waves propagate in the medium with different phase velocities and acquire the path difference depending on the optical path so that the polarisation plane of linearly polarised light propagated over the distance l in the magnetised medium rotates through the angle Φ, which linearly depends on the magnetic field strength H ($\Phi = VHl$, V is the Verde constant). The sign of the optical rotation angle in the case of Faraday effect (unlike the case of natural optical activity) is independent of the direction of light propagation. Therefore, after multiple forward and backward propagations of light through the medium placed in a magnetic field, the optical rotation angle increases by the corresponding factor. In the general case (taking into account absorption in the medium), the phase shift has the complex character and a linearly polarised wave propagated through a Faraday element becomes elliptically polarised.

In monolithic ring lasers, the magnetooptical properties of active media themselves are commonly used. This method to control the optical nonreciprocity in ring chip lasers was proposed in [19]. The optical nonreciprocity sufficient for obtaining stationary unidirectional lasing in such lasers upon their optimisation can be achieved in rather weak ($\sim$ 100 Oe) magnetic fields [35].

In the case of a nonplanar monolithic ring resonator, the reciprocal rotation of the polarisation planes of each of the counterpropagating waves takes place, and when an external magnetic field is applied, the nonreciprocal rotation of these waves also appears [36]. This gives rise both to the frequency and amplitude nonreciprocity because losses at the output mirror of the monolithic resonator depend on the polarisation of radiation. The properties of ring resonators were analysed in detail in [36] taking into account the influence of nonreciprocal effects appearing in monolithic ring lasers. In lasers consisting of discrete elements, the nonreciprocal Faraday elements are used, as a rule, in combination with other anisotropic elements performing the reciprocal rotation of the polarisation plane.

The ring resonator nonreciprocity in solid-state lasers can be also controlled by using acousto-optic nonreciprocal effects (see, for example, [32, 37]). In a transparent dielectric (which can be the active medium itself), a refractive-index grating is produced with the help of an ultrasonic wave. Upon diffraction of a light wave from this grating, a number of diffraction maxima appear at frequencies $\omega \pm m\Omega_s$ (ω is the incident light wave frequency, Ω_s is the ultrasonic frequency, and m is the diffraction order). In the case of Bragg diffraction, the light beam diffracts only to the first diffraction order, and the energy of the incident light beam is distributed between the beam propagated without diffraction and the diffracted beam. The amplitude nonreciprocity appears in the case of Bragg diffraction from a travelling ultrasonic wave. In this case, the conditions of the interaction of counterpropagating waves with a travelling refractive-index grating prove to be different. The asymmetry appears because one of the light waves interacts with the incident ultrasonic wave, while another interacts with the runaway ultrasonic wave. As a result, the Bragg angles for counterpropagating waves diffracted from a moving ultrasonic grating are not equal to each other, which leads to different diffraction losses, i.e., to the appearance of amplitude nonreciprocity. The efficiency of using acousto-optic effects to control optical nonreciprocity can be considerably increased by employing the optical feedback over the diffracted beam [37, 38].

Along with the amplitude nonreciprocity, the frequency nonreciprocity also can take place upon acousto-optic interaction in the Bragg regime. Its appearance can be qualitatively explained as follows. The phase of a light wave propagated through a cell without diffraction depends on the period of the moving ultrasonic grating. The grating periods for the counterpropagating light waves of a ring laser and, hence, phase shifts prove to be different. This circumstance gives rise to the frequency nonreciprocity of the ring resonator.

The frequency nonreciprocity also appears upon the rotation of the resonator around an arbitrary axis lying outside its plane (see, for example, [10, 39, 40]). The different conditions for the movement of counterpropagating waves appearing in this case form the basis of laser gyroscopes. Nonreciprocal effects appear in this case because the propagation time of a electromagnetic wave over a closed contour in the rotating noninertial coordinate system differs from its propagation time over the same contour in the inertial coordinate system (the Sagnac effect). The difference in the round-trip transit times for counterpropagating waves in the ring resonator gives rise to the difference in the phase shifts $\Delta\varphi = 8\pi S\Omega/\lambda c$ and to the difference in the resonator eigenfrequencies (frequency nonreciprocity) $\Omega = \Delta\varphi/T = 4\pi S\dot{\theta}\omega/Lc$, which is proportional to the angular rotational velocity θ of the resonator (here S and L are the area and period of the ring resonator, respectively; and T is the round-trip transit time for light in the resonator).

2.3 Coupling coefficients of counterpropagating waves

Coupling between counterpropagating waves via backward scattering is one of the main factors determining the lasing regime of a SRL. There exist two types of sources of such coupling. First of all, these are spatial microscopic inhomogeneities of the refractive indices and loss (conductivity) coefficients inside the active medium or other intracavity elements from which backward scattering of radiation occurs, and also the resonator mirrors and active element ends.

The value and phase of the coupling coefficients for counterpropagating waves depend on external parasitic couplings caused by backward scattering both from optical elements of the excitation system and elements of the reception channel [41]. Their contribution is especially large when the transmission of the output mirror is considerable (more than 1 %). To obtain the maximum stability and minimum width of the emission spectrum of chip lasers, efforts are taken to reduce the parasitic optical feedback between the monolithic element itself and a pump source, as well as between the active element and a detector. The presence of parasitic couplings and their instability give rise, on the one hand, to fluctuations of the coupling coefficients of counterpropagating waves in the ring chip laser itself and, on the other, to fluctuations of the spectrum and intensity of radiation of a diode pump laser, which in turn causes the instability of radiation from the chip laser. The influence of these couplings is reduced by means of an additional selective mirror and optimal focusing of pump radiation with respect to the spherical surface of a monolithic resonator [41].

The coupling coefficient of counterpropagating waves can be controlled by adjusting the ring resonator. The coupling coefficient in a monolithic ring laser can be varied upon small displacements of the axis of the chip-laser resonator with respect to the monoblock by changing the pump-beam direction. Such an adjustment leads to a change in the effective values of backscattering coefficients $r_{1,2}$. For fixed values of $r_{1,2}$, the coupling coefficients $m_{1,2}$ of counterpropagating waves in the ring laser monotonically decrease with increasing the resonator perimeter ($m_{1,2} = r_{1,2}c/L$). The values of $r_{1,2}$ for real lasers are determined by the quality of optical elements used and lie usually within $10^{-5} - 10^{-4}$. It is assumed in this case that backward scattering has no specular-reflection components and has a diffusion nature.

In the presence of specular reflections, the moduli of coupling coefficients can be rather high. Therefore, it is obvious that the existence of interfaces between two media with different values of the permittivity and with the normals coinciding with the propagation direction of the waves is inadmissible in ring lasers in most cases.

The coupling coefficient can be efficiently varied by using additional reflecting elements placed inside or outside the optical cavity. An intracavity reflecting element (for example, the ends of an active element with AR coatings) should have a low reflection coefficient (otherwise only reciprocal synchronisation of counterpropagating waves can occur in the laser). A more 'flexible' element for controlling the coupling of counterpropagating waves is an auxiliary external mirror returning radiation back to the resonator (to the oncoming wave). This method was used in [42, 43] to control the dynamics of monolithic chip lasers.

Although the coefficients of coupling of counterpropagating waves via backward scattering are rather important parameters determining the nonlinear dynamics of radiation, it is impossible to measure them directly in experiments. However, they can be measured indirectly, for example, from the frequency of self-modulation oscillations in the absence of frequency nonreciprocity.

2.4 Laser radiation frequency and its detuning from the gain-line centre

Because the gain linewidth in solid-state lasers usually greatly exceeds the frequency interval between adjacent axial modes, the relative detuning of the lasing frequency from the line centre $\delta = (\omega - \omega_0)T_2$ (T_2 is the relaxation time of the polarisation of the amplifying medium) in the case of single-mode lasing is always small. Nevertheless, the absolute value of detuning of the lasing frequency from the gain-line centre can vary in a broad range because the frequency interval between adjacent longitudinal modes of ring chip lasers is rather large and can achieve ten gigahertz.

The detuning from the gain-line centre is usually performed by varying the perimeter of a ring resonator because the gain-line frequency ω_0 weakly depends on external perturbations. Large detunings from the gain-line centre are achieved by using intracavity selective elements (see, for example, [26]).

If a laser consists of discrete elements, the perimeter of a ring resonator can be easily changed. The perimeter can be varied by different methods, for example, by displacing the resonator mirrors mechanically or by means of a piezoelectric element, by heating (cooling) an active element, and with the help of an intracavity electrooptical element. Frequency tuning by means of a piezoelectric element is often used in semi-monolithic ring chip lasers [28]. The tuning range in this case can achieve a few hundreds of megahertz.

In the case of monolithic ring chip lasers, whose geometrical parameters are specified by the monoblock configuration, the situation is more complicated. However, the perimeter of a ring resonator can be also changed, for example, by heating the monoblock. The thermal frequency tuning of garnet monolithic chip lasers is characterised by the coefficient equal to 3.2 GHz grad^{-1}. A monolithic chip laser can be also tuned by producing controllable mechanical stresses in the active element itself [44–46]. The tuning range in a static regime can achieve 100 MHz. When the resonator perimeter is changed, the effective values of coupling coefficients of counterpropagating waves also change, as a rule, due to a change in the distance between scattering centres.

2.5 Pump radiation power and polarisation

One of the controlling parameters, which can be quite simply varied and precisely controlled, is the pump-power excess η over the lasing threshold. This parameter considerably affects the lasing dynamics and output characteristics of the laser, in particular, the conditions of stability of stationary lasing regimes (travelling and standing wave regimes), the frequencies of relaxation and self-modulation oscillations, the number of excited modes, etc. By varying the pump-power excess over the lasing threshold, it is possible to control the regions of existence of parametric resonances between self-modulation and relaxation oscillations (see below). Single-mode lasing can be obtained in a SRL only within a limited range of variation of this parameter if the amplitude nonreciprocity is absent or sufficiently small. If the amplitude nonreciprocity of the resonator is sufficiently high, the single-mode travelling wave regime can be also obtained for large values of η. The polarisation characteristics and nonlinear radiation dynamics of solid-state lasers can be also controlled by using polarised pumping [47].

3. Basic equations of the semiclassical theory of a solid-state ring laser

3.1 Theoretical models of solid-state ring lasers

Despite the complexity of physical processes proceeding in ring lasers with the homogeneous gain line, the theoretical methods for studying their dynamics are well developed at present. There exist several theoretical models of SRLs which describe the properties of the active medium and radiation field by using various mathematical approximations.

The theoretical models of lasers can be divided into quantum and semiclassical models. Quantum models describe the radiation field and active medium based on the quantum theory. This is necessary, for example, for studying natural fluctuations of laser radiation. The quantum models of bidirectional ring lasers (see, for example, [48] and references therein) have been developed for lasers of class A characterised by a rapid relaxation of the inverse population. In the case of bidirectional ring lasers with a slow relaxation of the inverse population (lasers of class B), to which almost all SRLs belong, the quantum approach has not been applied, as far as we know.

The semiclassical theory of SRLs is based on Maxwell's equations for the intracavity field and the system of quantum-mechanical equations for the density matrix of active ions. Solid-state lasers are characterised by a rapid relaxation of the polarisation of the medium. As a result, polarisation follows quasi-statically the complex amplitudes of fields and the inverse population. This allows one to exclude it adiabatically, thereby reducing the equations of the semiclassical theory to a system of nonlinear equations in partial derivatives for the complex amplitudes of intracavity fields and inverse population. However, the problem proves to be so complicated that it can be solved only in some particular cases even in the one-dimensional approximation. Particular problems are usually solved in the semiclassical theory by expanding the intracavity field in the modes of a ring resonator, thereby reducing the problem to a system of ordinary differential equations. Nevertheless, in this case many simplifying assumptions should be also used, which although restrict the generality of analysis but take into account the main factors inherent in the problem under study.

Several simplified models of SRLs have been developed. The most popular among them is the so-called standard model in which the polarisation of counterpropagating waves is assumed specified and the same for both waves. In addition, the standard model assumes that the gain line is homogeneous. The dynamics of bidirectional SRLs was first studied within the framework of the standard model in papers [9, 49–54].

These studies have shown that the standard model correctly describes qualitatively (and in many cases, quantitatively) the nonlinear dynamics of SRLs. However, to obtain a more rigorous quantitative description of the observed effects and to achieve good quantitative agreement between the theory and experiment, the standard theoretical model should be complicated sometimes. For example, the real structure of the gain line in a ring Nd : YAG laser consisting of several homogeneously broadened components should be taken into account in a number of problems [55, 56]. The peculiarities of the radiation dynamics related to different polarisations of counterpropagating waves were analysed [57–59] by using SRL models with specified arbitrary polarisations of counterpropagating waves. In [60], the vector model of a ring laser with a nonplanar resonator was developed, in which the interaction between counterpropagating elliptically polarised waves was described taking into account the polarisation anisotropy induced during saturation of the inverse population.

Note also that the dynamics of unidirectional SRLs was studied in some papers, for example, in [61–64] not expanding the field in resonator modes (based on equations in partial derivatives). The unidirectional lasing was also investigated by using the model based on balance equations [26]. This model is rather rough because it neglects phase effects in ring lasers, and now it is not used in fact.

3.2 Derivation of basic equations of the standard model

The standard SRL model can describe all the lasing regimes observed in experiments. It is this model that will be used below (unless otherwise stated). The interaction of an active atom with the intracavity electromagnetic field $\boldsymbol{E}$ is described in the dipole approximation by the Hamiltonian $\hat{H} = \hat{H}_0 - \hat{\boldsymbol{d}}\boldsymbol{E}$ ($\hat{H}_0$ is the Hamiltonian of a free atom and $\hat{\boldsymbol{d}}$ is the dipole moment operator of the atom), and the system of quantum-mechanical equations for the density matrix of active atoms at rest can be written in the form [65]

$$\left[\frac{\partial}{\partial t} + \mathrm{i}\omega_0 + \frac{1}{T_2}\right]\rho_{\mathrm{ab}} = \frac{\mathrm{i}}{\hbar}\,\boldsymbol{d}_{\mathrm{ab}}(\rho_{\mathrm{b}} - \rho_{\mathrm{a}})\boldsymbol{E}, \tag{1}$$

$$\frac{\partial N}{\partial t} = W - \frac{N}{T_1} + \mathrm{i}\,\frac{2n_0}{\hbar}(\boldsymbol{d}_{\mathrm{ab}}\rho_{\mathrm{ba}} - \rho_{\mathrm{ab}}\boldsymbol{d}_{\mathrm{ba}})\boldsymbol{E}, \tag{2}$$

where ρ_{ab} and ρ_{ba} are the nondiagonal elements of the density matrix for the resonance levels a and b; ρ_{a} and ρ_{b} are the diagonal elements of the density matrix; $N = n_0(\rho_{\mathrm{a}} - \rho_{\mathrm{b}})$ is the inverse population density; $\boldsymbol{d}_{\mathrm{ab}}$ and $\boldsymbol{d}_{\mathrm{ba}}$ are the matrix elements of the dipole moment operator of an active atom; n_0 is the density of active atoms; ω_0 is the operating transition frequency corresponding to the centre of the homogeneously broadened gain line with the half-width $1/T_2$: W is the pump rate; and T_1 is the relaxation time of population inversion. Here, we used for simplicity the model in which the lower resonance level b rapidly relaxes, thereby being virtually unpopulated.

The intracavity electromagnetic field is described by a system of Maxwell's equations, which has the form

$$\mathrm{rot}\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\,\boldsymbol{J}, \quad \mathrm{rot}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\mathrm{div}\boldsymbol{B} = 0, \quad \mathrm{div}\boldsymbol{E} = -4\pi\mathrm{div}\boldsymbol{P} \tag{3}$$

in the inertial reference frame, and by the constitutive equations

$$\boldsymbol{J} = \sigma_{\mathrm{c}}\boldsymbol{E}, \quad \boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}, \quad \boldsymbol{B} = \boldsymbol{H}. \tag{4}$$

Here, $\boldsymbol{P}$ is the polarisation vector of a medium and σ_{c} is the electric conductivity which formally takes into account all intracavity losses.

The polarisation of the medium $\boldsymbol{P} = \boldsymbol{P}_1 + \boldsymbol{P}_{\mathrm{a}}$ is determined by two components: the nonresonance polarisation $\boldsymbol{P}_1 = \boldsymbol{E}(\varepsilon - 1)/4\pi$ (ε is the dielectric constant of the medium) and the resonance polarisation $\boldsymbol{P}_{\mathrm{a}}$ appearing upon interaction of active atoms with the field. In the semiclassical theory, the resonance polarisation is expressed in terms of the nondiagonal elements of the density matrix of active atoms:

$$\boldsymbol{P}_{\mathrm{a}} = n_0(\boldsymbol{d}_{\mathrm{ab}}\rho_{\mathrm{ba}} + \boldsymbol{d}_{\mathrm{ba}}\rho_{\mathrm{ab}}). \tag{5}$$

We will assume first that there are no intracavity nonreciprocal elements and the only source of optical nonreciprocity is the rotation of a ring cavity. By excluding $\boldsymbol{B}$ and $\boldsymbol{P}_1$ from Eqns (3), we obtain the wave equation

$$\varepsilon\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + c^2\mathrm{rot}\,\mathrm{rot}\boldsymbol{E} + 4\pi\sigma_{\mathrm{c}}\,\frac{\partial \boldsymbol{E}}{\partial t} = -4\pi\,\frac{\partial^2 \boldsymbol{P}_{\mathrm{a}}}{\partial t^2}. \tag{6}$$

For the transverse electromagnetic field, we have $\mathrm{rot}\,\mathrm{rot}\boldsymbol{E} = -\Delta\boldsymbol{E}$ and Eqn (6) takes the form

$$\varepsilon\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + 4\pi\sigma_{\mathrm{c}}\,\frac{\partial \boldsymbol{E}}{\partial t} - c^2\Delta\boldsymbol{E} = -4\pi\,\frac{\partial^2 \boldsymbol{P}_{\mathrm{a}}}{\partial t^2}. \tag{7}$$

In the case of a rotating ring laser, the coordinate system coupled to it is noninertial and, as a result, the constitutive equations take the form (see, for example, [40, 65])

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} + \frac{1}{c}[\boldsymbol{B}[\dot{\boldsymbol{\theta}}\boldsymbol{r}]], \quad \boldsymbol{H} = \boldsymbol{B} + \frac{1}{c}[\boldsymbol{E}[\dot{\boldsymbol{\theta}}\boldsymbol{r}]]. \tag{8}$$

Here, $\dot{\boldsymbol{\theta}}$ is the angular rotation velocity vector and $\boldsymbol{r}$ is the radius vector of a point inside the resonator. These relations are written in the first approximation in υ/c ($\upsilon = [\dot{\theta}\boldsymbol{r}]$ is the linear velocity at the point $\boldsymbol{r}$ appearing due to rotation). Taking (8) into account, the wave equation (7) in the rotating reference frame takes the form

$$\varepsilon\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + 4\pi\sigma_{\mathrm{c}}\,\frac{\partial \boldsymbol{E}}{\partial t} - c^2\Delta\boldsymbol{E} - 2([\dot{\boldsymbol{\theta}}\boldsymbol{r}]\mathrm{grad})\frac{\partial \boldsymbol{E}}{\partial t} = -4\pi\,\frac{\partial^2 \boldsymbol{P}_{\mathrm{a}}}{\partial t^2}. \tag{9}$$

Spatial microscopic inhomogeneities of the dielectric constant and conductivity caused by absorbing centres and defects in the active medium, resonator mirrors, and intracavity elements are sources of backscattering resulting in a linear coupling of counterpropagating waves. To take this coupling into account, we will assume that ε and σ depend on coordinates.

Let us expand the intracavity field in the modes of an ideal (without losses) ring resonator filled with a medium with a constant refractive index $n_0 = \sqrt{\varepsilon}$. In the plane-wave approximation, the spatial distribution of the intracavity fields of counterpropagating waves for the nth axial mode can be written in the form

$$U_{1,2}^{n} = \exp(\mp \mathrm{i} k_n z), \tag{10}$$

where $k_n = 2\pi n n/L$ is the wave number. The expansion of the field in the modes in the case of single-mode lasing in each direction has the form

$$\boldsymbol{E} = \mathrm{Re}\Big[\sum_{1,2} \boldsymbol{e}\tilde{E}_{1,2}(t)\exp(\mathrm{i}\omega_n t)U_{1,2}^{n}\Big], \tag{11}$$

where $\tilde{E}_{1,2}(t) = E_{1,2}\exp(\mathrm{i}\varphi_{1,2})$; $\tilde{E}_{1,2}$, $E_{1,2}$, and $\varphi_{1,2}$ are the complex amplitudes, moduli, and phases of counterpropagating waves, respectively; $\omega_n = k_n c/\sqrt{\varepsilon}$ is the cyclic frequency of the axial mode; c is the speed of light in vacuum; and $\boldsymbol{e}$ is the unit polarisation vector.

By multiplying the wave equation (9) by functions $U_{1,2}^{n}$ and integrating over the volume V of the ring resonator, we can obtain in the slowly varying amplitude approximation $\tilde{E}_{1,2}(t)$ the truncated equations for complex amplitudes

$$\frac{\mathrm{d}\tilde{E}_{1,2}}{\mathrm{d}t} = -\frac{\omega_n}{2Q}\tilde{E}_{1,2} + \frac{\mathrm{i}\tilde{m}_{1,2}}{2}\tilde{E}_{2,1} \pm \frac{\mathrm{i}\Omega}{2}\tilde{E}_{1,2} + 4\pi\mathrm{i}\omega_n\tilde{P}_{1,2}^{\mathrm{a}}. \tag{12}$$

The frequency Ω in these equations determines the splitting of the eigenfrequencies of the ring resonator for counterpropagating waves caused by the Sagnac effect [10, 39, 40]:

$$\Omega = \frac{4\pi\dot{\boldsymbol{\theta}}\boldsymbol{S}\omega}{Lc}, \tag{13}$$

where $\boldsymbol{S}$ and L are the area vector and perimeter of the ring resonator. Hereafter in this section, we will omit the subscript n at ω_n. The splitting of the resonator eigenfrequencies for counterpropagating waves can be caused not only by rotation but also by other nonreciprocal effects. The resonance contribution of the polarisation of the active medium to the amplification of counterpropagating waves is

$$\tilde{P}_{1,2}^{\mathrm{a}} = \frac{1}{L}\int_0^L \boldsymbol{e}\boldsymbol{P}_{\mathrm{a}}\exp[-\mathrm{i}(\omega t \mp kz)]\mathrm{d}z. \tag{14}$$

The complex coefficients of linear coupling of counterpropagating waves via backward scattering are determined by the expressions

$$\tilde{m}_{1,2} = \frac{4\pi}{V}\int\left(\mathrm{i}\sigma_{\mathrm{c}} + \frac{\omega}{4\pi}\varepsilon\right)\exp(\mp 2\mathrm{i}kz)\mathrm{d}\boldsymbol{r}. \tag{15}$$

Let us represent these coefficients in the form

$$\tilde{m}_{1,2} = m_{1,2}\exp(\pm\mathrm{i}\vartheta_{1,2}), \tag{16}$$

where $m_{1,2}$ and $\vartheta_{1,2}$ are the moduli and phases of coupling coefficients. The linear coupling appears due to the backward scattering of counterpropagating waves by the inhomogeneities of the dielectric constant ε and electric conductivity σ_{c} of the elements of the optical resonator, which always occurs in reality, and due to diffraction from the elements of the resonator. The first term in (15) describes coupling due to the scattering of waves by the conductivity inhomogeneities and the second one – due to scattering by the dielectric-constant inhomogeneities. In the general case the moduli and phases of the coupling coefficients for counterpropagating waves are different. If backward scattering occurs only from the dielectric-constant inhomogeneities (refractive-index inhomogeneities), the coupling coefficients are, as one can easily see, are complex conjugated ($\vartheta_1 = \vartheta_2$). Such coupling is called sometimes conservative, unlike the dissipative coupling appearing due to the backward scattering of counterpropagating waves by conductivity inhomogeneities (absorbing centres). In the case of dissipative coupling, the coupling coefficients are anticomplex conjugate (the phase difference of the coupling coefficients is close to π). The moduli of coupling coefficients are related to the effective backscattering amplitude coefficients $r_{1,2}$ by the expression

$$m_{1,2} = \frac{c}{L}r_{1,2}. \tag{17}$$

As mentioned above, the system of equations of the semiclassical theory in the case of solid-state lasers can be solved by assuming that the polarisation of the active medium (and, therefore, ρ_{ab}) follows quasi-statically the complex field amplitudes $\tilde{E}_{1,2}$ and the inverse population. The solution of Eqn (1) in the quasi-static approximation is

$$\rho_{\mathrm{ab}} = -\frac{\mathrm{i}\boldsymbol{d}_{\mathrm{ab}}\boldsymbol{e}}{2\hbar\gamma_{\mathrm{ab}}}\frac{\rho_{\mathrm{a}} - \rho_{\mathrm{b}}}{1 - \mathrm{i}\delta}$$

$$\times\left[\tilde{E}_1^{*}\exp(\mathrm{i}kz) + \tilde{E}_2^{*}\exp(-\mathrm{i}kz)\right]\exp(-\mathrm{i}\omega t). \tag{18}$$

By using (18) and expression (14) for the polarisation vector, we can write the system of equations of the standard model for a single-mode SRL in the form

$$\frac{\mathrm{d}\tilde{E}_{1,2}}{\mathrm{d}t} = -\frac{\omega}{2Q_{1,2}}\tilde{E}_{1,2} \pm \mathrm{i}\frac{\Omega}{2}\tilde{E}_{1,2} + \frac{\mathrm{i}}{2}\tilde{m}_{1,2}\tilde{E}_{2,1} + \frac{\sigma l}{2T}(1 - \mathrm{i}\delta)$$

$$\times\int N\left[\tilde{E}_{1,2} + \tilde{E}_{2,1}\exp(\pm\mathrm{i}2kz)\right]\mathrm{d}z, \tag{19}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = W - \frac{N}{T_1}\left\{1 + \frac{a}{1+\delta^2}\left[E_1^2 + E_2^2\right.\right.$$

$$\left.\left. + 2\mathrm{Re}\left[\tilde{E}_1\tilde{E}_2^{*}\exp(-\mathrm{i}2kz)\right]\right]\right\}, \tag{20}$$

where l is the active medium length; $\sigma = \sigma_0/(1+\delta^2)$; σ_0 is the laser transition cross section at the gain line centre; and $a = |\boldsymbol{d}_{\mathrm{ab}}|^2 T_1 T_2/2\hbar^2$ is the saturation parameter. Equations (19) take into account that in the general case the resonator losses (and, therefore, $Q_{1,2}$ factors of the resonator) can be different for counterpropagating waves. The difference of losses for counterpropagating waves determines the amplitude nonreciprocity of the ring resonator

$$\Delta = \frac{1}{2}\left(\frac{\omega}{Q_2} - \frac{\omega}{Q_1}\right).$$

A specific property of bidirectional SRLs is a nonlinear coupling of counterpropagating waves on spatial inverse-population gratings in the active medium. The appearance of these gratings is qualitatively explained as follows. Due to the interference of counterpropagating waves in the optical resonator, the energy density of the light field periodically changes in space (along the resonator axis) and in time (if the frequencies of counterpropagating waves are not equal). Due to the inverse population saturation by the intracavity field, dynamic periodic structures (gratings) are induced in the active medium, which are analogous to gratings produced upon hologram writing. The presence of such gratings is manifested in variations in the gain of counterpropagating waves (amplitude gratings) and in the refractive index of the active medium (phase gratings). The self-diffraction of radiation from induced gratings leads to a nonlinear coupling (competition) of counterpropagating waves. The competitive interaction between counterpropagating waves proves to be the strongest in the case of the homogeneous broadening of the gain line, which is typical for most solid-state lasers.

Equations (19) and (20) represent the system of integro-differential equations. This system can be reduced to a system of ordinary differential equations by expanding the expression for the population inversion in a series in spatial harmonics:

$$N(z) = N_0 + N_+ \exp(\mathrm{i}2kz) + N_- \exp(-\mathrm{i}2kz) + \ldots \quad (21)$$

In the case of a small excess of the pump over the lasing threshold, we can retain in (21) only two spatial harmonics of the inverse population: the zero (N_0) and second harmonics ($N_\pm$). The consideration of higher harmonics does not result in the appearance of any qualitative peculiarities in the lasing dynamics [26]. By using expression (21) we obtain from (19) and (20) the system of ordinary differential equations

$$\frac{\mathrm{d}\tilde{E}_{1,2}}{\mathrm{d}t} = -\frac{\omega}{2Q_{1,2}}\tilde{E}_{1,2} \pm \mathrm{i}\,\frac{\Omega}{2}\,\tilde{E}_{1,2} + \frac{\mathrm{i}}{2}\,\tilde{m}_{1,2}\tilde{E}_{2,1}$$

$$+ \frac{\sigma l}{2T}(1 - \mathrm{i}\delta)\big(N_0\tilde{E}_{1,2} + N_\mp\tilde{E}_{2,1}\big),$$

$$T_1\,\frac{\mathrm{d}N_0}{\mathrm{d}T} = N_{\mathrm{th}}(1+\eta) - N_0 - \frac{1}{1+\delta^2} \quad (22)$$

$$\times\big[N_0 a\big(|E_1|^2 + |E_2|^2\big) + N_+ aE_1E_2^* + N_- aE_1^*E_2\big],$$

$$T_1\,\frac{\mathrm{d}N_+}{\mathrm{d}T} = -N_+ - \frac{1}{1+\delta^2}\big[N_+ a\big(|E_1|^2 + |E_2|^2\big) + N_0 aE_1^*E_2\big],$$

where

$$N_0 = \frac{1}{L}\int_0^L N\mathrm{d}z; \quad N_\pm = \frac{1}{L}\int_0^L N\exp(\mp\mathrm{i}2kz)\mathrm{d}z. \quad (23)$$

The system of equations (22) represents the standard mathematical model of a SRL. A somewhat different derivation of the basic equations of the standard model is presented in paper [66].

Let us emphasise that the equations of the standard model were derived by using the following assumptions:

(i) only one type of oscillations is generated in each direction;

(ii) the plane wave approximation is used;

(iii) spatial inhomogeneities of the pump radiation and inverse population in the transverse (with respect to the resonator axis) direction are neglected;

(iv) diffraction effects (diffraction frequency splitting, diffraction nonreciprocity, etc.) are neglected;

(v) polarisation of counterpropagating waves is assumed the same and linear.

A solid-state ring laser is a rather complicated nonlinear dynamic system with characteristics depending on many controlling parameters. The nonlinear dynamics of the SRL is described in the standard model by a system of ordinary differential equations (22) for complex variables (complex amplitudes of counterpropagating waves and complex harmonics of inverse population). In real variables, this system consists of seven first-order differential equations. All the parameters (coefficients in equations) of an autonomous laser are independent of time. The number of degrees of freedom of this model shows that an autonomous SRL (even single-mode one) is a complicated dynamic system admitting the appearance of dynamic chaos regimes.

It will be shown below that, depending on the parameters of a SRL and their combination, the following lasing regimes are possible:

(i) stationary regimes with constant intensities and equal frequencies of counterpropagating waves (standing wave regimes and unidirectional lasing);

(ii) periodic nonstationary regimes (the self-modulation regime of the first kind and in-phase modulation regime);

(iii) regime of beatings of counterpropagating waves;

(iv) quasi-periodic regimes (self-modulation regime of the second kind, regimes with the self-modulation period doubling);

(v) dynamic chaos regimes.

4. Interaction of counterpropagating waves in stationary lasing regimes

4.1 Peculiarities of the nonlinear dynamics

The diversity of lasing regimes in SRLs is determined by the nonlinear coupling of counterpropagating waves in the active medium on inverse-population gratings and by the linear coupling between counterpropagating waves, which is described by the complex coupling coefficients. The linear and nonlinear couplings of counterpropagating waves substantially affect the existence and stability of various lasing regimes, resulting in the appearance of self-modulation oscillations and other nonstationary regimes.

We consider first the peculiarities of the nonlinear dynamics in stationary lasing regimes, when counterpropagating waves have identical frequencies and their amplitudes (intensities) are independent of time. In ring lasers, the stationary states of two types are possible. For the states of the first type, the intracavity field forms a standing wave (the intensities of counterpropagating waves are comparable, $E_1 \approx E_2$). The states of the second type correspond to unidirectional lasing, when the intensity of one wave greatly exceeds that of the counterpropagating wave (for example, $E_1 \gg E_2$).

Consider qualitatively the main physical factors determining the stability of stationary states. The nonlinear coupling of counterpropagating waves on inverse-population gratings causes a strong competition between these waves: the self-diffraction of counterpropagating waves on induced gratings leads to the inequality of their gains. In the case of small detunings of the lasing frequency from the gain-line centre ($|\delta| \ll 1$), which can be neglected, the gains of counterpropagating waves in the presence of self-diffraction of radiation from gratings induced in the active medium, are

$$\kappa_{1,2} = \frac{\sigma l}{2T}\left(N_0 + N_{\mp}\frac{E_2}{E_{1,2}}\right). \tag{24}$$

Expression (24) in the weak-field approximation, when the pump excess η over the lasing threshold is small ($\eta \ll 1$), can be written in the form

$$\kappa_{1,2} = \kappa_0\left(1 - \alpha a E_{1,2}^2 - \beta a E_{2,1}^2\right),$$

where κ_0 is the unsaturated gain; α and β are the self- and cross-saturation coefficients. In the case of the homogeneously broadened gain line and the identical frequencies of counterpropagating waves, we have $\alpha = 1$ and $\beta = 2$. The difference of the gains for counterpropagating waves is determined by the expression

$$\kappa_1 - \kappa_2 = (\beta - \alpha)a\left(E_1^2 - E_2^2\right)\kappa_0. \tag{25}$$

For $\beta > \alpha$, the gain of the more intense wave is higher. This leads to the instability of the standing wave regime and competitive suppression of one of the counterpropagating waves in the case of a sufficiently weak linear coupling and a small detuning of the lasing frequency from the gain-line centre ($|\delta| \ll 1$). Then, the unidirectional lasing regime proves to be stable. The linear coupling of counterpropagating waves via backward scattering destabilises the unidirectional lasing regime. As the linear coupling increases, unidirectional lasing becomes unstable and the nonstationary regime of self-modulation oscillations of the first kind appears, which in the case of a sufficiently strong linear coupling is changed by the stable stationary standing-wave regime. Therefore, the linear coupling of counterpropagating waves stabilises the standing-wave regime.

In the presence of detuning from the gain-line centre satisfying the condition

$$|\delta| > \delta_{\rm cr} = (1 + \eta)\left(\frac{T_1\eta\omega}{Q}\right)^{-1/2}, \tag{26}$$

the type of nonlinear coupling changes [53, 54]. In the absence of detuning δ, the induced inverse-population gratings are purely amplitude gratings, while in the presence of detuning the phase shift of the scattered wave occurs, which leads to the instability of unidirectional lasing even in the absence of linear coupling [53, 54].

The stability of stationary regimes is usually studied by considering weak perturbations with respect to stationary states and linearising the dynamic equations with respect to these perturbations. As a result, a system of linear differential equations with constant coefficients is obtained. This system of equations gives the characteristic equation for λ for solutions depending on $\exp(\lambda t)$. In the case of lasers with a slow relaxation of the population inversion, the weak perturbations of dynamic variables with respect to their stationary values experience relaxation oscillations, whose frequency is determined by the imaginary part of λ. The condition for stability of stationary solutions is the negative value of the real parts of the roots of the characteristic equation. In the general case, three relaxation frequencies can be observed in a SRL because the order of the characteristic equation is six.

In the unidirectional lasing regime, transient processes in a SRL are characterised by three relaxation frequencies [67–69]. The main relaxation frequency

$$\omega_{\rm r} = \left(\frac{\eta\omega}{QT_1}\right)^{1/2} \tag{27}$$

is independent of the frequency nonreciprocity of the resonator, whereas the two other frequencies depend on Ω and are described by the expression

$$\omega_{\rm r}^{(1,2)} = \left(\frac{\omega_{\rm r}^2}{2} + \frac{\Omega^2}{4}\right)^{1/2} \pm \frac{\Omega}{2}. \tag{28}$$

In the absence of the frequency nonreciprocity of the resonator, the relaxation frequencies $\omega_{\rm r}^{(1,2)}$ are degenerate: $\omega_{\rm r}^{(1)} = \omega_{\rm r}^{(2)} = \omega_{\rm r}/\sqrt{2}$. The degeneracy also takes place when the frequency nonreciprocity Ω of the resonator is equal to $\omega_{\rm r}/2$. In this case, the frequency $\omega_{\rm r}^{(1)}$ coincides with the main relaxation frequency. In the case of the degeneracy of frequencies, a parametric resonance appears between two branches of relaxation oscillations [69–72]. The critical value of a linear coupling resulting in the instability of unidirectional lasing proves to be minimal in regions of parametric resonance [69].

Along with the coupling of counterpropagating waves via backward scattering, the optical nonreciprocity of the ring resonator plays an important role in the lasing dynamics of the SRL. Consider qualitatively the influence of the frequency (phase) nonreciprocity on the SRL radiation dynamics. In the standard model of a ring laser described by the system of equations (22), the optical frequencies of counterpropagating waves are determined by the expression

$$\omega_{1,2} = \omega_n + \frac{{\rm d}\varphi_{1,2}}{{\rm d}t}, \tag{29}$$

where ${\rm d}\varphi_{1,2}/{\rm d}t$ are the frequency shifts of counterpropagating waves with respect to ω_n (the eigenfrequencies of a ring resonator in the absence of frequency nonreciprocity). These shifts can be found by solving the system of equations (22). In an empty resonator (in the absence of an active medium and coupling between counterpropagating waves), the frequencies $\omega_{1,2}$ prove to be equal to the eigenfrequencies of the optical resonator: $\omega_{1,2} = \omega_n \pm \Omega/2$. In the presence of a linear optical coupling, the frequency synchronisation of counterpropagating waves appears in the ring laser and, as a result, both counterpropagating waves oscillate at the frequency $\omega = (\omega_1 + \omega_2)/2$. Frequency synchronisation can appear both in the bidirectional lasing regime (standing-wave regime) and in regimes with substantially different intensities of counterpropagating waves (unidirectional lasing regime). The optical nonreciprocity noticeably affects the intensity ratio of counterpropagating waves in stationary lasing regimes. In the presence of the amplitude nonreciprocity Δ of the ring resonator, even rather small difference $\Delta Q = Q_1 - Q_2$ in the resonator Q factors for counterpropagating waves considerably expands the region of SRL parameters at which the stable travelling-wave regime exists. Experiments show that the difference in the Q factors of a few percent is sufficient for obtaining stable unidirectional lasing [35].

The synchronisation band in a single-mode SRL in the synchronisation regime can be unlimited. If in the absence of the frequency nonreciprocity, the standing-wave regime is stable, then the synchronisation regime is preserved with increasing the frequency nonreciprocity Ω for all its values (unlimited synchronisation band) [73]. As Ω is increased, one of the counterpropagating waves is gradually suppressed and the standing-wave regime passes to the unidirectional lasing regime. The frequencies of counterpropagating waves can be different (see below) in nonstationary lasing regimes (self-modulation regimes, beating regime, and dynamic chaos regimes). In the presence of frequency and amplitude nonreciprocities, the dependence of the intensity of counterpropagating waves on Ω considerably changes (loop dependences of the intensity of counterpropagating waves on the frequency nonreciprocity appear) [27, 74].

4.2 Unidirectional lasing in the absence of nonreciprocity

One of the practically important stationary regimes, which can exist at a sufficiently weak linear coupling between counterpropagating waves, is the unidirectional lasing regime (travelling-wave regime). It is this regime that allows one to provide the maximum stability of the output parameters of the laser and to obtain high-power single-frequency lasing.

The stationary travelling-wave regime in a SRL can be analysed with the help of the system of equations (22). It is convenient to pass from complex variables to the real ones by using the expressions

$$\tilde{E}_{1,2} = E_{1,2}\exp(\mathrm{i}\varphi_{1,2}), \quad n_0 = \frac{\sigma l}{T} N_0 \left(\frac{\omega}{Q}\right)^{-1},$$

$$\frac{\sigma l}{T} N_{\pm}\left(\frac{\omega}{Q}\right)^{-1} = n_2 \exp(\pm \mathrm{i}\psi_N).$$

Consider the case when the optical nonreciprocities of a ring resonator are absent ($\Delta = 0,\ \Omega = 0$), the lasing frequency coincides with the gain-line centre ($\delta = 0$), and the moduli of the linear-coupling coefficients of counterpropagating waves are identical ($m_1 = m_2 = m$). The system of equations (22) in new variables $E_{1,2}$, n_0, n_2, $\Phi = \varphi_2 - \varphi_1$, and $\Psi = \Phi + \psi_N$ takes the form

$$\frac{\mathrm{d}E_{1,2}}{\mathrm{d}t} = \frac{\omega}{2Q}[(n_0 - 1)E_{1,2} + n_2\cos\Psi E_{2,1}]$$

$$\mp \frac{m}{2}\sin(\Phi + \vartheta_{1,2})E_{2,1},$$

$$T_1\frac{\mathrm{d}n_0}{\mathrm{d}t} = 1 + \eta - n_0\left(1 + aE_1^2 + aE_2^2\right) - 2n_2 aE_1E_2\cos\Psi,$$

$$T_1\frac{\mathrm{d}n_2}{\mathrm{d}t} = -n_2\left(1 + aE_1^2 + aE_2^2\right) - n_2 aE_1E_2\cos\Psi,$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\omega}{2Q}n_2\sin\Psi\left(\frac{E_2}{E_1} + \frac{E_1}{E_2}\right) \tag{30}$$

$$+\frac{m}{2}\left[\frac{E_1}{E_2}\cos(\Phi + \vartheta_2) - \frac{E_2}{E_1}\cos(\Phi + \vartheta_1)\right],$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{n_0}{T_1 n_2}E_1E_2\sin\Psi.$$

The order of this system of equations is smaller by unity than that of the initial system in complex variables. This is explained by the fact that the phase of any of the travelling waves can be chosen arbitrarily, and only the phase difference of counterpropagating waves is important in analysis of stationary regimes.

In the case of weak coupling ($m \ll \omega/Q$), the stationary solutions of equations (30) and their stability can be studied by the method of successive approximations in a small parameter $\rho = m(\omega/Q)^{-1}$ [9, 68–72, 75]. In the zero-order approximation, corresponding to $m = 0$, the stationary solutions describing the travelling-wave regime, can be represented in the form (we assume for definiteness that $E_1 \gg E_2$)

$$\sqrt{a}E_1^{(0)} = \sqrt{\eta}, \quad E_2^{(0)} = 0, \quad n_0^{(0)} = 1, \quad n_2^{(0)} = 0. \tag{31}$$

Taking into account the terms of the order ρ^2 in the next approximation, we write the stationary solution in the form

$$aE_1^2 = \eta - \eta_1\rho^2, \quad \eta_1 = \frac{1+\eta}{\eta}[2 + (1+\eta)\cos\vartheta],$$

$$\vartheta = \vartheta_2 - \vartheta_1,$$

$$aE_2^2 = \frac{(1+\eta)^2}{\eta}\rho^2, \; n_2 = \rho, \; \sin(\Phi - \vartheta_1) = 1, \tag{32}$$

$$\cos\Psi = -1,$$

$$n_0 = 1 + \eta_2\rho^2, \; \eta_2 = \frac{1+\eta}{\eta}(1 + \cos\vartheta).$$

Note that because the backward scattering of counterpropagating waves always takes place in real lasers, bidirectional lasing in the travelling-wave regime occurs, strictly speaking, with substantially different intensities of counterpropagating waves. The necessary condition for the existence of the travelling-wave regime is a weak coupling of counterpropagating waves due to their backward scattering ($\rho \ll 1$).

In the case under study, the characteristic equation splits into the quadratic and fourth-order equations (see [68, 70]). The quadratic equation has the roots

$$\lambda_{1,2} = -\frac{1+\eta}{2T_1}\left[1 - \frac{1+\eta}{2(2+\eta)}\frac{\rho^2}{\rho_{\mathrm{cr}}^2}\right] \pm \mathrm{i}\left(\frac{\eta\omega}{2QT_1}\right)^{1/2}, \tag{33}$$

while the approximate values of the roots of the second equation are determined by the expressions

$$\lambda_{3,4} = -\frac{1+\eta}{2T_1}\left(1-\frac{\rho^2}{\rho_{cr}^2}\right) \pm \mathrm{i}\left(\frac{\eta\omega}{2QT_1}\right)^{1/2},$$
$$\lambda_{5,6} = -\frac{1+\eta}{2T_1}\left[1+\frac{\rho^2}{\rho_{cr}^2}(2+\eta)\right] \pm \mathrm{i}\left(\frac{\eta\omega}{QT_1}\right)^{1/2}, \qquad (34)$$

where

$$\rho_{cr}^2 = \frac{(1+\eta)\eta}{(2+\eta)[1+\cos(\vartheta_1-\vartheta_2)]}\,\frac{Q}{\omega T_1}.$$

The roots $\lambda_{5,6}$ correspond to the main relaxation frequency, and the roots $\lambda_{1,2}$ and $\lambda_{3,4}$ to the degenerate frequencies $\omega_r/\sqrt{2}$. The strong wave oscillates at the frequency ω_r, while the weak wave and the phase difference of counterpropagating waves oscillate at the frequency $\omega_r/\sqrt{2}$. It follows from expressions (33) and (34) that the necessary and sufficient condition for the stability of the travelling-wave regime is the inequality

$$m^2 \leqslant (m_{cr}^0)^2 = \frac{(1+\eta)\eta}{2+\eta}\,\frac{\omega}{QT_1}\,\frac{1}{1+\cos(\vartheta_1-\vartheta_2)}. \qquad (35)$$

One can see that the stability of the travelling-wave regime depends not only on the moduli of coupling coefficients $m_{1,2}$ but also on the phase difference $\vartheta_1-\vartheta_2$. In the case of dissipative linear coupling, when the condition $\cos(\vartheta_1-\vartheta_2) = -1$ is fulfilled, the travelling-wave regime in a SRL is stable in the entire region of its existence. It is reasonable to assume that a linear coupling in monolithic solid-state lasers is mainly determined by backward scattering for inhomogeneities of the dielectric constant of the active element. In this case, the feedback coefficients prove to be close to the complex conjugate ones ($\vartheta_1-\vartheta_2 \ll 1$), and for a small pump excess of the lasing threshold ($\eta \ll 1$), the condition (35) for stability of unidirectional lasing can be written in a simple form $m \leqslant \omega_r/2$. This condition, as a rule, is not fulfilled in monolithic SRLs (because of a small perimeter and, hence, a large linear coupling coefficient m). In this case, the instability of unidirectional lasing gives rise to self-modulation oscillations of the first kind. Nevertheless, the stationary unidirectional regime can be easily obtained in such lasers in the presence of the optical nonreciprocity of the resonator (see below).

4.3 Stationary standing-wave regime

The linear coupling of counterpropagating waves via backward scattering stabilises the standing-wave regime [8, 9, 76]. Consider the influence of the linear coupling on the standing-wave regime in the absence of the frequency nonreciprocity of the resonator ($\Omega = 0$) and for a small detuning of the lasing frequency from the gain-line centre ($\delta = 0$). In the case of a small excess of the lasing threshold ($\eta \ll 1$) and equal moduli of the coupling coefficients ($m_1 = m_2 = m$), the stationary amplitudes of the intracavity field in the standing-wave regime are independent of the linear coupling and are determined by the expression

$$aE_1^2 = aE_2^2 = \frac{\eta}{3}. \qquad (36)$$

Two standing-wave regimes can exist in a SRL, for which the phase differences for counterpropagating waves differ by π:

$$\Phi_1 = \frac{\vartheta_1+\vartheta_2}{2}, \quad \Phi_2 = \frac{\vartheta_1+\vartheta_2}{2} + \pi. \qquad (37)$$

The positions of the nodes and antinodes of a standing wave in these regimes are displaced by one fourth of the wavelength. The characteristic equation for the standing-wave regime splits into three equations: linear

$$\lambda T_1 + 1 = 0, \qquad (38)$$

quadratic

$$\lambda(\lambda T_1+1) + \eta\left(\frac{\omega}{Q} - m\left|\sin\frac{\vartheta_1-\vartheta_2}{2}\right|\right) = 0, \qquad (39)$$

and cubic

$$(\lambda T_1+1)\left[(\lambda-\Pi)^2 + m^2\cos^2\frac{\vartheta_1-\vartheta_2}{2}\right] + (\lambda-\Pi)$$
$$\times\left(\frac{\omega}{Q} - m\left|\sin\frac{\vartheta_1-\vartheta_2}{2}\right|\right)\frac{\eta}{3} = 0, \qquad (40)$$

where Π is the stationary value $(\sigma l/T)N_0 - \omega/Q$. For the two standing-wave regimes [see (37)], the values of Π prove to be different:

$$\Pi_{1,2} = \frac{\sigma l}{T}N_0 - \frac{\omega}{Q} = \frac{\omega\eta}{3Q} - \frac{m}{3}\left|\sin\frac{\vartheta_1-\vartheta_2}{2}\right|$$
$$\pm\frac{2m}{3}\sin\frac{\vartheta_1-\vartheta_2}{2}. \qquad (41)$$

The roots of linear and quadratic equations (38) and (39) have the negative real parts. According to the Routh–Hurwitz criterion for the roots of cubic equation (40), we obtain the stability condition

$$m\sin\left|\frac{\vartheta_1-\vartheta_2}{2}\right| > \frac{\omega}{3Q}\eta \qquad (42)$$

for the standing-wave regime. Only one of the two stationary standing-wave regimes proves to be stable.

From the physical point of view, the standing wave becomes stable due to a change in the effective gains of counterpropagating waves in the presence of feedback:

$$(\chi_{1,2})_{\mathrm{eff}} = \frac{\sigma l}{2}\left(N_0 + N_\pm\frac{\tilde{E}_{2,1}}{\tilde{E}_{1,2}}\right)$$
$$\pm m_{1,2}T\sin(\Phi-\vartheta_{1,2})\,\frac{E_{2,1}}{2E_{1,2}}, \qquad (43)$$

where Φ is the phase difference of counterpropagating waves. While in the absence of coupling, a more intense wave had a higher gain, in the presence of a sufficiently strong coupling satisfying condition (42), the situation changes and a more intense wave has a lower efficient gain, resulting in the stability of a standing wave.

Therefore, by increasing coupling between counterpropagating waves, the competitive suppression of counterpropagating waves can be eliminated and a stable standing wave can be generated. The feedback stabilises the standing-wave regime most efficiently in the case of anti-complex conjugate coupling coefficients. In this case, the stability condition for a standing wave has a simple form $m > \eta\omega/(3Q)$.

The inverse population in the standing-wave regime is burnt spatially inhomogeneously. This favours excitation of many axial modes in a solid-state laser with increasing pump power. Single-mode lasing in the standing-wave regime can be obtained only by introducing special selecting devices into the resonator. In the absence of selection, the multimode standing-wave regime appears. In the multimode standing-wave regime, the inverse population is spatially burnt more homogeneously than in the single-mode regime. As shown in [76], this reduces the rise increment of the standing-wave perturbations. This can be qualitatively explained as follows. As the inverse population is spatially smoothed, the reflection of waves from its inhomogeneities decreases, reducing the difference in the gains of counterpropagating waves, which causes the standing-wave instability.

Due to a decrease in the rise increment of perturbations, the multimode standing-wave regime becomes stable at a weaker coupling than in the single-mode regime. The stability condition for the two-mode standing-wave regime has the form [76]

$$m\left|\sin\frac{\vartheta_1^n - \vartheta_2^n}{2}\right| > \frac{\omega}{Q_n}\,\eta\left(1 - \frac{4}{5 - \sin x/x}\right). \qquad (44)$$

Here, $x = 2\pi l/L$; $\vartheta_{1,2}^n$ are the phases of the coupling coefficients of the counterpropagating waves for the nth mode; and l/L is the filling factor of the resonator.

Note that the linear coupling in monolithic ring lasers, as a rule, is insufficient for condition (42) to be fulfilled, and the stationary standing-wave regime is unstable in such lasers. In lasers consisting of discrete elements, the standing-wave regime can be readily realised if the ends of the active element are oriented normally to the resonator axis [9, 27]. It has been shown experimentally in [77] that, according to (44), upon excitation of several longitudinal modes, the standing-wave regime can be stabilised at a weak linear coupling.

4.4 Influence of the resonator nonreciprocity on stationary lasing regimes

The study of the influence of the optical nonreciprocity of a ring resonator on stationary lasing regimes is both of scientific and practical interest. It is important, for example, to investigate the width of the frequency-locking region for counterpropagating waves in bidirectional lasing regimes, the conditions for the beating-regime existence, and the conditions for considerable suppression of a weak wave in the unidirectional lasing regime.

Consider first the influence of the frequency nonreciprocity on the standing-wave regime. The analytic study of the stability of the stationary bidirectional regime in the presence of the frequency nonreciprocity of a ring laser can be performed in the case of a strong coupling between counterpropagating waves via backward scattering with the coupling coefficients close to the complex conjugate ones [78]. We assume for simplicity that the moduli of coupling coefficients are equal ($m_1 = m_2 = m$). In this case, the method of successive approximations in small parameters

$$\varepsilon = \frac{\omega\eta}{Qm} \ll 1, \quad \mu = \frac{\vartheta_1 - \vartheta_2}{2} \ll 1 \qquad (45)$$

can be used.

The intensities of counterpropagating waves in the stationary bidirectional lasing regime under study depend on the frequency nonreciprocity Ω as

$$aE_1^2 = \frac{\eta}{2}\,\frac{v^2}{2v^2 + m^2}\left[1 + \mathrm{sign}\mu\,\frac{\Omega}{v}\right],$$
$$aE_2^2 = \frac{\eta}{2}\,\frac{v^2}{2v^2 + m^2}\left[1 - \mathrm{sign}\mu\,\frac{\Omega}{v}\right], \qquad (46)$$

where $v = (m^2 + \Omega^2)^{1/2}$.

In the absence of the frequency nonreciprocity, these expressions describe the standing-wave regime considered above. In the presence of the frequency nonreciprocity Ω, the phase difference of counterpropagating waves changes, which in the case of a feedback via scattering leads to the inequality of their intensities. The dependence of the intensities of counterpropagating waves on Ω has the characteristic x-shaped form (Fig. 3). As Ω is increased, one of the counterpropagating waves is suppressed, and the standing-wave regime passes to unidirectional lasing at large Ω. In the absence of the frequency nonreciprocity, the intensities of counterpropagating waves are different if the moduli of coupling coefficients are not equal. In this case, the intensities of counterpropagating waves become equal (a pure standing wave) when the frequency nonreciprocity is $\Omega_0 = (m_2 - m_1)/2$.

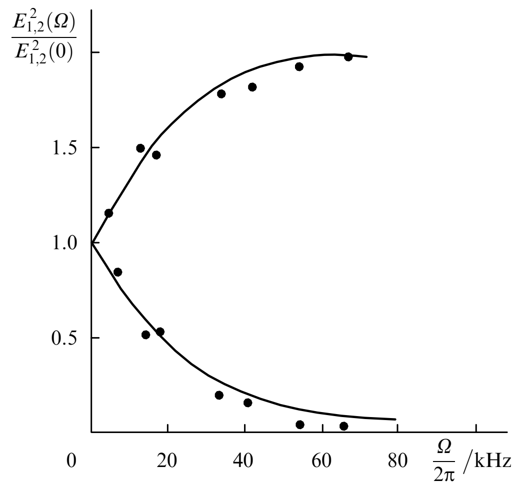

Figure 3. Theoretical (solid curves) and experimental (circles) dependences of the relative intensities of counterpropagating waves on the frequency nonreciprocity of a ring resonator in the stationary regime of synchronisation of counterpropagating waves [73] in the case of stable standing wave for $mQ/\omega = 0.8$, $\vartheta_1 - \vartheta_2 = 0.1$, and $\eta = 0.1$.

The stability condition for the stationary bidirectional lasing regime under study (as the stability condition for the standing-wave regime for $\Omega = 0$) is determined by the inequality

$$\Pi = \frac{\sigma l}{T}N_0 - \frac{\omega}{Q} < 0.$$

In the case of equal moduli of linear coupling coefficients ($m_1 = m_2 = m$), it has the form

$$m\left|\frac{\sin(\vartheta_1-\vartheta_2)}{2}\right| \geqslant m\left(m^2+\Omega^2\right)^{1/2}\frac{\omega\eta}{Q\left(3m^2+2\Omega^2\right)}. \quad (47)$$

In the absence of the frequency nonreciprocity ($\Omega = 0$), this condition coincides with the stability condition (42) for a standing wave. It follows from (47) that in the case of a stable standing wave [i.e., when inequality (42) is fulfilled], the bidirectional stationary lasing is stable for any values of Ω. Therefore, the frequency locking regime for counterpropagating waves exists in a SRL which does not pass to the beating regime with increasing frequency nonreciprocity. In this case, one of the waves is suppressed with increasing Ω without leaving the frequency-locking region. This conclusion well agrees with experimental results [8, 9, 73].

If the linear-coupling strength is insufficient to stabilise a standing wave at $\Omega = 0$, i.e., the inequality

$$m\sin\left|\frac{\vartheta_1-\vartheta_2}{2}\right| < \frac{\omega}{3Q}\,\eta \quad (48)$$

is fulfilled, then for frequency nonreciprocities satisfying the inequality $|\Omega| \leqslant \Omega_1$, the stationary bidirectional lasing is unstable and the self-modulation lasing regime of the first kind appears in a SRL, which passes for $|\Omega| > \Omega_1$ to the stable stationary bidirectional (or unidirectional) lasing regime. The boundary value Ω_1 can be found from the equation

$$m\left|\frac{\sin(\vartheta_1-\vartheta_2)}{2}\right| = m\left(m^2+\Omega_1^2\right)^{1/2}\frac{\omega\eta}{Q\left(3m^2+2\Omega_1^2\right)}. \quad (49)$$

The propagation direction of a strong wave is determined by the sign of Ω and the sign of the phase difference for coupling coefficients: if $\Omega > 0$ and $\vartheta_1 - \vartheta_2 > 0$, then $E_1^2 > E_2^2$.

Consider now the influence of the frequency nonreciprocity of a ring resonator on the nonlinear dynamics for an arbitrary linear coupling. The asymptotic study of stationary lasing in the region of large frequency nonreciprocities (for $|\Omega| \gg m_{1,2}$, $\eta\omega/Q$) was performed in [79]. In this region, a stable stationary regime exists with substantially different intensities of counterpropagating waves. The regime with a strong wave E_1 ($E_1^2 \gg E_2^2$) is stable when $\Omega\sin(\vartheta_2 - \vartheta_1) < 0$. In the case $\Omega\sin(\vartheta_2 - \vartheta_1) > 0$, the regime with a strong wave E_2 is stable ($E_2^2 \gg E_1^2$). Therefore, the propagation direction of radiation in a SRL operating in the travelling-wave regime can be switched by changing the sign of Ω.

Consider now the influence of the frequency nonreciprocity on the stability of unidirectional lasing in the case of a weak linear coupling. As mentioned above, in the absence of the frequency nonreciprocity ($\Omega = 0$) and frequency detuning from the gain-line centre, the travelling-wave regime is stable at a rather weak coupling, $m < m_{cr}$ [see (35)]. The critical value of coupling m_{cr} depends nonmonotonically on the frequency nonreciprocity of the resonator [69]. Figure 4 shows the dependence of $\rho_{cr} = m_{cr}Q/\omega$ on Ω. For $\rho > \rho_{cr}$, the stationary regimes of unidirectional lasing are unstable. The values of ρ_{cr} are minimal for the frequency nonreciprocities corresponding to the degenerate relaxation frequencies [as mentioned above, the degeneracy appears for $\Omega = 0$ ($\rho_{cr} = \rho_{cr}^0$) and $\Omega = \pm\omega_r/2$]. For $\rho < \rho_{cr}$, both travelling-wave regimes, corresponding to the opposite propagation directions of radiation, are stable, i.e., bistability takes place. In the presence of such bistability, the propagation direction of radiation in a ring laser can be spontaneously switched due to the technical fluctuations of laser parameters, which is inadmissible in practical applications. This disadvantage can be easily eliminated by producing unequal intracavity losses for counterpropagating waves. In this case, the unidirectional lasing regime for which losses are minimal in the propagation direction of a strong wave becomes stable.

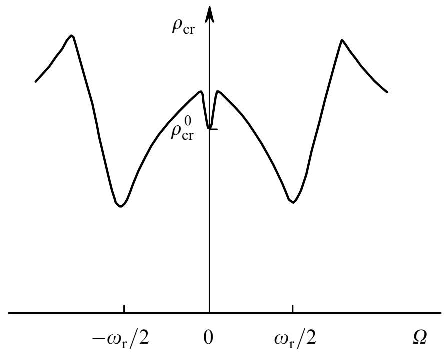

Figure 4. Theoretical dependence of the critical value ρ_{cr} of a linear coupling on Ω in the unidirectional lasing regime in the absence of the frequency detuning from the gain-line centre for $T_1\omega/Q = 5000$ and $\eta = 0.4$ [69].

The unidirectional lasing regime in the presence of the amplitude and frequency nonreciprocities was studied theoretically and experimentally in papers [69–72]. The dependences of the relaxation frequencies and their decay decrements on the frequency nonreciprocity of a ring resonator were investigated. Figure 5 shows the dependences of the relaxation frequencies on Ω in the absence of a linear coupling. The presence of a linear coupling leads to a considerable change in the relaxation frequencies and decay decrements in the vicinity of the point $\Omega = \omega_r/2$ (Fig. 6) where the main relaxation frequency coincides with the frequency $\omega_r^{(1)}$. In this case, the degeneracy of the relaxation

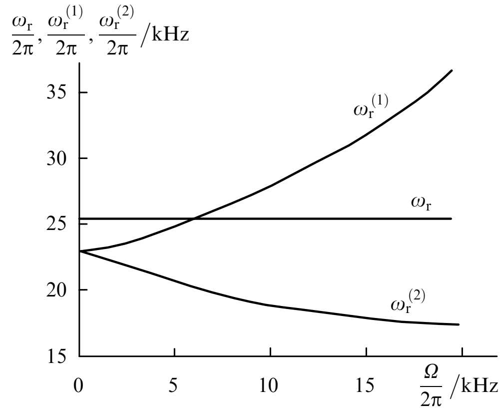

Figure 5. Theoretical dependences of the relaxation frequencies ω_r, $\omega_r^{(2)}$, and $\omega_r^{(1)}$ on Ω in the unidirectional lasing regime in the absence of the linear coupling for $T_1\omega/Q = 5000$ and $\eta = 0.2$ [72].

frequencies [$\omega_r = \omega_r^{(1)}$] takes place within a finite interval of values of Ω, and we can speak about the mutual synchronisation (frequency locking) of relaxation oscillations. The width of the frequency-locking band increases with increasing m. One can see from Fig. 6 that the decay decrements [δ_1 at the main relaxation frequency ω_r and δ_2 at the frequency $\omega_r^{(1)}$] change their sign in the vicinity of the point $\Omega = \omega_r/2$, which means that the travelling-wave regime becomes unstable. In the locking region of relaxation frequencies in the presence of the amplitude nonreciprocity, the chaotic self-modulation of the lasing intensity can appear [71].

The investigations of fluctuations of the radiation intensity and relaxation frequencies in the regime of stationary unidirectional lasing [56, 70–72, 80] have shown that all the experimental results well agree with the theoretical results obtained within the framework of the standard model.

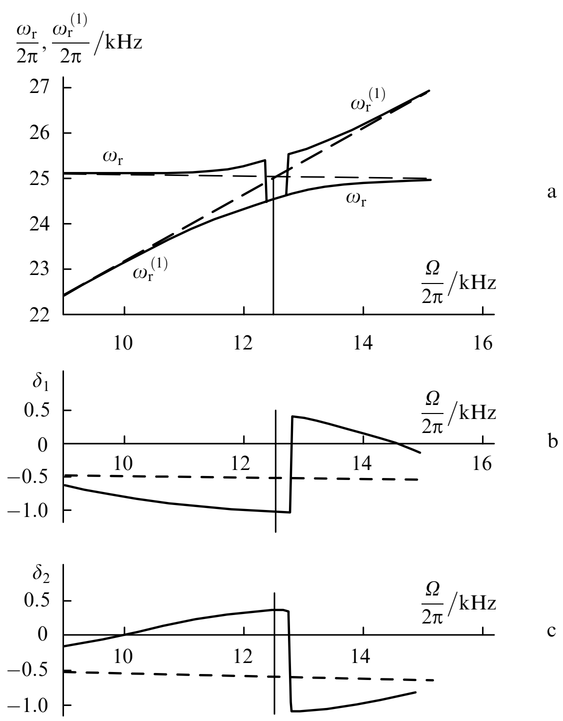

Figure 6. Theoretical dependences of the relaxation frequencies ω_r, $\omega_r^{(1)}$ (a) and decay decrements δ_1 (at the frequency ω_r) and δ_2 [at the frequency $\omega_r^{(1)}$] (b) on Ω in the region of the parametric resonance at $\Omega = \omega_r/2$ in the unidirectional lasing regime for $T_1\omega/Q = 5000$ and $\eta = 0.2$, $\rho = 2\times10^{-3}$ and $\vartheta_1 - \vartheta_2 = 0$ [71]. The dashed straight lines show the decay decrements δ_1 and δ_2 away from the parametric resonance region.

The stationary regime of unidirectional lasing in cw Nd : YAG lasers was experimentally studied in many papers (see, for example, reviews [14, 25, 27] and references therein). The detailed analysis of the technical characteristics of SRL radiation in the stationary unidirectional lasing regime (the radiation amplitude and frequency stability, the generation of high-power single-frequency radiation, etc.) is beyond the scope of this review. We consider here only several papers illustrating different methods for obtaining stationary unidirectional lasing.

The travelling-wave regime in a laser consisting of discrete elements is usually obtained by introducing a nonreciprocal element into the resonator [6, 70, 81] or by applying a magnetic field directly to the active element [82]. One of the methods is based on the use of an external retroreflecting mirror [83]. Unidirectional lasing can be also obtained by employing acousto-optic nonreciprocal elements [37, 84]. Unidirectional lasing in a ring resonator in the presence of frequency nonreciprocity produced by the rotation of the resonator was demonstrated in [73].

The travelling-wave regime in monolithic ring chip lasers is usually achieved by applying a magnetic field to the active element [18–25]. Unidirectional single-frequency lasing appears in this case due to the use of a nonplanar resonator providing the reciprocal rotation of polarisation planes. The spatiotemporal and polarisation characteristics of radiation from a ring laser with a nonplanar resonator were studied in [85, 86].

4.5 Influence of the gain-line structure on the unidirectional lasing regime

As mentioned above, in the presence of the laser-frequency detuning from the gain-line centre, the type of nonlinear coupling of counterpropagating waves on inverse-population gratings changes, resulting in a change in the stability conditions for stationary lasing regimes. In the case of sufficiently large relative detunings ($\delta \sim 1$), even in the absence of a linear coupling between counterpropagating waves, bifurcations can appear which cause the instability of unidirectional lasing and produce regular and chaotic self-oscillations of the intensities of counterpropagating waves [87]. Such detunings, however, are possible only after introducing special selecting elements into the ring resonator.

The unidirectional lasing regime in the absence of a linear coupling between counterpropagating waves is unstable for the relative detunings [53, 54]

$$|\delta| < \delta_{cr} = (1+\eta)\left(\frac{T_1\eta\omega}{Q}\right)^{-1/2}. \qquad (50)$$

If inequality (50) is fulfilled ($|\delta| < \delta_{cr}$), then in the presence of detuning, the critical linear coupling $\rho_{cr} = m_{cr}Q/\omega$ decreases, resulting in the instability of stationary unidirectional lasing [69]. In addition, regimes that differ in the propagation direction of a strong wave have different values of ρ_{cr} (in the presence of detuning and frequency nonreciprocity). Figure 7 shows the dependences of ρ_{cr} on Ω for two stationary lasing regimes for the relative

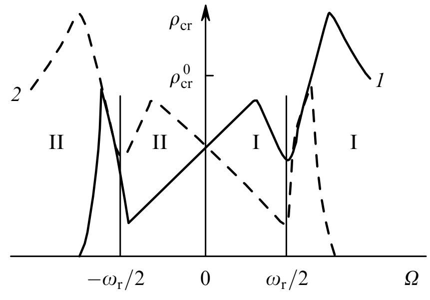

Figure 7. Theoretical dependences of the critical values of the linear coupling on Ω in two unidirectional lasing regimes in the case of the frequency detuning from the gain-line centre for $\delta = 0.1$, $T_1\omega/Q = 5000$, and $\eta = 0.4$ [69].

frequency detuning $\delta = 0.1$. In regions I and II between curves (*1*) and (*2*), only unidirectional lasing is stable: in region I, the wave E_1 is strong, while in region II, the wave E_2 is strong. Below regions I and II, both unidirectional lasing regimes are stable (bistability).

The numerical solution of the system of equations of the standard model (22) shows that the nonstationary regime of low-frequency switching of the propagation direction of radiation (the self-modulation regime of the second kind) appears in a SRL at the relative frequency detunings $|\delta| > \delta_{\rm cr}$. Although this condition is usually not fulfilled in the case of single-mode lasing, such a stationary regime was observed during single-mode lasing in ring Nd : YAG lasers [7, 9]. This contradiction was explained in the SRL model taking into account the real structure of the gain line [55, 56]. The luminescence spectrum of Nd : YAG crystals has two closely spaced lines: a strong line at 1.0639 μm and a weaker line at 1.0643 μm. The distance between these lines at room temperature is of the order of their half-width and their intensities differ approximately by a factor of three. The weak component causes the asymmetry to the total gain line, which is equivalent to the wave frequency detuning from the gain-line centre.

The system of equations describing the nonlinear dynamics of the SRL with two closely spaced gain lines has the form [55, 56, 68, 71]

$$\frac{{\rm d}\tilde{E}_{1,2}}{{\rm d}t} = -\frac{\omega}{2Q_{1,2}}\tilde{E}_{1,2} \pm {\rm i}\,\frac{\Omega}{2}\tilde{E}_{1,2} + \frac{\rm i}{2}\tilde{m}_{1,2}\tilde{E}_{2,1} + \frac{\sigma l}{2T}\big[(N_0 + L'N_0')\tilde{E}_{1,2} + (N_{\mp} + L'N_{\mp}')\tilde{E}_{2,1}\big],$$

$$T_1\frac{{\rm d}N_0}{{\rm d}T} = N_{\rm th}(1+\eta) - N_0\big[1 + a\big(|E_1|^2 + |E_2|^2\big)\big] - N_+aE_1E_2^* - N_-aE_1^*E_2,$$

$$T_1\frac{{\rm d}N_+}{{\rm d}T} = -N_+\big[1 + a\big(|E_1|^2 + |E_2|^2\big)\big] - N_0aE_1^*E_2, \qquad (51)$$

$$T_1\frac{{\rm d}N_0'}{{\rm d}T} = N_{\rm th}(1+\eta') - N_0' - {\rm Re}L'\big[N_0'a\big(|E_1|^2 + |E_2|^2\big) + N_+'aE_1E_2^* + N_-'aE_1^*E_2\big],$$

$$T_1\frac{{\rm d}N_+'}{{\rm d}T} = -N_+' - {\rm Re}L'\big[N_+'a\big(|E_1|^2 + |E_2|^2\big) + N_0'aE_1^*E_2\big].$$

Along with variables used in equations (22) of the standard model, these equations also contain variables N_0' and $N_\pm'$, which are the constant component and the complex amplitudes of the spatial harmonics of the inverse population related to the weak line, $L' = 1/(1 + {\rm i}\delta')$, and the relative distance between the centres of the lines $\delta' = (\omega_0 - \omega_0')/T_2$. The different intensities of luminescence lines are simulated by different pump rates $N_{\rm th}\alpha/T_1$ and $N_{\rm th}\alpha'/T_1$.

In the absence of the linear coupling of counterpropagating waves, the stationary solution of this system of equations corresponding to unidirectional lasing is described by the expressions

$$aE_2^2 = \frac{1+\delta'^2}{2}\left\{\left[\frac{4}{1+\delta'^2}(\alpha + \alpha'{\rm Re}L' - 1) + \big[1 - (\alpha + \alpha' - 1){\rm Re}L'\big]^2\right]^{1/2} - \big[1 - (\alpha + \alpha' - 1){\rm Re}L'\big]\right\},$$

$$aR_1^2 = 0,$$

$$n_0 \equiv \frac{\sigma l N_0}{T}\left(\frac{\omega}{Q}\right)^{-1} = \frac{\alpha}{1 + aE_2^2}, \qquad (52)$$

$$n_0' \equiv \frac{\sigma l N_0'}{T}\left(\frac{\omega}{Q}\right)^{-1} = \frac{\alpha'}{1 + {\rm Re}L'aE_2^2},$$

where $\alpha = 1 + \eta$ and $\alpha' = 1 + \eta'$.

The study of the stability of unidirectional lasing in the absence of linear coupling between counterpropagating waves leads to the characteristic equation [68]

$$\lambda^3T_1^3 + \big[2 + aE_2^2(1 + {\rm Re}L')\big]\lambda^2T_1^2 + \lambda T_1 \times\left[\big(1 + aE_2^2\big)\big(1 + {\rm Re}L'aE_2^2\big) + \frac{\omega T_1}{2Q}\,aE_2^2\big(n_0 + n_0'L'{\rm Re}L'\big)\right] + \frac{\omega T_1}{2Q}\,aE_2^2\big[n_0\big(1 + {\rm Re}L'aE_2^2\big) + n_0^2L'{\rm Re}L'\big(1 + aE_2^2\big)\big] = 0. \qquad (53)$$

The unidirectional lasing regime (52) proves to be unstable (${\rm Re}\lambda > 0$) if the condition

$$\frac{\omega T_1}{2Q}\,aE_2^2n_0'\big({\rm Re}L'\big)^2F\delta' \geqslant F^2\big[2 + aE_2^2\big(1 + {\rm Re}L'\big)\big] - \frac{\omega T_1}{2Q}\,aE_2^2\big[n_0\big(1 + {\rm Re}L'aE_2^2\big) + n_0'\big({\rm Re}L'\big)^2\big(1 + aE_2^2\big)\big] \qquad (54)$$

is fulfilled, where

$$F^2 = \frac{\omega T_1}{2Q}\big[n_0 + n_0'\big({\rm Re}L'\big)^2\big]aE_2^2.$$

Figure 8 shows the dependence of the stability-region boundary for stationary unidirectional lasing (solid curve) on the parameters α' and δ' calculated in [68] from (52) and (54). The point R with the coordinates $\delta' = 1$ and $\alpha' = \alpha/3$ corresponding to the parameters of a weak

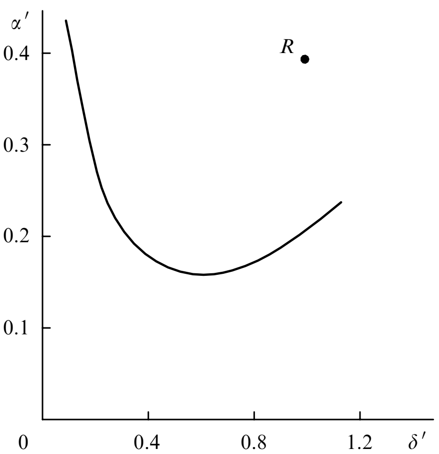

Figure 8. Stability boundaries of unidirectional lasing in the (δ', α') plane of the SRL model parameters with the two-component gain line for $T_1\omega/Q = 5000$, $\eta = 0.2$, $\rho = 10^{-3}$, and $\vartheta_1 - \vartheta_2 = 0$ [68].

line in a Nd : YAG laser is located in the instability region. Therefore, if the real structure of the gain line is taken into account, the unidirectional lasing in a ring Nd : YAG laser proves to be unstable in the absence of optical nonreciprocity and linear coupling. In the standard SRL model, such instability takes place for tunings $|\delta| > \delta_{cr}$. In the instability region (above the boundary), the self-modulation regime of the second king appears (see below).

5. Periodic self-modulation lasing regimes

The analysis performed in the previous section shows that there exist the regions of laser parameters where all the stationary lasing regimes prove to be unstable. For example, in the case of lasing at the gain-line centre ($\delta = 0$) and a small pump excess over the lasing threshold ($\eta \ll 1$), all the stationary regimes are unstable if the inequalities

$$m\left|\cos\frac{\vartheta_1 - \vartheta_2}{2}\right| > \frac{1}{2}\left(\frac{\eta\omega}{QT_1}\right)^{1/2},$$
$$m|\sin(\vartheta_1 - \vartheta_2)| < \frac{\omega}{3Q}\,\eta \tag{55}$$

are fulfilled. Recall that these inequalities are written in a particular case of equal moduli of the linear coupling coefficients ($m = |\tilde{m}_1| = |\tilde{m}_2|$) in the absence of the amplitude and frequency nonreciprocities ($\omega/Q = \omega/Q_1 = \omega/Q_2$, $\Omega = \omega_1 - \omega_2 = 0$). In the case of sufficiently large frequency detunings satisfying the condition $|\delta| > \delta_{cr}$, all the stationary states are unstable even in the absence of linear coupling between counterpropagating waves (see Section 4.5). In these regions, nonstationary lasing regimes appear, among which the most interesting are self-modulation regimes of the first and second kinds, quasi-periodic regimes, beatings of counterpropagating waves, and the dynamic chaos regime.

5.1 Self-modulation of intensity of counterpropagating waves

The simplest nonstationary regimes are periodic regimes, among which the most important is the self-modulation regime of the first kind, in which the intensities of counterpropagating waves experience out-of-phase sinusoidal oscillations. This regime was first studied by solving numerically equations (22) of the standard model in [52] and was experimentally observed in [9]. The frequency ω_m of self-modulation oscillations is determined by the strength of the linear coupling of counterpropagating waves and the nonreciprocity of the ring resonator. In the absence of optical nonreciprocities, the frequency ω_m can vary from tens of kilohertz to a few megahertz.

Consider first the features of the nonlinear dynamics in the self-modulation regime of the first kind in the absence of the amplitude nonreciprocity of the resonator ($\Delta = 0$). The region of existence of the self-modulation regime of the first kind in the plane of parameters $mQ/\eta\omega$ and $\vartheta_1 - \vartheta_2$ characterising the value and phase of the linear coupling coefficients is shown in Fig. 9. This region is widest when the coupling coefficients are close to the complex conjugate ones ($|\vartheta_1 - \vartheta_2| \ll 1$). The region narrows down with increasing the phase difference of the coupling coefficients, and the existence of self-modulation oscillations of the first kind becomes impossible when $|\vartheta_1 - \vartheta_2| \to \pi$.

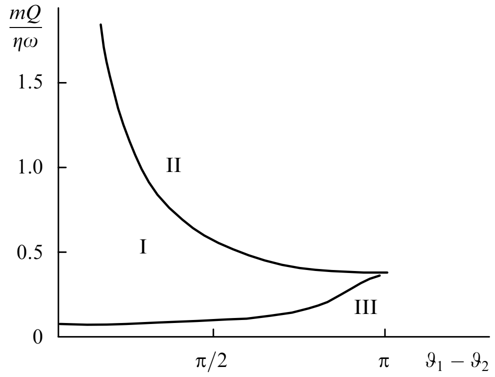

Figure 9. Regions of existence of stationary standing-wave (II) and unidirectional lasing (III) regimes and the self-modulation regime of the first kind (I) in the plane of parameters $\vartheta_1 - \vartheta_2$, $mQ/\omega\eta$ [27].

The system of equations of the standard model in the case of periodic regimes can be solved analytically [74, 78, 88–90] assuming that the self-modulation frequency ω_m is high compared to the main relaxation frequency $\omega_r = (\omega\eta/QT_1)^{1/2}$. In this case, the method of successive approximations in a small parameter

$$\varepsilon = \frac{\omega_r}{\omega_m} \ll 1 \tag{56}$$

can be used.

In the zero-order approximation, the inverse-population modulation at the frequency ω_m is neglected and only the constant components $\bar{N}_0$ and $\bar{N}_\pm$ of the spatial harmonics N_0 and $N_\pm$ of the inverse population are taken into account. In this approximation, the complex amplitudes are described by the system of two first-order differential equations with constant coefficients

$$\frac{d\tilde{E}_{1,2}}{dt} = -\frac{\omega}{2Q}\,\tilde{E}_{1,2} \pm i\,\frac{\Omega}{2}\,\tilde{E}_{1,2} + \frac{i}{2}\,\tilde{m}_{1,2}\tilde{E}_{2,1}$$
$$+ \frac{\sigma l}{2T}(1 - i\delta)\big(\bar{N}_0\tilde{E}_{1,2} + \bar{N}_\mp\tilde{E}_{2,1}\big). \tag{57}$$

The constant components $\bar{N}_0$ and $\bar{N}_\pm$ can be determined from Eqns (22) for the spatial harmonics N_0 and $N_\pm$ of the inverse population, which are reduced to algebraic equations in the zero-order approximation. The complex fields of counterpropagating waves in the zero-order approximation in ε can be written in the form

$$\tilde{E}_1\exp(i\omega_n t) = A_1\exp(i\omega_1 t) + B_1\exp(i\omega_2 t),$$
$$\tilde{E}_2\exp(i\omega_n t) = A_2\exp(i\omega_1 t) + B_2\exp(i\omega_2 t), \tag{58}$$

where A_i and B_i ($i = 1, 2$) are constant coefficients and ω_n is the eigenfrequency of the ring resonator in the absence of the linear coupling between counterpropagating waves. According to (58), the emission spectrum of each of the counterpropagating waves contains two spectral components with frequencies ω_1 and ω_2. The self-modulation oscillations of the first kind can be treated as beatings between two eigenfrequencies appearing in the ring resonator in the presence of the linear coupling between counterpropagating waves [57, 58]. The self-modulation frequency ω_m proves to be equal to the difference between these eigenfrequencies.

The frequencies ω_1 and ω_2 of the spectral components in the established periodic regime should be real quantities. By solving the zero-order equations and taking into account that ω_1 and ω_2 are real, we can obtain the following expressions for these frequencies and the self-modulation oscillation frequency ω_m [88]:

$$\omega_1 = \omega_n - \frac{\delta\omega}{Q} - \frac{\omega_m}{2}, \quad \omega_2 = \omega_n - \frac{\delta\omega}{Q} + \frac{\omega_m}{2}, \tag{59}$$

$$\omega_m = \left(\omega_{m0}^2 + \Omega^2\right)^{1/2}, \tag{60}$$

where ω_{m0} is the self-modulation oscillation frequency in the absence of the optical nonreciprocity of the resonator, which is described by the expression

$$\omega_{m0}^2 = m_1 m_2 \cos(\vartheta_1 - \vartheta_2) + \frac{(1+\delta^2) m_1^2 m_2^2 \sin^2(\vartheta_1 - \vartheta_2)}{m_1^2 + m_2^2 + 2m_1 m_2 \cos(\vartheta_1 - \vartheta_2)}$$
$$-\delta m_1 m_2 \sin(\vartheta_1 - \vartheta_2). \tag{61}$$

According to the above expressions, the self-modulation frequency ω_m depends on the moduli and phases of the linear coupling coefficients of counterpropagating waves, the relative detuning of the lasing frequency from the gain-line centre $\delta = (\omega - \omega_0)T_2$, and the frequency nonreciprocity Ω of the ring resonator. As Ω is increased, the self-modulation frequency ω_m increases, approaching Ω. In the case of a symmetric feedback, when $m_1 = m_2 = m$, the expression for ω_{m0} is considerably simplified:

$$\omega_{m0} = m\left|\cos\frac{\vartheta_1 - \vartheta_2}{2} - \delta \sin\frac{\vartheta_1 - \vartheta_2}{2}\right|. \tag{62}$$

By taking into account small terms of the order of ε^2, we obtain the correction to the self-modulation frequency, and the expression for ω_m takes the form [89, 90]

$$\omega_m = \left(\omega_{m0}^2 + \Omega^2\right)^{1/2} + \Delta\omega_m, \tag{63}$$

where

$$\Delta\omega_m = \frac{\omega_r^2\left(\Omega^2 + \omega_{m0}^2\right)}{4\omega_{m0}^3}. \tag{64}$$

According to (63) and (64), the self-modulation oscillation frequency depends on the pump excess η over the lasing threshold.

Taking into account expressions (58) for the fields, we can represents the dimensionless intensities $I_{1,2} = a\tilde{E}_{1,2}\tilde{E}_{2,1}^*$ of counterpropagating waves in the form

$$I_{1,2} = I_{1,2}^0 \pm I_{1,2}^m \cos(\omega_m t + \varphi_{1,2}), \tag{65}$$

where $I_{1,2}^0$ are the constant components (average values) and $I_{1,2}^m$ are the intensity modulation amplitudes of counterpropagating waves. In the absence of the amplitude nonreciprocity of the ring laser in the self-modulation regime of the first kind, the intensities of counterpropagating waves are modulated strictly out of phase, which corresponds to $\varphi_1 = \varphi_2$ in (65). In the presence of the amplitude nonreciprocity, the additional phase shift $\Delta\varphi \equiv \varphi_1 - \varphi_2$ of self-modulation oscillations appears, which is described by the expression [89]

$$\sin\Delta\varphi = \frac{2\omega_m \Delta}{\left[\left(\Omega^2 - \omega_m^2 + \Delta^2\right)^2 + 4\Delta^2\omega_m^2\right]^{1/2}}, \tag{66}$$

where Δ is the amplitude nonreciprocity of the ring resonator. In the case of a sufficiently small amplitude nonreciprocity ($\Delta \ll \omega_m$), expression (66) is simplified to

$$\sin\Delta\varphi = \frac{2\omega_m \Delta}{\omega_{m0}^2}. \tag{67}$$

For $\Delta\varphi \neq 0$, the intensity modulation is no longer strictly out of phase (Fig. 10). By measuring the phase shift $\Delta\varphi$ of self-modulation oscillations, we can determine from (66) the amplitude nonreciprocity Δ because the self-modulation frequencies ω_m and ω_m^2 are measured directly in experiments. This was demonstrated in [91]. The amplitude and frequency nonreciprocities were controlled in [91] by applying an external magnetic field to the single crystal of a ring chip laser. It is reasonable to assume that the frequency and amplitude nonreciprocities of the ring laser are related to its parameters and the magnetic field strength H as

$$\Omega = k_1 H, \tag{68}$$

$$\Delta = \frac{1}{2}\left(\frac{\omega_1}{Q_1} - \frac{\omega_2}{Q_2}\right) = k_2 H + \Delta_0, \tag{69}$$

where ω_1 and ω_2 are the lasing frequencies for counterpropagating waves; k_1 and k_2 are the coefficients depending on the magnetic-field orientation with respect to the resonator contour, its nonplanarity, and other parameters; and Δ_0 is the amplitude nonreciprocity of the ring resonator for $H = 0$. Note that the coefficients k_1, k_2, and Δ_0 for a particular resonator can be calculated by using the formalism of Jones matrices.

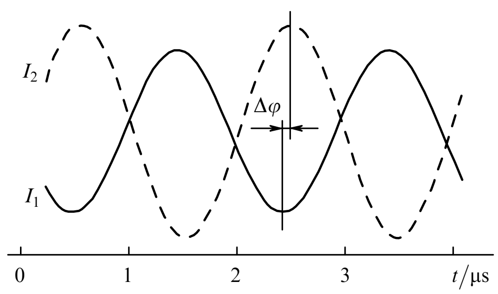

Figure 10. Oscillograms of the intensities of counterpropagating waves in the self-modulation regime of the first kind in the presence of the amplitude nonreciprocity of the resonator [91].

It follows from (67) that

$$\frac{\sin\Delta\varphi}{\omega_m} = \frac{2(\Delta_0 + k_2 H)}{\omega_{m0}^2}, \tag{70}$$

i.e., the ratio $(\sin\Delta\varphi)/\omega_m$ linearly depends on H. A comparison of the experimental and calculated dependences of $\sin\Delta\varphi$ on the magnetic field strength in Fig. 11 demonstrates their good agreement.

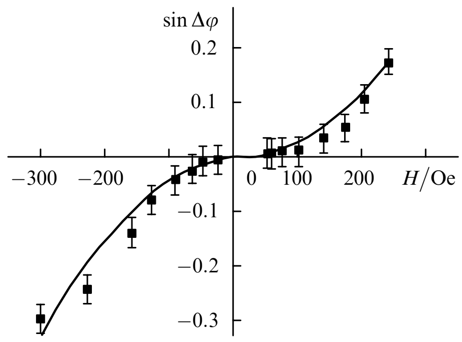

Figure 11. Experimental (squares) and theoretical (solid curve) dependences of $\Delta\varphi$ on the magnetic field strength H [91].

The average values $I_{1,2}^{0}$ and the intensity modulation amplitudes $I_{1,2}^{\mathrm{m}}$ for counterpropagating waves depend on the linear coupling coefficients, the pump excess over the threshold, and the optical nonreciprocity of the ring resonator. Because in the general case the expressions determining this dependence are rather cumbersome (see, for example, [89, 92]), they are not presented here. As the frequency nonreciprocity Ω is increased, the difference of the average intensities of counterpropagating waves increases and the modulation depth monotonically decreases. As a result, for sufficiently large values of Ω satisfying the inequality $|\Omega| > \Omega_1$, where Ω_1 is determined by expression (49), the self-modulation regime of the first kind passes to the stationary regime with unequal intensities of counterpropagating waves.

Figure 12 shows the average intensities of counterpropagating waves and the region of intensity variation during self-modulation oscillations as functions of the frequency nonreciprocity of the resonator. These dependences were calculated in [92] for the typical parameters of ring Nd : YAG lasers. Note that Fig. 12 also shows the dependences of the average intensities on Ω when polarisations of counterpropagating waves are different. Figure 13 presents the experimental dependences [60] of the average intensities and the depth of out-of-phase intensity modulation of counterpropagating waves on the solenoid current producing a magnetic field (the frequency nonreciprocity of the resonator) in the monolithic ring Nd : YAG laser. In the presence of the amplitude nonreciprocity of the ring resonator, more complicated (loop) dependences of the average intensities of counterpropagating waves on the frequency nonreciprocity of the resonator are observed [27, 74].

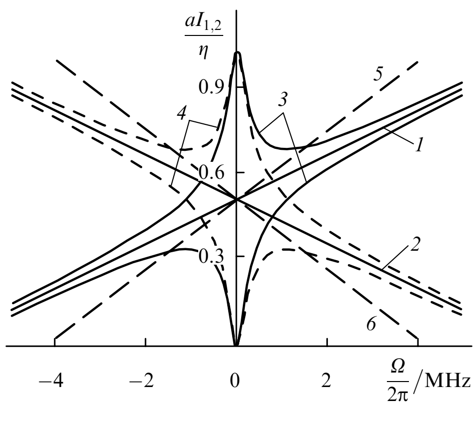

Figure 12. Theoretical dependences of the average intensities of counterpropagating waves I_1 (*1*) and I_2 (*2*) and the ranges of their variation $I_1 \pm I_1^{\mathrm{m}}$ (*3*) and $I_2 \pm I_2^{\mathrm{m}}$ (*4*) during self-modulation oscillations on the frequency nonreciprocity Ω of the resonator for $m/2\pi = 200$ kHz, $\vartheta_1 - \vartheta_2 = 0.3$, $\eta = 0.1$, $\delta = 0$, and $\Delta = 0$ in the case of the same polarisation of counterpropagating waves ($\boldsymbol{e}_1 = \boldsymbol{e}_2$) and similar dependences for I_1 (*5*) and I_2 (*6*) for $\boldsymbol{e}_1\boldsymbol{e}_2 = 0.8$ [92].

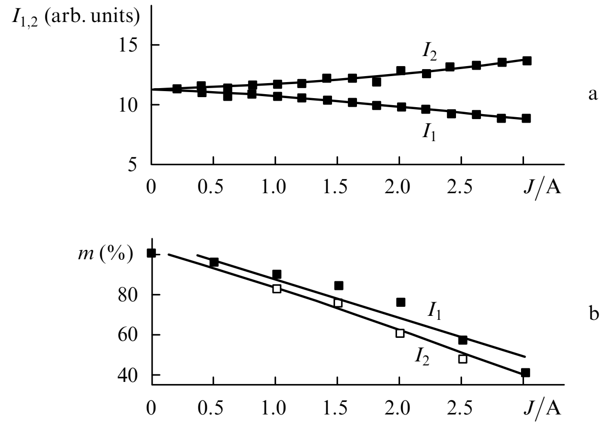

Figure 13. Influence of the solenoid current (the frequency nonreciprocity of a ring resonator) on the average intensities (a) and depths (b) of the out-of-phase modulation of counterpropagating waves in the self-modulation regime of the first kind for $\eta = 0.06$ [60].

Aside from the self-modulation regime of the first kind, another periodic lasing regime is possible in a SRL, in which, unlike the self-modulation regime of the first kind, the in-phase intensity modulation of counterpropagating waves takes place [88, 93]. In the absence of the amplitude nonreciprocity of the resonator, the in-phase self-intensity modulation is possible when the inequality

$$\Omega_0^2 = -m_1 m_2 \cos(\vartheta_1 - \vartheta_2) - \frac{(1+\delta^2) m_1^2 m_2^2 \sin^2(\vartheta_1 - \vartheta_2)}{m_1^2 + m_2^2 + 2 m_1 m_2 \cos(\vartheta_1 - \vartheta_2)}$$

$$+ \delta m_1 m_2 \sin(\vartheta_1 - \vartheta_2) > 0 \qquad (71)$$

is fulfilled. For $\delta = 0$, this inequality can be fulfilled only when the moduli of the linear coupling coefficients are different ($m_1 \neq m_2$) and the phase difference for the coupling coefficients is $|\vartheta_1 - \vartheta_2| > \pi/2$. The presence of the amplitude nonreciprocity of the resonator considerably alleviates the appearance of the in-phase regime. The in-phase self-intensity modulation exists within a limited region of frequency nonreciprocities $|\Omega| > \Omega_0$. The self-modulation frequency in this regime is determined by the expression

$$\omega_{\mathrm{m}} = \left(\Omega^2 - \Omega_0^2\right)^{1/2}. \qquad (72)$$

The difference between the coupling coefficients of counterpropagating waves appearing in the scheme with an additional external mirror leads under certain conditions to the passage from the self-modulation regime of the first kind to the periodic regime with in-phase self-modulation [93]. As a rule, the average intensities of counterpropagating waves in this regime are substantially different because of the inequality of the moduli of coupling coefficients or

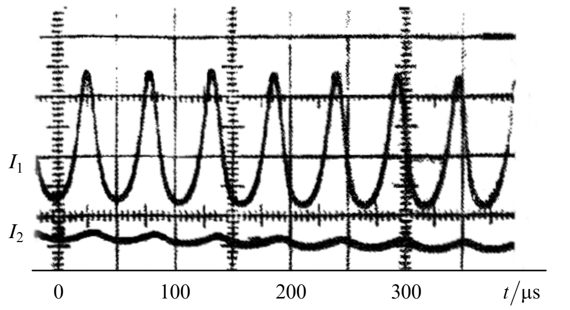

Figure 14. Oscillograms of the intensities I_1 and I_2 of counterpropagating waves in the periodic in-phase modulation regime [27].

resonator Q factors (Fig. 14). Note that the in-phase self-modulation regime remains inadequately studied at present.

5.2 Self-modulation of optical phases of counterpropagating waves

In most of the papers on the nonlinear dynamics of SRLs, the temporal and spectral characteristics of the intensity of counterpropagating waves were studied. However, the phase dynamics of counterpropagating waves was studied inadequately. This is explained by the fact that the direct measurement of optical phases of counterpropagating waves and their difference is a rather complicated technical problem. Nevertheless, the phase dynamics of ring lasers in nonstationary lasing regimes was experimentally studied in some papers. Such studies were performed by two methods: either by mixing (heterodyning) the radiation under study with radiation from another laser [94] or by mixing the optical fields of counterpropagating waves [95, 96].

The phase dynamics of SRLs in periodic nonstationary lasing regimes can be analysed by using standard model (22). Consider the dynamics of the phase difference $\Phi = \varphi_2 - \varphi_1$ for counterpropagating waves in the self-modulation regime of the first kind. It follows directly from the equation for the phase difference of counterpropagating waves that the instant phase difference $\mathrm{d}\Phi/\mathrm{d}t$ of counterpropagating waves depends on the optical nonreciprocity of the ring resonator and the linear and nonlinear couplings of counterpropagating waves. In the self-modulation regime of the first kind, both linear and nonlinear couplings lead to the self-modulation of the phase difference of counterpropagating waves. In the case of a sufficiently strong linear coupling, when the self-modulation oscillation frequency is much higher than the relaxation oscillation frequency ω_r, the phase modulation is mainly determined by the linear coupling of counterpropagating waves. In this case, by neglecting the nonlinear coupling on inverse-population gratings and assuming that the linear coupling coefficients are close to the complex conjugate ones ($m_1 = m_2 = m,\ |\vartheta_1 - \vartheta_2| \ll 1$), we obtain from (22) the equation for Φ

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Omega + \frac{m}{2}\frac{E_1^2 - E_2^2}{E_1E_2}\cos(\Phi + \vartheta), \tag{73}$$

where $\vartheta = (\vartheta_1 + \vartheta_2)/2$.

In the self-modulation regime of the first kind, we can obtain from (22) in the approximation used here the following expressions [78]

$$E_{1,2}^2 = \frac{B}{2} \pm \left[\frac{\Omega}{2\omega_m}\left(B^2 - A^2\right)^{1/2} + \frac{m}{2\omega_m}A\sin(\omega_m t + \varphi)\right],$$
$$E_1E_2\sin(\Phi + \vartheta) = \frac{A}{2}\cos(\omega_m t + \varphi). \tag{74}$$

Here, $\omega_m = (m^2 + \Omega^2)^{1/2}$; φ is an arbitrary constant, and constants A and B are determined from the expressions

$$aB = 2\eta(1 - K), \quad a\left(B^2 - A^2\right)^{1/2} = 4\eta K, \tag{75}$$

where

$$K = \frac{m|\vartheta_1 - \vartheta_2|}{2(\omega/Q)\eta}.$$

By substituting (74) into (73), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Bigg\{A\Omega - m\left(B^2 - A^2\right)^{1/2}\sin(\omega_m t + \varphi)$$
$$\times\left\{B^2 - \left[\frac{\Omega}{\nu}\left(B^2 - A^2\right)^{1/2} + \frac{m}{\nu}A\sin(\omega_m t + \varphi)\right]^2\right\}^{-1}\Bigg\}. \tag{76}$$

According to (76), $\mathrm{d}\Phi/\mathrm{d}t$ is an oscillating function of time. By averaging expression (76) in time, we obtain the average difference of frequencies of counterpropagating waves $\langle \mathrm{d}\Phi/\mathrm{d}t\rangle$ (beat frequencies) in the form

$$\langle \mathrm{d}\Phi/\mathrm{d}t\rangle = 0, \text{ if } |\Omega| \leqslant \omega_m\left(1 - \frac{A^2}{B^2}\right)^{1/2},$$
$$\langle \mathrm{d}\Phi/\mathrm{d}t\rangle = \omega_m \operatorname{sign}\Omega, \text{ if } |\Omega| > \omega_m\left(1 - \frac{A^2}{B^2}\right)^{1/2}. \tag{77}$$

One can see from the above expressions that in the case of sufficiently small frequency nonreciprocities in the self-modulation regime of the first kind, the average frequencies of counterpropagating waves are synchronised. In this case, the phase difference of counterpropagating waves is an oscillating function of time and changes within a finite range. As the frequency nonreciprocity is increased, the nonzero average frequency difference increases jumpwise, while the phase difference of counterpropagating waves increases infinitely with time. If the coupling coefficients are close to the complex conjugate ones, the critical value Ω_{cr} at which the average frequency difference experiences a jump is determined by the expression

$$\Omega_{cr} = 2\,\frac{m^2|\vartheta_1 - \vartheta_2|}{(\omega/Q)\eta}. \tag{78}$$

Figure 15 shows the time dependences of the intensity and phase difference of counterpropagating waves in the regime of self-modulation oscillations of the first kind for $\Omega = 0$. One can see that the intensity and phase difference of counterpropagating waves are periodic functions of time. The phase difference Φ remains constant during a greater part of the period of self-modulation oscillations and drastically changes by π (or $-\pi$) in the time interval when the intensity of one of the waves is close to zero. Figure 16 shows the dependences of the intensity and phase difference of counterpropagating waves for the frequency nonreciprocity Ω exceeding Ω_{cr}.

Figure 17 presents the dependence of the average frequency difference $\langle \mathrm{d}\Phi/\mathrm{d}t\rangle$ for counterpropagating waves on

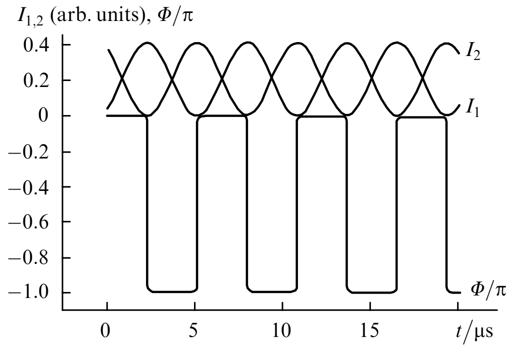

Figure 15. Theoretical time dependences of the intensities I_1 and I_2 and the optical phase difference $\Phi = \varphi_1 - \varphi_2$ for counterpropagating waves in the self-modulation oscillation regime of the first kind for $\Omega = 0$ (numerical simulation).

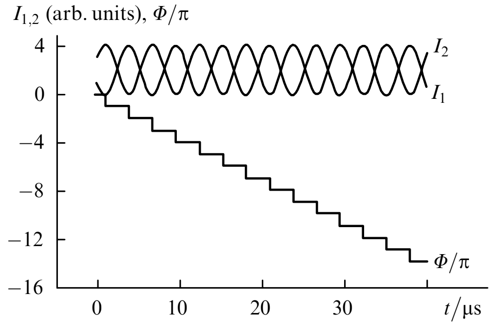

Figure 16. Theoretical time dependences of the intensities I_1 and I_2 and the optical phase difference $\Phi = \varphi_1 - \varphi_2$ for counterpropagating waves in the self-modulation oscillation regime of the first kind in the presence of the frequency nonreciprocity Ω exceeding Ω_{cr} (numerical simulation).

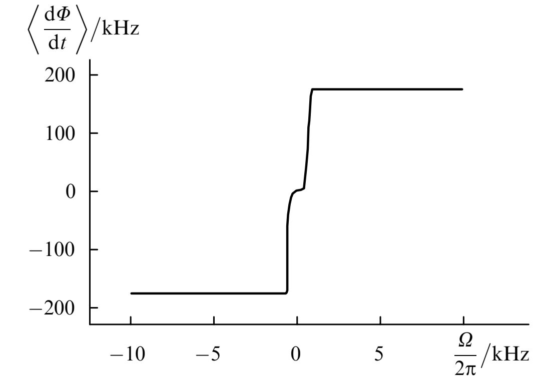

Figure 17. Dependence of the average frequency difference $\langle d\Phi/dt \rangle$ of counterpropagating waves on the frequency nonreciprocity (numerical simulation).

the frequency nonreciprocity Ω obtained by solving numerically the system of equations of the standard model. One can see from these results that the jump of the frequency difference is observed at the frequency $\Omega_{cr}/2\pi$ approximately equal to 100 Hz, in good agreement with expression (78).

5.3 Relaxation oscillations in the self-modulation regime of the first kind

Because of a slow relaxation of the inverse population, the establishment of self-modulation oscillations has an oscillatory character. Relaxation oscillations in a SRL operating in the self-modulation regime of the first kind noticeably differ from relaxation oscillations during stationary unidirectional lasing.

To analyse relaxation processes in the self-modulation regime, it is necessary to consider the dynamics of small perturbations with respect to the established periodic regime. In the linear approximation in small perturbations, we obtain a system of linear differential equations with periodic coefficients. These equations can be solved by the method of successive approximations in a small parameter $\varepsilon = \omega_r/\omega_m \ll 1$. In this case, as shown in [97, 98], the periodic coefficients of the equations can be replaced by their average values to obtain a system of equations with constant coefficients. It follows from the characteristic equation for the system of equations obtained in this way that there exist three characteristic frequencies in the case under study. One of the frequencies is equal to the frequency of self-modulation oscillations and the two others to the relaxation frequencies. One of the relaxation frequencies (main) is determined by the expression

$$\omega_{r0} = \left(\frac{\omega}{Q} \frac{\eta}{T_1} \right)^{1/2} \tag{79}$$

and coincides with the main relaxation frequency in the stationary unidirectional lasing regime. The second relaxation frequency is determined by the expression

$$\omega_{r1}^2 = \frac{1}{2} \left[\omega_m^2 + \omega_{r0}^2 - \left(\omega_m^4 + 2\omega_{r0}^2 \Omega^2 \right)^{1/2} \right]. \tag{80}$$

The frequency ω_{r1} is always lower than the main relaxation frequency ω_r. In the absence of the frequency nonreciprocity ($\Omega = 0$), the frequency ω_{r1} has the maximum value $(\omega_{r1})_{max} = \omega_r/\sqrt{2}$.

Note that in the SRL model [60] taking into account the peculiarities of the interaction of elliptically polarised counterpropagating waves in lasers with a nonplanar resonator (the so-called vector model), the expression for the relaxation frequency ω_{r1} is somewhat different. In particular, the maximum value $(\omega_{r1})_{max}$ achieved in the absence of the frequency nonreciprocity of the resonator ($\Omega = 0$) depends on the polarisation of radiation, and according to [60], $(\omega_{r1})_{max} = \omega_r/1.6$ for elliptically polarised waves. Figure 18 presents the theoretical and experimental dependences of the relaxation frequencies ω_r and ω_{r1} on the frequency nonreciprocity of the ring laser [60]. One can see that the dependence of ω_{r1} on Ω in the vector model well

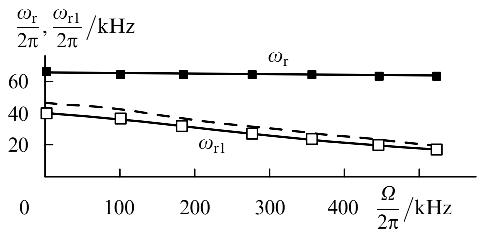

Figure 18. Theoretical and experimental dependences [60] of the relaxation oscillation frequencies ω_r and ω_{r1} on the frequency nonreciprocity of a ring resonator. The dashed curve shows the dependence of ω_{r1} on Ω described by expression (80).

agrees with dependence (80) obtained within the framework of the standard model.

The relaxation frequencies ω_r and ω_{r1} are usually observed in the emission spectra of SRLs operating in the self-modulation regime of the first kind. Figure 19 presents the output emission spectra in the presence and absence of the frequency nonreciprocity Ω. In the absence of the frequency nonreciprocity, the frequency ω_{r1} is not observed in the spectra, as a rule, and the spectrum (Fig. 19a) exhibits in this case three peaks at the frequencies ω_r, ω_m, and the combination frequency $\omega_m - \omega_r$. The frequency ω_{r1} in the absence of the frequency nonreciprocity can be observed in the emission spectrum upon the periodic modulation of laser parameters. In the presence of the frequency nonreciprocity, the frequency ω_{r1} can be also observed without modulation of laser parameters, which is demonstrated in Fig. 19b, where the emission spectrum is shown in a narrower frequency range and exhibits peaks only at the frequencies ω_{r1} and ω_r.

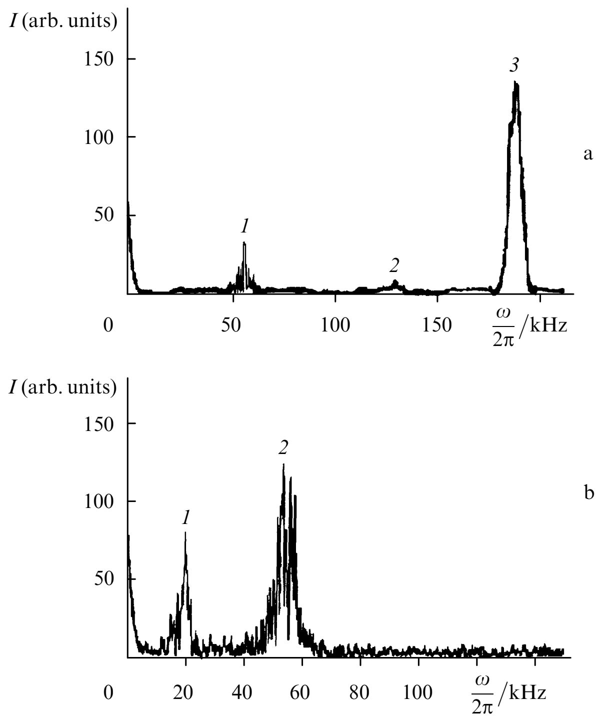

Figure 19. Output emission spectra in the absence (a) and presence (b) of the frequency nonreciprocity. Peak (*1*) in Fig. 19a corresponds to the main relaxation frequency ω_r, peak (*2*) – to the combination frequency $\omega_m - \omega_r$, and peak (*3*) – to the self-modulation oscillation frequency ω_m. Peak (*1*) in Fig. 19b corresponds to the frequency ω_{r1} and peak (*2*) – to the frequency ω_r [97].

The theoretical and experimental dependences of the self-modulation frequency ω_m and relaxation frequency ω_{r1} on the frequency nonreciprocity of the resonator are presented in Fig. 20. Note that expressions (79) and (80) for relaxation frequencies are approximate because they were obtained in the limiting case of $\varepsilon = \omega_r/\omega_m \ll 1$. For an arbitrary relation between ω_m and ω_r, the expression for the main relaxation frequency has the form [99]

$$\omega_r^2 = \frac{1}{2}\left\{\omega_m^2 + \omega_{r0}^2 - \left[\left(\omega_m^2 + \omega_{r0}^2\right)^2 + \omega_{r0}^4 - 4\omega_{r0}^2\omega_m^2\right]^{1/2}\right\}. \quad (81)$$

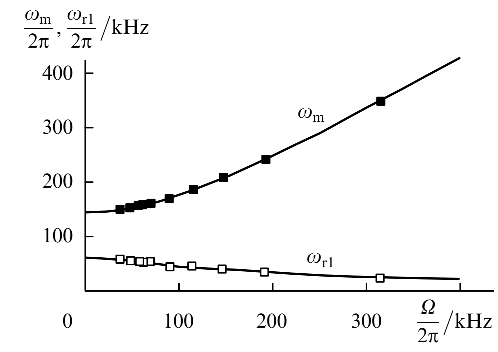

Figure 20. Theoretical and experimental dependences of the self-modulation oscillation frequency ω_m and relaxation frequency ω_{r1} on the frequency nonreciprocity of the resonator [97].

According to this expression, the frequency ω_r depends on the self-modulation frequency ω_m. In the case $\varepsilon = \omega_r/\omega_m \ll 1$, it follows from (81) that $\omega_r = \omega_{r0}$, in agreement with expression (79).

The theoretical and experimental dependences of the main relaxation frequency on the self-modulation oscillation frequency [99] are compared in Fig. 21. Two series of data obtained for two different pump levels are presented. One can see that expression (81) well describes the dependence of the main relaxation frequency on the self-modulation oscillation frequency over the entire region of ω_m, except two regions of parametric resonances where the self-modulation regime of the first kind proves to be unstable. One of the regions of the parametric resonance is observed for $\omega_m = 2\omega_{r0}$. In this region, the parametric synchronisation of the frequencies of relaxation and self-modulation oscillations takes place at the frequency $\omega_r = \omega_m/2$ and self-excitation of relaxation oscillations occurs at the main relaxation frequency [curve (*6*)]. In the second region, the instability of self-modulation oscillations appears upon the parametric resonance $\omega_m = 2\omega_{r1} = 2\omega_{r0}/\sqrt{2}$, which also causes the synchronisation and self-excitation of relaxation oscillations. In this case, the main relaxation frequency is determined by the expression $\omega_r = \omega_m\sqrt{2}/2$ [curve (*5*)].

As the linear coupling coefficients decrease, the self-modulation regime of the first kind passes to the stationary unidirectional lasing in the region $m < m_{cr}$. In this case, relaxation oscillations in the self-modulation regime at

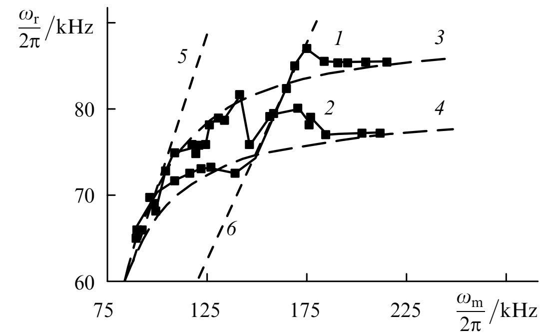

Figure 21. Experimental (*1*, *2*) and theoretical (*3*, *4*) dependences of the main relaxation frequency on the self-modulation oscillation frequency for $\eta = 0.46$ (*1*, *3*) and 0.56 (*2*, *4*) and the dependences $\sqrt{2}\omega_r = \omega_m$ (*5*) and $2\omega_r = \omega_m$ (*6*) [99].

frequencies ω_r and ω_{r1} pass to the relaxation oscillations at frequencies ω_r and $\omega_r^{(1)}$ in the travelling-wave regime, while the self-modulation oscillation frequency passes to the relaxation frequency $\omega_r^{(2)}$.

5.4 Frequency stabilisation and narrowing of the self-modulation oscillation spectrum

The flactuations of the frequency ω_m of self-modulation oscillations of the first kind are mainly due to the instability of the linear coupling coefficients of counterpropagating waves. The fluctuations of ω_m can be substantially reduced by eliminating feedbacks appearing upon parasitic reflections from the elements of a photoreception channel and focusing lenses of a pump system [100]. According to expressions (63) and (64), ω_m depends on the relaxation oscillation frequency, so that fluctuations of the self-modulation frequency are also caused by the pump instability (variations in the power, spectral, and spatial characteristics of a pump diode laser). The presence of several spectral components at the relaxation frequencies in the emission spectrum allows the stabilisation of the self-modulation oscillation frequency by stabilising the frequency of relaxation oscillations. This method was used to stabilise ω_m in paper [101], where the main relaxation frequency was stabilised by means of an external highly stable radio engineering generator. Figure 22 presents the time dependences of the self-modulation oscillation frequency in the absence and presence of stabilisation.

Experimental studies show that the width of the main peak at the frequency ω_m in the self-modulation oscillation spectrum is rather large, of the order of one kilohertz. Note

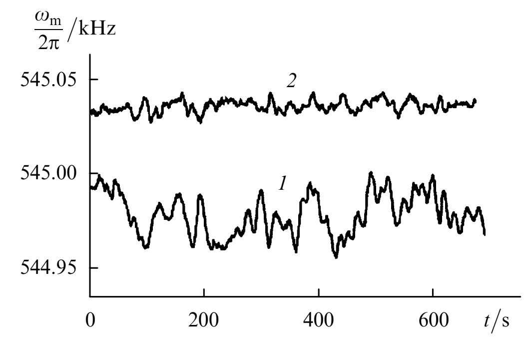

Figure 22. Time dependences of the self-modulation oscillation frequency in the absence (*1*) and presence (*2*) of stabilisation [101].

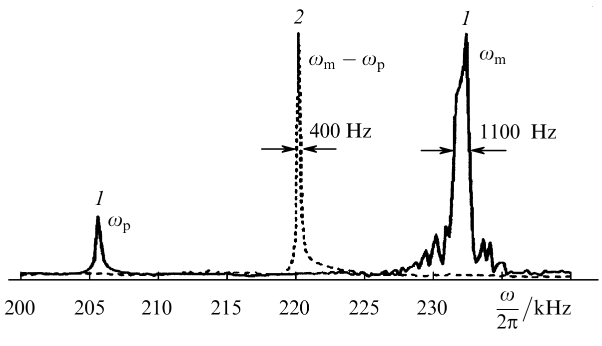

Figure 23. Self-modulation oscillation spectra of a ring chip laser in the absence (*1*) and presence (2) of frequency locking in the case of pump modulation at the frequency ω_p [102].

that the mechanism of this broadening has not been studied. We can assume that the broadening of the peak is caused by fluctuations of inverse-population gratings in the self-modulation regime of the first kind. The considerable narrowing (almost by a factor of three) of the self-modulation oscillation spectrum caused by periodic modulation of the pump at the frequency close to ω_m was demonstrated in paper [102]. In this case, self-modulation oscillations are synchronised (locked) by an external signal. This is clearly illustrated in Fig. 23 where the emission spectra of a ring chip laser are shown in the absence and presence of frequency locking.

6. Beating regime and methods for its stabilisation

6.1 Beating-regime-stability condition

We considered above periodic lasing regimes appearing in a SRL in the case of the homogeneous gain line and the presence of a strong competition between counterpropagating waves caused by their nonlinear coupling on inverse-population gratings. By introducing into the ring resonator the additional devices (elements) reducing the competition, we can expand the region of parameters in which periodic lasing regimes are observed and obtain the beating regime of counterpropagating waves important for practical applications.

In the beating regime of counterpropagating waves with different frequencies, their average intensities prove to be virtually equal, while the intensity modulation amplitude at the beat frequency (the frequency difference of counterpropagating waves) is small compared to the average intensity. The beating regime is similar in many respects to the self-modulation regime of the first kind. In both regimes, the intensity and phase difference of counterpropagating waves are periodically modulated at the beat frequency (self-modulation frequency).The main difference between these regimes is that the radiation intensity and the instant difference of frequencies of counterpropagating waves in the self-modulation regime of the first kind are strongly modulated, while such a periodic modulation in the beating regime is rather weak and the intensity and frequency of the waves can be considered virtually constant.

The methods for reducing the competition of counterpropagating waves (methods for stabilising the beating regime) are based on the introduction of additional intracavity losses depending on the intensity of counterpropagating waves so that the losses for a stronger wave should exceed those for the counterpropagating wave. The difference of the additional losses proportional to the difference in the intensities of counterpropagating waves can be written in the form

$$\Delta = qa(E_1^2 - E_2^2)\omega/Q, \qquad (82)$$

where q is the proportionality coefficient and $aE_{1,2}^2$ are the dimensionless intensities of counterpropagating waves. Additional losses should compensate for the inequality of the gains $\kappa_1 - \kappa_2 = (\beta - \alpha)a(E_1^2 - E_2^2)\kappa_0$ of counterpropagating waves [see expressions (24) and (25)] appearing due to their competitive interaction. Under the condition

$$|\Delta| > |\kappa_1 - \kappa_2| \qquad (83)$$

the effective gain for a weak wave proves to be higher than that for a strong wave, and the competitive suppression is eliminated. The condition for stabilisation (appearance) of the beating regime (83) can be written in the form

$$q > \frac{\kappa_0}{1+\Omega^2 T_1^2}\frac{Q}{\omega}. \qquad (84)$$

Consider some methods for reducing the competition between counterpropagating waves to stabilise the beating regime.

6.2 Methods for stabilising the beating regime

The use of a feedback circuit. The beating regime can be stabilised by using a feedback circuit [103, 104] producing inside the resonator the difference of losses for counterpropagating waves, which is proportional to the difference of their intensities (82). Such intracavity losses can be introduced by using a nonreciprocal amplitude Faraday element controlled by a signal proportional to the difference of the intensities of the waves. The radiation dynamics of a SRL with a feedback circuit can be described by the system of equations of the standard model assuming that losses per time unit are described by the expressions

$$\frac{\omega}{Q_{1,2}} = \frac{\omega}{Q} \pm \frac{qa\left(E_1^2 - E_2^2\right)}{2}\frac{\omega}{Q}, \qquad (85)$$

where ω/Q are losses per time unit in the absence of a feedback circuit. These equations can be solved by the method of successive approximations assuming that the laser operates in the beating regime with approximately equal intensities of counterpropagating waves and a weak amplitude modulation of each of the waves at the beat frequency. The approximate solution in the case of sufficiently large frequency nonreciprocities $|\Omega| \gg 1/T_1$ and the equal moduli of coupling coefficients has the form

$$I_1 + I_2 = \eta\left(1 + \frac{1}{1+\Omega^2 T_1^2}\right)^{-1}, \qquad (86)$$

$$I_1 - I_2 = m^2 \sin(\vartheta_1 - \vartheta_2)\left[\frac{\omega}{Q}\,\Omega\left(2q - \frac{1}{1+\Omega^2 T_1^2}\right)\right]^{-1}, \qquad (87)$$

$$\left\langle \frac{\mathrm{d}\Phi}{\mathrm{d}t} \right\rangle = \Omega + m^2\cos(\vartheta_1 - \vartheta_2)\frac{1}{\Omega} + \frac{\omega_r^2}{2\Omega}, \qquad (88)$$

where $I_{1,2} = aE_{1,2}^2$ are the dimensionless intensities of counterpropagating waves; $\langle \mathrm{d}\Phi/\mathrm{d}t \rangle$ is the average beat frequency; and $\omega_r^2 = \eta\omega/QT_1$ is the square of the relaxation oscillation frequency. This solution is valid if $|I_1 - I_2| \ll I_1 + I_2$. The stability condition for the beating regime in the presence of an inertialless feedback circuit has the form

$$2q > \frac{1}{1+\Omega^2 T_1^2}. \qquad (89)$$

According to the above expressions, a sufficiently strong feedback can equate the average intensities of counterpropagating waves and stabilise the beating regime. If in the absence of a feedback circuit ($q = 0$) one of the counterpropagating waves is suppressed in the region of rather large frequency nonreciprocities, the switching of the feedback equates the average intensities of counterpropagating waves.

Due to the inertia of the feedback circuit, the instability of bidirectional lasing can appear with increasing its transfer coefficient, which causes the out-of-phase oscillations of the intensities of counterpropagating waves. The conditions for the appearance of such instability were studied in [103].

The use of the intracavity second harmonic generation. The beating regime can be also stabilised by using nonlinear losses appearing upon the intracavity second harmonic generation (SHG). The SRL operation regimes upon intracavity SHG were studied theoretically and experimentally in [105]. The system of equations (22) in this case changes as follows: instead of the resonator bandwidth ω/Q describing linear loses in the resonator, we introduce the complex quantities

$$\frac{\omega}{\tilde{Q}_{1,2}} = \left[1 + (q_r + \mathrm{i}q_i)aE_{1,2}^2\right]\frac{\omega}{Q}. \qquad (90)$$

The expression $q_r aE_{1,2}^2\omega/Q$ determines nonlinear losses per time unit due to SHG. The imaginary part $\omega/\tilde{Q}_{1,2}$ equal to $q_i aE_{1,2}^2\omega/Q$ describes the additional phase nonreciprocity of a ring resonator produced upon SHG. The explicit expressions for the parameters q_r and q_i are presented in [105].

The average intensities of counterpropagating waves in the beating regime in the case of intracavity SHG in the asymptotic region of large frequency nonreciprocities ($|\Omega \gg m_{1,2}|$) are determined by expressions (86) and (87) in which q is replaced by q_r. The beating regime is stable when the inequality

$$q_r > \frac{1}{2}\left(1+\Omega^2 T_1^2\right)^{-1} \qquad (91)$$

is fulfilled. The averaged intensities of counterpropagating waves ($|I_1 - I_2| \ll I_1 + I_2$) are equated when

$$|\Omega| \gg \frac{m^2|\sin(\vartheta_1 - \vartheta_2)|T}{\kappa}, \qquad (92)$$

where κ are losses of the fundamental radiation for each of the waves (for the round-trip transit time T) due to SHG. When conditions (91) and (92) are fulfilled, the beating regime with almost equal intensities of the waves appears in a SRL with the intracavity SHG.

Stabilisation of the beating regime in a laser with a nonlinear absorber. In the presence of a nonlinear absorbing element in the resonator, the difference between the absorption coefficients of counterpropagating waves is described by the expression

$$\chi_1 - \chi_2 = (\beta_a - 1)\chi_0\frac{I_1 - I_2}{I_{sa}}, \qquad (93)$$

where χ_0 is the absorption coefficient of the unbleached absorber; $\beta_a = 1 + 1/(1+\Omega^2 T_a^2)$; and I_{sa} and T_a are the bleaching intensity and the relaxation time of the absorber. It follows from this expression that losses introduced by the nonlinear absorber are higher for a stronger wave. If the difference of the absorption coefficients $|\chi_1 - \chi_2|$ exceeds the difference of the gains $|\kappa_1 - \kappa_2|$, a weaker wave has a higher gain, which results in the equating of intensities of the waves and in the stability of the beating regime. The condition $|\chi_1 - \chi_2| > |\kappa_1 - \kappa_2|$ can be written in the form [106, 107]

$$\frac{I_{sa}}{I_s} < \frac{\Omega^2 T_1^2 \chi_0}{\kappa_0}. \quad (94)$$

In this case, it is assumed that the inequalities $\Omega T_1 \gg 1$ and $\Omega T_a \ll 1$ are fulfilled, i.e., the absorber is inertialless. It follows from (94) that the use of even a weakly bleaching nonlinear absorber ($I_a \gg I_s$) can provide the stability of the beating regime in the region of a sufficiently large difference of the frequencies Ω of counterpropagating waves.

The fulfilment of condition (94) is necessary but not sufficient for the existence of the beating regime with the equal amplitudes of counterpropagating waves. To eliminate the suppression of one of the waves, it is also necessary to compensate the inequality of losses caused by their coupling. The beating regime with virtually equal intensities of counterpropagating waves $[|I_1 - I_2| \ll (I_1 + I_2)]$ will exist if the inequality

$$m_1 m_2 |\sin(\vartheta_1 - \vartheta_2)| \frac{T I_a}{I_s \Omega \chi_0 \eta} \ll 1 \quad (95)$$

is fulfilled [106, 107]. Inequalities (94) and (95) determine the parameters of a laser with a nonlinear absorber at which the intensities of counterpropagating waves are efficiently equated and the beating regime is stable. Therefore, by using a nonlinear absorber, we can obtain in a certain range of coupling coefficients the self-modulation and beating regimes existing for any values of the frequency nonreciprocity Ω. The stability condition for the beating regime in a SRL with a nonlinear absorber has the form [107]

$$\frac{\eta \chi_0 T_1 I_s}{I_a T} < \frac{2}{3}. \quad (96)$$

If condition (96) is violated, stationary lasing in the SRL becomes impossible due to instability with respect to excitation of relaxation oscillations. A similar instability also takes place in linear solid-state lasers with a nonlinear absorber.

Except the methods for stabilising the beating regimes considered above, there also exist other methods, in particular, the use of self-illumination waves [27, 108]. The self-illumination waves are produced after the return of a part of the output radiation to the active medium at an angle to the resonator axis, which can be achieved without introducing any additional elements into the resonator. The studies performed in [27] have shown that this method is efficient when a laser operates in the mode locking regime, however, it has not been realised in the free running regime so far.

6.3 Anomalies in the frequency characteristics of a solid-state ring laser

The beating regime in a SRL has a number of specific features, one of which is the anomalous behaviour of the beat frequency in the region of small frequency nonreciprocities. This feature is caused by the nonlinear coupling of counterpropagating waves on moving inverse-population gratings. In the region of sufficiently large Ω, the dependence of the beat frequency on the frequency nonreciprocity of the resonator (frequency characteristic) is determined by expression (88). According to this expression, the deviation of the frequency characteristic from the ideal one ($\langle \mathrm{d}\Phi/\mathrm{d}t \rangle = \Omega$) is determined by the linear and nonlinear coupling of counterpropagating waves. The nonlinear coupling changes the frequencies of counterpropagating waves due to the Doppler frequency shifts from moving inverse-population gratings. In the case of a sufficiently small linear coupling, when $m^2 \ll \omega_r^2$, the main contribution to the deviation of the frequency characteristic from the ideal one is introduced by the nonlinear coupling. In this case, the frequency characteristics of a SRL can have anomalies in the region of sufficiently small frequency nonreciprocities.

Because of nonlinearity, the dependence of the beat frequency on Ω becomes ambiguous, and three branches of the frequency characteristic appear in the region of sufficiently small frequency nonreciprocities ($\Omega \ll \omega_r$). The two of them are determined by the expressions [109]

$$\left\langle \frac{\mathrm{d}\Phi}{\mathrm{d}t} \right\rangle = \mathrm{sign}\Omega \left[\frac{|\Omega|}{2} \pm \left(\frac{\Omega^2}{4} + \frac{\omega_r^2}{2} \right)^{1/2} \right]. \quad (97)$$

The beat frequencies on these branches in the region of small frequency nonreciprocities are close to the relaxation oscillation frequency. The third branch of the frequency characteristic is approximately described by the equation

$$\left\langle \frac{\mathrm{d}\Phi}{\mathrm{d}t} \right\rangle = k\Omega, \quad (98)$$

where the coefficient $k = -2(T_1 \omega \eta / Q)^{-1}$ proves to be anomalously small and has the negative sign. Figure 24 presents the typical dependence of the frequency difference of counterpropagating waves on the frequency nonreciprocity Ω for a weak linear coupling. In the region of small frequency nonreciprocities, the beat frequency changes jumpwise and the frequency characteristic exhibits the hysteresis. As the competition between counterpropagating waves is weakened by means a nonlinear absorber, the additional distortions of the frequency characteristic appear, which are caused by the Doppler frequency shifts on moving gratings induced in the nonlinear absorber [110]. The anomalous behaviour of the frequency characteristics

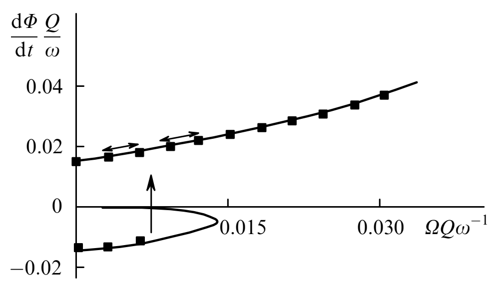

Figure 24. Dependence of the normalised frequency difference $\mathrm{d}\Phi/\mathrm{d}t$ (Q/ω) of counterpropagating waves in the beating regime on the normalised frequency nonreciprocity $\Omega Q \omega^{-1}$ in the case of the reduction of competition between counterpropagating waves by introducing additional nonlinear losses $\omega/Q_{1,2} = \omega/(Q\gamma a E_{1,2}^2)$ for $m = 10^{-3}\omega/Q$, $\vartheta_1 - \vartheta_2 = 0.08$, $\eta = 0.1$, $\gamma = 0.1$, and $T_1\omega/Q = 200$ (numerical simulation) [109].

was studied experimentally in a ring Nd : YAG by reducing the competition between counterpropagating waves with the help of a Nd : YAG crystal inserted into the resonator [109].

7. Quasi-periodic and chaotic lasing regimes

7.1 Regimes with self-modulation oscillations of a complicated shape

Aside from periodic lasing regimes, a number of stationary regimes with a more complicated type of radiation modulation can exist in SRLs. One of such regimes is the self-modulation lasing regime of the second kind in which spontaneous quasi-periodic variations in the propagation direction of laser radiation occur at frequencies not exceeding a few kilohertz (low-frequency out-of-phase self-intensity modulation of counterpropagating waves). Such low-frequency switchings of the radiation-propagation direction are usually accompanied by a faster intensity modulation of counterpropagating waves at the relaxation oscillation frequency. The typical intensity oscillogram for counterpropagating waves in the self-modulation regime of the second kind is presented in Fig. 25.

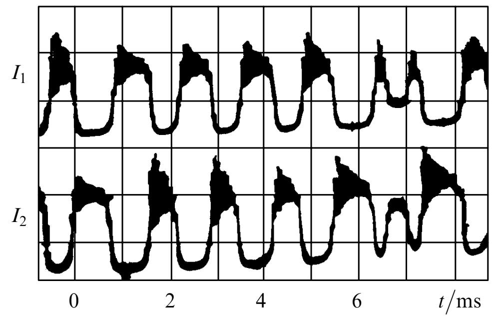

Figure 25. Oscillogram of the intensity of counterpropagating waves in the self-modulation regime of the second kind [9].

Even more complicated and irregular self-modulation of radiation takes place in dynamic chaos regimes. In these regimes, radiation is emitted as a series of pulses with amplitudes and repetition periods changing irregularly in time. The irregularity appears in the absence of technical or quantum noises and is caused by the specificity of the phase space of a nonlinear dynamic system. The possibility (or impossibility) of existence of chaotic regimes in a nonlinear dynamic system (such as a SRL) is determined by the dimensionality of a mathematical model describing the nonlinear system or by the dimensionality of the corresponding phase space [111 – 113]. In the case of single-mode SRLs with bidirectional lasing, the dimensionality of the phase space (a system of the seven-order differential equations in the standard model) admits the existence of not only periodic and quasi-periodic but also chaotic oscillations.

The self-oscillations established in the phase space are represented geometrically by a phase attractor – a trajectory (or a set of trajectories) located within a limited region of the phase space and attracting all the neighbouring trajectories. The attractor is a limiting set of trajectories, which attracts all the trajectories located in some vicinity of the phase space. If the limiting set corresponds to a stable stationary regime, the attractor is a fixed point. If the lasing regime is periodic, the attractor is a closed curve called the limiting cycle. In the case of dynamic chaos excitation, an attractor also takes place; all the trajectories in the phase space are located within a limited region, where, however, neither stable states nor limiting cycles are present. Such an attractor is called a strange attractor [111 – 113]. It represents an attracting set of trajectories, each of them being unstable. The strange attractor has two substantial distinctions, its trajectories being nonperiodical and unclosed.

It is sometimes difficult to distinguish the quasi-periodic regimes from the dynamic chaos regime by their time realisations. However, a number of methods have been developed in the nonlinear dynamics which allow the unique identification of lasing regimes. Consider some criteria allowing the identification of the dynamical chaos regime in SRLs. An important characteristic used for the classification of operating regimes of ring lasers is the radiation power spectrum of counterpropagating waves. While the power spectrum in the periodic or quasi-periodic regime exhibits a set of discrete spectral components, the power spectrum in the dynamic chaos regime is characterised by a relatively broad band. The power spectrum of chaotic oscillations may contain intense discrete components against a broad noise background. In some cases, the power spectrum can be completely of the 'noise' type. Note that the power spectra of counterpropagating waves can be both identical and different (i.e., the spectral nonreciprocity can take place). When the spectrum contains discrete components, the components having the maximum intensity may have different frequencies in counterpropagating waves.

Lasing regimes can be also classified by using phase portraits and Poincare sections. The criterion indicating the presence of chaotic oscillations is also the shape of the correlation function $K = \langle I(t+\tau)I(t)\rangle$, which exponentially decays with increasing τ in the case of dynamic chaos.

One of the important characteristics allowing one to find out the existence of a strange attractor is the presence of the positive Lyapunov coefficients [111 – 113], whose spectrum can give quantitative information on the average stability of a phase trajectory. The type of lasing can be determined by calculating Lyapunov coefficients from experimental data.

Lasing regimes with a complicated shape of self-modulation oscillations appear in the regions of parametric resonances between different characteristic frequencies (relaxation frequencies and self-modulation oscillation frequencies). One of the regions of a parametric resonance exists only in the case of a weak linear coupling ($m < m_{\rm cr}$) and in the absence of the frequency nonreciprocity. In this case, the resonance appears between two relaxation frequencies: $\omega_{\rm r}^{(1)} = \omega_{\rm r}^{(2)} = \omega_{\rm r}\sqrt{2}$ [see (27), (28)]. In the vicinity of this parametric resonance, the self-modulation regime of the second kind and dynamic chaos can appear.

7.2 Self-modulation regime of the second kind

The self-modulation regime of the second kind can exist in a ring laser only in the case of a sufficiently weak linear coupling ($m < m_{\rm cr}$) and sufficiently large detunings of the laser frequency from the gain-line centre. This regime, as the self-modulation regime of the first kind, is excited in the instability region of stationary unidirectional lasing. However, the instability mechanisms resulting in the appearance of these regimes are different. In the case of the self-modulation regime of the first kind, stationary lasing becomes unstable with increasing coupling via backward scattering, while the instability mechanism for the self-modulation regime of the second kind is determined by the nonlinear coupling of counterpropagating waves in an amplifying medium. In this case, the phase shift upon Bragg reflections of counterpropagating waves from inverse-population gratings plays an important role. The necessary condition for the appearance of instability is the nonzero real part of the susceptibility of the active medium at the lasing frequency. This can take place when the laser-frequency detuning from the gain-line centre exceeds the critical value determined by expression (26). The same inequality can be obtained in the case of the asymmetric gain line (see Section 4.5).

The features of the nonlinear radiation dynamics of a SRL in this regime can be analysed by using either the standard model [the system of equations (22)] or a system of equations taking into account the real structure of the gain line in a Nd : YAG laser. Note that attempts to obtain the analytic solution in the case of the self-modulation regime of the second kind have failed, and this regime was studied only by numerical calculations.

The influence of the linear-coupling strength on the self-modulation regime of the second kind in the absence of the optical nonreciprocity of a ring resonator was analysed in papers [71, 72] by using the system of equations (52). Figure 26 shows the dependence of the leading Lyapunov index $\Lambda = \lambda T_1$ on the dimensionless coupling parameter $\rho = m(\omega/Q)^{-1}$ calculated in [71]. One can see that in the case of a weak linear coupling, the self-modulation regime of the second kind is quasi-periodic ($\Lambda = 0$). The time dependences of the intensities of counterpropagating waves in the self-modulation regime of the second kind, calculated by integrating numerically the system of equations (52), are shown in Fig. 27a in the absence of linear coupling and in

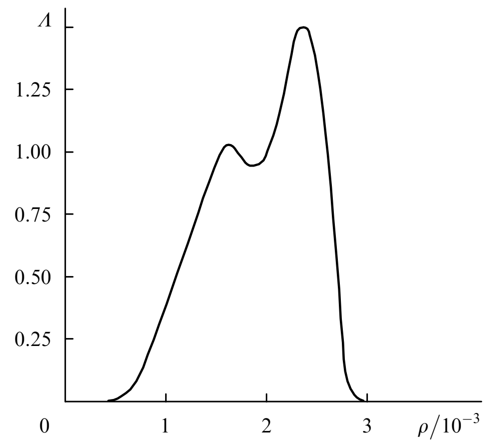

Figure 26. Dependence of the leading Lyapunov index $\Lambda = \lambda T_1$ on the dimensionless coupling parameter $\rho = m/(\omega/Q)$ [72].

Fig. 27b in the presence of linear coupling with $\rho = 10^{-3}$. As the coupling strength is increased, the period of low-frequency switchings of the propagation direction of radiation decreases and a periodic chaotic modulation of the low-frequency envelopes of the intensity of counterpropagating waves appears. As the coupling strength is further increased (Fig. 28), the self-modulation regime of the second kind is replaced by dynamic chaos. In the region of linear coupling coefficients $\rho > \rho_{cr}^0$, the dynamic chaos regime passes to the self-modulation regime of the second kind.

Self-modulations of intensity of counterpropagating waves in self-modulation regimes of the second kind are accompanied by the modulation of the phase and frequency difference of counterpropagating waves. The frequency dynamics of a ring laser operating in the self-modulation regime of the second kind was studied in papers [114, 115].

The linear coupling coefficients in monolithic SRLs prove to be quite high due to a small perimeter of the

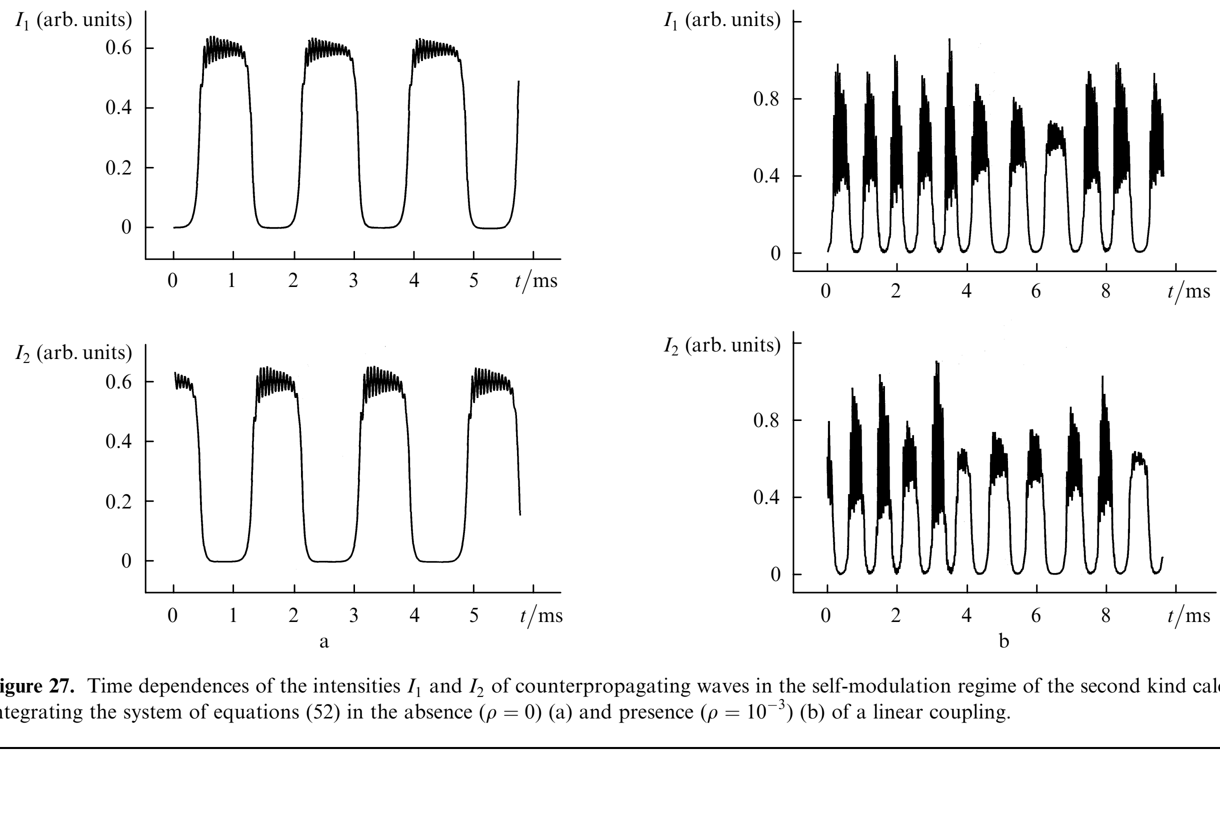

Figure 27. Time dependences of the intensities I_1 and I_2 of counterpropagating waves in the self-modulation regime of the second kind calculated by integrating the system of equations (52) in the absence ($\rho = 0$) (a) and presence ($\rho = 10^{-3}$) (b) of a linear coupling.

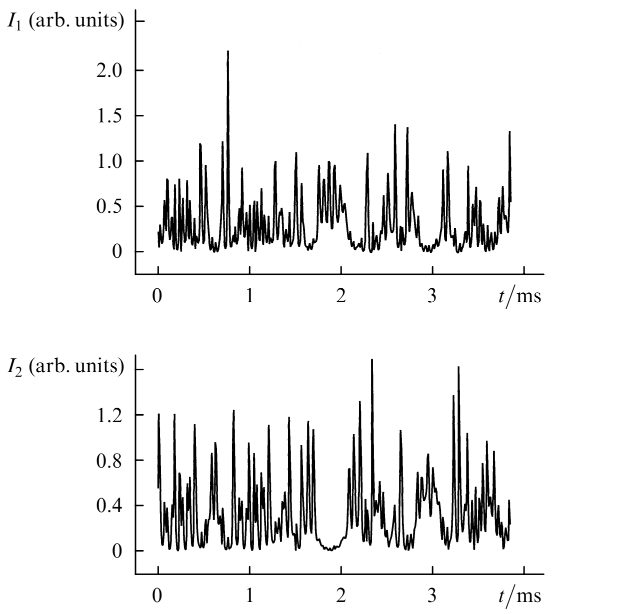

Figure 28. Time dependences of the intensities I_1 and I_2 of counterpropagating waves in the dynamic chaos regime calculated by integrating the system of equations (52) for the linear coupling $\rho = 2 \times 10^{-3}$.

ring resonator. The coupling strength in such lasers, as a rule, satisfies the condition $\rho > \rho_{cr}^0$, which corresponds to the right region with $\Lambda = 0$ in Fig. 26. In this case, according to the theory, the self-modulation regime of the second kind is absent.

7.3 Parametric interactions between self-modulation and relaxation oscillations

Apart from the parametric resonance between relaxation frequencies $\omega_r^{(1)}$ and $\omega_r^{(2)}$, the resonance between the main relaxation frequency and frequency $\omega_r^{(2)}$ is also possible. It takes place when the optical nonreciprocity Ω of the resonator is equal to $\omega_r/2$. The possibility of the appearance of dynamic chaos in this region was demonstrated in papers [70, 71].

In the case of a sufficiently strong linear coupling ($m > m_{cr}$), when a laser operates in the self-modulation regime of the first kind, the parametric resonances of another type can appear, which are caused by the interaction between self-modulation and relaxation oscillations [99, 116, 117]. Such parametric resonances can appear when the frequencies of self-modulation and relaxation oscillations are related by the expression $i\omega_m = j\omega_r + f\omega_{r1}$ (i, j, $f = 0, 1, 2\ldots$).

Let us explain qualitatively the mechanism of appearance of the parametric resonance at the self-modulation oscillation frequency close to the doubled relaxation oscillation frequency ($i = 1$, $j = 2$, $f = 0$). In the presence of perturbations, the amplitude of self-modulation oscillations proves to be modulated at the relaxation oscillation frequency ω_r, and spectral components appear in the emission spectrum at frequencies ω_r and the combination frequency $\omega_m - \omega_r$. When these frequencies coincide ($\omega_m = 2\omega_r$), the parametric resonance appears, which was studied theoretically and experimentally in [99].

Experimental studies were performed with a monolithic ring Nd : YAG laser. As a rule, the self-modulation oscillation frequency in such lasers greatly exceeds the relaxation oscillation frequency, and the self-modulation oscillation regime is stable (Fig. 29a). To obtain the conditions for appearing parametric resonances at a fixed relaxation oscillation frequency, the self-modulation oscillation frequency was tuned by varying the amplitude and phase of the coefficient of backward scattering from an auxiliary external mirror. As the self-modulation frequency was decreased, the passage to the parametric resonance region was observed, which was accompanied by the oscillation frequency locking at the frequency $\omega_m/2$ and by a change in the decrement sign (the decay of relaxation oscillations was changed by their rise).

The self-excitation of relaxation oscillations in the parametric-resonance region leads to the passage of the self-modulation regime of the first kind to the quasi-periodic lasing regime, in which the period of self-modulation oscillations is doubled and the low-frequency envelope appears (Fig. 29b). As the self-modulation frequency was further decreased, the frequency came out from the parametric-resonance region, and the quasi-periodic regime passed again to the self-modulation regime of the first kind (Fig. 29c). Another region of the parametric resonance, in which the self-modulation oscillation frequency is close to the doubled frequency ω_{r1}, was observed for $\omega_m/2\pi <$ 90 kHz. In this case, the frequency locking at the frequency $\omega_{r1} = \omega_m/2$ also occurred and the sign of the decrement of relaxation oscillations changed, which was accompanied by passing to the dynamic chaos regime (Fig. 29d). These experimental results agree with theoretical predictions of the standard SRL model [98, 99, 118].

8. Conclusions

Let us sum up the results presented above. Solid-state ring lasers attract the considerable attention of the researchers in the field of laser physics due to their applications in fundamental laser physics (the search for gravitational waves, the verification of the fundamental concepts of quantum electronics and the relativity theory, fundamental quantum metrology, etc.) and in laser technologies (Doppler measuring systems, optical communication, laser gyroscopes, etc.). Solid-state ring lasers offer wide functional possibilities because they can operate in a variety of stationary and nonstationary regimes [117, 118].

The travelling-wave regime and self-modulation lasing regime of the first kind are especially interesting from the practical point of view. It is the travelling-wave regime that provides the highly stable amplitude and frequency of output radiation and the laser linewidth close to the quantum limit. The use of solid-state ring lasers in optical frequency standards and very precise measurements opens up a new page in fundamental metrology. The radiation dynamics of such lasers is characterised by a high sensitivity to the frequency nonreciprocity of the resonator, which allows precision investigations to be made of various nonreciprocity effects.

Solid-stage ring lasers are a complicated nonlinear system in which the linear and nonlinear interactions of counterpropagating waves lead to a rather intricate radiation dynamics. The investigation of the nonlinear dynamics of such lasers attract great interest because the obtained results can be used for studying the general properties of the behaviour of complex nonlinear systems of different types, in particular, the conditions and reasons for the appearance

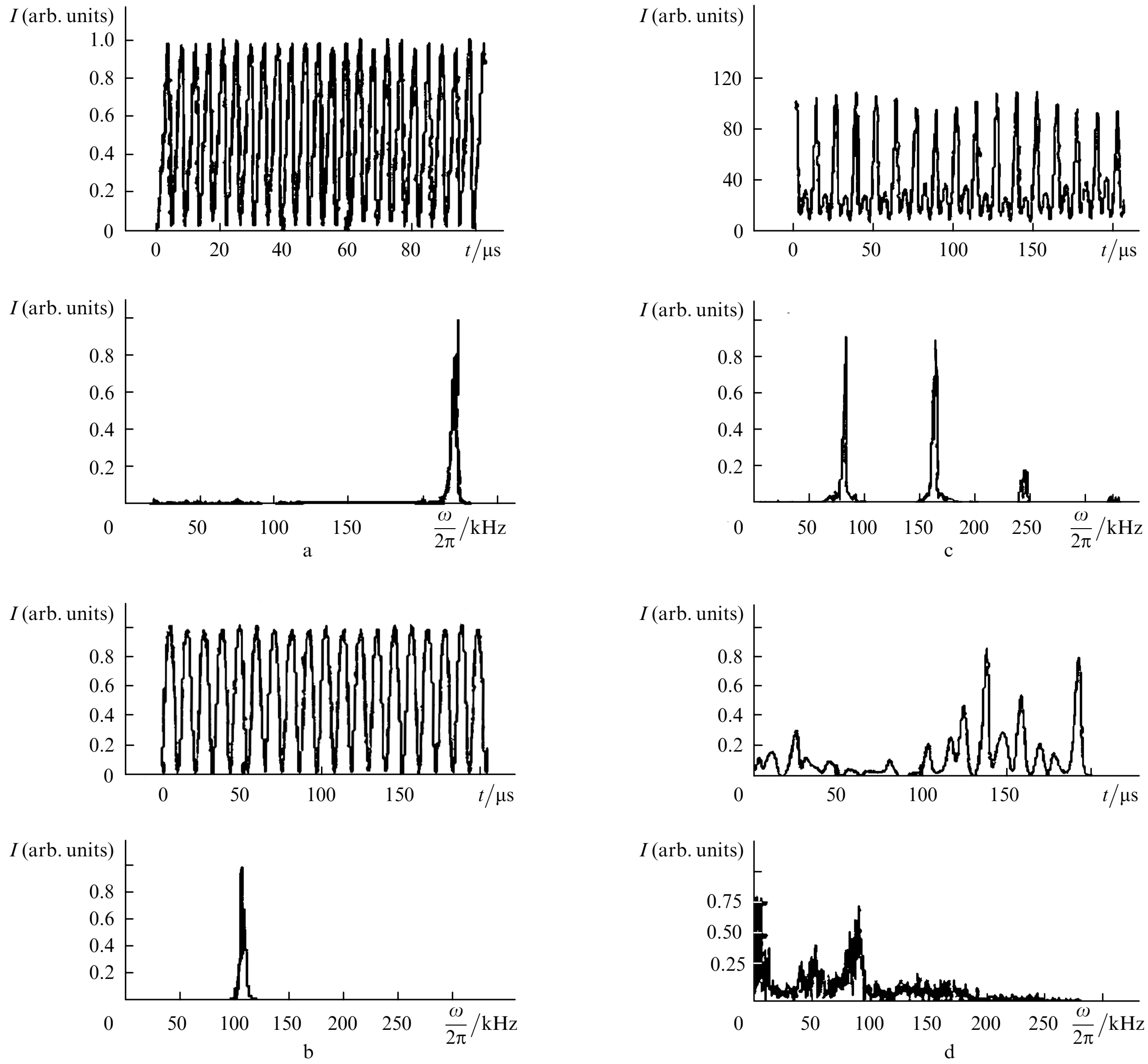

Figure 29. Temporal and spectral characteristics of radiation for different linear coupling coefficients (different self-modulation oscillation frequencies) in the self-modulation regime of the first kind (a, c), the double-period regime (b), and the dynamic chaos regime (d) [99].

of self-oscillations, various parametric processes, and dynamic chaos in them.

We have analysed the state of the art in the theoretical and experimental studies of the nonlinear dynamics of solid-state ring lasers. It has been shown that the standard model well describes all its basic properties, which make it possible to use modern solid-state ring lasers not only as an object for fundamental studies but also as a very efficient tool for precision measurements in various fields of physics.

Despite the apparent completeness of investigations performed up to now, we believe that a number of inadequately studied problems still remain in the nonlinear dynamics of solid-state ring lasers. These are the radiation dynamics with arbitrarily polarised counterpropagating waves, a very interesting case of orthogonal polarisations, when both the nonlinear and linear couplings of counterpropagating waves can be weakened, and also the lasing dynamics in regions of parametric resonances related to relaxation and self-modulation oscillations. In addition, the further development of the methods for reducing the competition between counterpropagating waves and stabilising periodic lasing regimes is undoubtedly important.

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References

1. Tang C.L., Statz H., de Mars G.A. *Appl. Phys. Lett.*, **2**, 222 (1963).
2. Tang C.L., Statz H., de Mars G.A., Wilson D.T. *Phys. Rev.*, **136A**, 1 (1964).
3. Kulevsky L.A., Pashinin P.P., Prokhorov A.M. *Proc. III Intern. Conf. Quantum Electronics* (Paris – New York, 1964) Vol. 2, p. 1065.
4. Krasyuk I.K., Pashinin P.P., Prokhorov A.M. *Pis'ma Zh. Eksp. Teor. Fiz.*, **7**, 117 (1968).
5. Kornienko L.S., Kravtsov N.V., Naumkin N.I., Prokhorov A.M. *Zh. Eksp. Teor. Fiz.*, **58**, 541 (1970).
6. Globes A.R., Brienza M.J. *Appl. Phys. Lett.*, **21**, 265 (1972).
7. Kornienko L.S., Kravtsov N.V., Shelaev A.N. *Opt. Spektr.*, **35**, 775 (1973).
8. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Pis'ma Zh. Eksp. Teor. Fiz.*, **17**, 405 (1973).

9. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Zh. Eksp. Teor. Fiz.*, **65**, 1344 (1973).
10. Stedman G.E. *Rep. Prog. Phys.*, **60**, 615 (1997).
11. Mashhoon B., Neutze R. J., Hannam M.D., Stedman G.E. *Phys. Lett. A*, **249**, 161 (1998).
12. Stedman G.E., Schreiber K.U., Bilger H.R. *Class. Quantum Grav.*, **20**, 2527 (2003).
13. Frede M., Wilhelm R., Kracht D., Fallnich C. *Opt. Express*, **13**, 7516 (2005).
14. Rikken G.L., Rizzo C. *Phys. Rev. A*, **63**, 012107 (2000).
15. Denisov V.I., Kravtsov N.V., Krivchenkov I.V. *Kvantovaya Elektron.*, **33**, 938 (2003) [*Quantum Electron.*, **33**, 938 (2003)].
16. Hall J.L., Ma L.Sh., Taubman M., Tiemann B., Hong F.L., Pfister O., Ye J. *IEEE Trans. Instrum. Meas.*, **48**, 586 (1999).
17. Skvortsov M.N., Okhapkin M.V., Nevskii A.Yu., Bagaev S.N. *Kvantovaya Elektron.*, **34**, 1101 (2004) [*Quantum Electron.*, **34**, 1101 (2004)].
18. Fan T.Y., Byer R.L. *IEEE J. Quantum Electron.*, **24**, 895 (1988).
19. Kane T.J., Byer R.L. *Opt. Lett.*, **10**, 65 (1985).
20. Hughes D.O., Barr J.R. *J. Appl. Phys.*, **25**, 563 (1992).
21. Garbuzov D.Z., Dedysh V.V., Kochergin A.V., Kravtsov N.V., Nanii O.E., Nadtocheev V.E., Strugov N.A., Firsov V.V., Shelaev A.N. *Kvantovaya Elektron.*, **16**, 2423 (1989) [*Sov. J. Quantum Electron.*, **18**, 1557 (1989)].
22. Garbuzov D.Z., Dedysh V.V., Kochergin A.V., Kravtsov N.V., Nanii O.E., Nadtocheev V.E., Strugov N.A., Firsov V.V., Shelaev A.N. *Izv. Akad. Nauk. SSSR, Ser. Fiz.*, **54**, 2397 (1990).
23. Bogatov A.G., Golyaev Yu.D., Dedysh V.V., et al. *Izv. Akad. Nauk. SSSR, Ser. Fiz.*, **56**, 170 (1992).
24. Kravtsov N.V., Nanii O.E. *Kvantovaya Elektron.*, **20**, 322 (1993) [*Quantum Electron.*, **23**, 272 (1993)].
25. Kravtsov N.V. *Kvantovaya Elektron.*, **31**, 661 (2001) [*Quantum Electron.*, **31**, 661 (2001)].
26. Khanin Ya.I. *Osnovy dinamiki lazerov* (Fundamentals of the Laser Dynamics) (Moscow: Nauka, 1999).
27. Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Laser Phys.*, **3**, 21 (1993).
28. Trutna W.R., Donald D.K. *Opt. Lett.*, **15**, 369 (1990).
29. Chen D., Fincher C.L., et al. *Opt. Lett.*, **20**,1203 (1995).
30. Baer T. *Opt. Lett.*, **12**, 392 (1987).
31. Sandoghdar V., Treussart F., Hare J., Lefevre-Seguin V., Raimond J.M., Haroche S. *Phys. Rev. A*, **54**, 1777, (1996).
32. Kravtsov N.V., Kravtsov N.N. *Kvantovaya Elektron.*, **27**, 98 (1999) [*Quantum Electron.*, **29**, 378 (1999)].
33. Andronova I.A. *Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.*, **47**, 982 (2004).
34. Kravtsov N.V., Lariontsev E.G., Pashinin P.P., Sidorov S.S., Chekina S.N. *Kvantovaya Elektron.*, **34**, 325 (2004) [*Quntum Electron.*, **34**, 325 (2004)].
35. Trutna W.R., Donald D.K., Nazarathy M. *Opt. Lett.*, **12**, 248 (1987).
36. Nilsson A.C., Gustafson E.K., Byer R.L. *IEEE J. Quantum Electron.*, **25**, 767 (1989).
37. Clarkson W.A., Neilson A.B., Hanna D.C. *IEEE J. Quantum Electron.*, **32**, 311 (1996).
38. Veselovskaya T.V., Klochan E.L., Lariontsev E.G. *Radiotekh. Elektron.*, **34**, 2509 (1989).
39. Post E.I. *Rev. Mod. Phys.*, **39**, 475 (1967).
40. Heer C.V. *Phys. Rev.*, **134A**, 799 (1964).
41. Boiko D.L., Golyaev Yu.D., Dmitriev V.G., Kravtsov N.V. *Kvantovaya Elektron.*, **24**, 653 (1997) [*Quantum Electron.*, **27**, 635 (1997)].
42. Kravtsov N.V., Lariontsev E.G. *Laser Phys.*, **7**, 196 (1997).
43. Kravtsov N.V., Makarov A.A. *Kvantovaya Elektron.*, **25**, 786 (1998) [*Quantum Electron.*, **28**, 765 (1998)].
44. Owyoung A, Esherick P. *Opt. Lett.*, **12**, 999 (1987).
45. Kane T.J., Cheng A.P. *Opt. Lett.*, **13**, 970 (1988).
46. Kravtsov N.N., Shabat'ko N.M. *Kvantovaya Elektron.*, **22**, 793 (1995) [*Quantum Electron.*, **25**, 762 (1995)].
47. Goidin R.V., Kichuk V.S., Kravtsov N.V., Laptev G.D., Lariontsev E.G., Firsov V.V. *Kvantovaya Elektron.*, **25**, 358 (1998) [*Quantum Electron.*, **28**, 347 (1998)].
48. Singh S. *Phys. Rev. A*, **23**, 837 (1981).
49. Zhelnov B.L., Kazantsev A.P., Smirnov V.S. *Fiz. Tverd. Tela*, **7**, 2816 (1965).
50. Morozov V.N. *Fiz. Tverd. Tela*, **8**, 2256 (1966).
51. Belenov E.M. *Zh. Tekh. Fiz.*, **38**, 871 (1967).
52. Efanova I.P., Lariontsev E.G. *Zh. Eksp. Teor. Fiz.*, **55**, 1532 (1967).
53. Zhelnov B.L., Smirnov V.S., Fadeev A.P. *Opt. Spektr.*, **28**, 744 (1970).
54. Krivoshchekov G.V., Makukha V.K., Semibalamut V.M., Smirnov V.S. *Kvantovaya Elektron.*, **3**, 1782 (1976) [*Sov. J. Quantum Electron.*, **6**, 965 (1976)].
55. Polushkin N.I., Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **10**, 1461 (1983) [*Sov. J. Quantum Electron.*, **13**, 950 (1983)].
56. Khandokhin P.A., Khanin Ya.I., Koryukin I.V. *Opt. Commun.*, **81**, 297 (1991).
57. Nanii O.E. *Kvantovaya Elektron.*, **19**, 762 (1992) [*Quantum Electron.*, **22**, 703 (1992)].
58. Nanii O.E., Paleev M.R. *Kvantovaya Elektron.*, **20**, 699 (1993) [*Quantum Electron.*, **23**, 605 (1993)].
59. Boiko D.L., Kravtsov N.V. *Kvantovaya Elektron.*, **25**, 880 (1998) [*Quantum Electron.*, **28**, 856 (1998)].
60. Mamaev Yu.A., Milovskii N.D., Turkin A.A., Khandokhin P.A., Shirokov E.Yu. *Kvantovaya Elektron.*, **27**, 228 (1999) [*Quantum Electron.*, **29**, 505 (1999)].
61. Risken H., Nummedal K. *Phys. Lett. A*, **26**, 275 (1968).
62. Risken H., Nummedal K. *J. Appl. Phys.*, **39**, 4662 (1968).
63. Lugiato L.A., Narducci L.M. *Phys. Rev. A*, **32**, 1563 (1985).
64. Samson A.M., Kotomtseva L.A., Loiko N.A. *Avtokolebaniya v lazerakh* (Self-oscillations in Lasers) (Minsk: Nauka i Tekhnika, 1990).
65. Zeiger S.G., Klimontovich Yu.L., Landa P.S., Lariontsev E.G., Fradkin E.E. *Volnovye i fluktuatsionnye protsessy v lazerakh* (Wave and Fluctuation Processes in Lasers) (Moscow: Nauka, 1974).
66. Zeghlache H., Mandel P., Abraham N.B., Hoffer L.M., Lippi G.L., Mello T. *Phys. Rev. A*, **37**, 470 (1988).
67. Perevedentseva G.V., Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **7**, 128 (1980) [*Sov. J. Quantum Electron.*, **10**, 71 (1980)].
68. Khandohin P.A., Khanin Ya.I. *J. Opt. Soc. Am. B*, **2**, 226 (1985).
69. Khandokhin P.A., Khanin Ya.I. *Kvantovaya Elektron.*, **23**, 36 (1996) [*Quantum Electron.*, **26**, 34 (1996)].
70. Parfenov V.A., Khandokhin P.A., Khanion Ya.I. *Kvantovaya Elektron.*, **15**, 1985 (1988) [*Sov. J. Quantum Electron.*, **18**, 1243 (1988)].
71. Khandokhin P.A. Khanin Ya.I. *Kvantovaya Elektron.*, **15**, 1993 (1988) [*Sov. J. Quantum Electron.*, **18**, 1248 (1988)].
72. Khanin Ya.I. *J. Opt. Soc. Am. B*, **5**, 889 (1988).
73. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Pis'ma Zh. Eksp. Teor. Fiz.*, **21**, 30 (1975).
74. Dotsenko A.V., Lariontsev E.G. *Kvantovaya Elektron.*, **8**, 1504 (1981) [*Sov. J. Quantum Electron.*, **11**, 907 (1981)].
75. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Dokl. Akad. Nauk SSSR.*, **215**, 313 (1974).
76. Klochan E.L., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Shelaev A.N. *Radiotekh. Elektron.*, **10**, 2096 (1974).
77. Ustyugov V.I. *Pis'ma Zh. Tekh. Fiz.*, **1**, 362 (1975).
78. Dotsenko A.V., Laiontsev E.G. *Kvantovaya Elektron.*, **4**, 1099 (1977) [*Sov. J. Quantum Electron.*, **7**, 616 (1977)].
79. Dotsenko A.V., Klochan E.L. Lariontsev E.G., Fedorovich O.V. *Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.*, **8**, 1132 (1978).
80. Khandokhin P.A. *Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.*, **22**, 813 (1979).
81. Nanii O.E., Shelaev A.N. *Kvantovaya Elektron.*, **11**, 943 (1984) [*Sov. J. Quantum Electron.*, **14**, 638 (1984)].
82. Scheps R., Myers J. *IEEE J. Quantum Electron.*, **26**, 413 (1990).
83. Mak A.A., Ustyugov V.I., Fomzel' F.A., Khaleev M.M. *Zh. Tekh. Fiz.*, **44**, 868 (1974).
84. Golyaev Yu.D., Zadernovskii A.A., Livintsev A.L. *Kvantovaya Elektron.*, **14**, 917 (1987) [*Sov. J. Quantum Electron.*, **17**, 950 (1987)].

85. Smyslyaev S.P., Kaptsov L.N., Evtyukhov KN., Golyaev Yu.D. *Pis'ma Zh. Tekh. Fiz.*, **5**, 1493 (1979).
86. Golyaev Yu.D., Evtyukhov K.N., Kaptsov L.N., Smyshlyaev S.P. *Kvantovaya Elektron.*, **8**, 2321 (1981) [*Sov. J. Quantum Electron.*, **11**, 1421 (1981)].
87. Vladimirov A.G. *Opt. Commun.*, **149**, 67 (1998).
88. Zolotoverkh I.I., Lariontsev E.G. *Kvantovaya Elektron.*, **20**, 67 (1993) [*Quantum Electron.*, **23**, 56 (1993)].
89. Zolotoverkh I.I., Lariontsev E.G. *Kvantovaya Elektron.*, **23**, 620 (1996) [*Quantum Electron.*, **26**, 604 (1996)].
90. Boiko D.L., Kravtsov N.V. *Kvantovaya Elektron.*, **25**, 361 (1998) [*Quantum Electron.*, **28**, 350 (1998)].
91. Kravtsov N.V., Lariontsev E.G., Naumkin N.I., Sidorov S.S., Firsov V.V., Chekina S.N. *Kvantovaya Elektron.*, **31**, 649 (2001) [*Quantum Electron.*, **31**, 649 (2001)].
92. Boiko D.L., Kravtsov N.V. *Kvantovaya Elektron.*, **27**, 27 (1999) [*Quantum Electron.*, **29**, 309 (1999)].
93. Kravtsov N.V., Kravtsov N.N., Makarov A.A., Firov V.V. *Kvantovaya Elektron.*, **23**, 195 (1996) [*Quantum Electron.*, **26**, 189 (1996)].
94. Abraham N.B., Weiss C.O. *Opt. Commun.*, **68**, 437 (1988).
95. Kotomtseva L.A., Kravtsov N.V., Lariontsev E.G., Chekina S.N. *Kvantovaya Elektron.*, **32**, 654 (2002) [*Quantum Electron.*, **32**, 654 (2002)].
96. Kotomtseva L.A., Kravtsov N.V., Lariontsev E.G., Chekina S.N. *Chaos*, **13**, 279 (2003).
97. Zolotoverkh I.I., Kravtsov N.V., Lariontsev E.G., Makarov A.A., Firsov V.V. *Opt. Commun.*, **113**, 249 (1994).
98. Zolotoverkh I.I., Lariontsev E.G. *Kvantovaya Elektron.*, **22**, 1171 (1995) [*Quantum Electron.*, **25**, 1133 (1995)].
99. Zolotoverkh I.I., Kravtsov N.V., Kravtsov N.N., Lariontsev E.G., Makarov A.A. *Kvantovaya Elektron.*, **24**, 638 (1997) [*Quantum Electron.*, **27**, 621 (1997)].
100. Boiko D.L., Golyaev Yu.D., Dmitriev V.G., Kravtsov N.V. *Kvantovaya Elektron.*, **24**, 653 (1997) [*Quantum Electron.*, **27**, 635 (1997)].
101. Boiko D.L., Golyaev Yu.D., Dmitriev V.G., Kravtsov N.V. *Kvantovaya Elektron.*, **25**, 366 (1998) [*Quantum Electron.*, **28**, 355 (1998)].
102. Kravtsov N.V., Lariontsev E.G., Pashinin P.P., Sidorov S.S., Firsov V.V. *Laser Phys.*, **13**, 305 (2003).
103. Dotsenko A.V., Lariontsev E.G. *Kvantovaya Elektron.*, **11**, 176 (1984) [*Sov. J. Quantum Electron.*, **14**, 117 (1984)].
104. Dotsenko A.V., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Nanii O.E., Shelaev A.N. *Kvantovaya Elektron.*, **13**, 86 (1986) [*Sov. J. Quantum Electron.*, **16**, 51 (1986)].
105. Dotsenko A.V., Kornienko L.S., Kravtsov N.V., Lariontsev E.G., Nanii O.E., Shelaev A.N. *Dokl. Akad. Nauk SSSR*, **255**, 339 (1980).
106. Dotsenko A.V., Lariontsev E.G. *Pis'ma Zh. Tekh. Fiz.*, **3**, 899 (1977).
107. Dotsenko A.V., Lariontsev E.G. *Kvantovaya Elektron.*, **6**, 979 (1979) [*Sov. J. Quantum Electron.*, **9**, 576 (1979)].
108. Klimenkova E.V., Lariontsev E.G. *Kvantovaya Elektron.*, **13**, 430 (1986) [*Sov. J. Quantum Electron.*, **16**, 283 (1986)].
109. Dotsenko A.V., Lariontsev E.G., Shelaev A.N. *Pis'ma Zh. Tekh. Fiz.*, **10**, 20 (1984).
110. Klochan E.L., Lariontsev E.G., Nanii O.E., Shelaev A.N. *Kvantovaya Elektron.*, **14**, 1385 (1987) [*Sov. J. Quantum Electron.*, **17**, 877 (1987)].
111. Anishchenko V.S. *Znakomstvo s nelineinoi dinamikoi* (Introduction to Nonlinear Dynamics) (Moscow – Izhevsk: Institute of Computer Studies, 2002).
112. Shuster G. *Determiniriovannyi khaos. Vvedenie* (Introduction to Determinate Chaos) (Moscow: Mir, 1988).
113. Anishchenko V.S. *Splozhnye kolebaniya v prostykh sistemakh* (Complex Oscillations in Simple Systems) (Moscow: Nauka, 1990).
114. Hoffer L.M., Lippi G.L., Abraham N.B., Mandel P. *Opt. Commun.*, **66**, 219 (1988).
115. Koryukin I.V., Khandokhin P.A., Khanon Ya.I. *Kvantovaya Elektron.*, **17**, 978 (1990) [*Sov. J. Quantum Electron.*, **20**, 895 (1990)].
116. Zolotoverkh I.I., Kravtsov N.V., Lariontsev E.G., Makarov A.A., Firsov V.V. *Kvantovaya Elektron.*, **22**, 213 (1995) [*Quantum Electron.*, **25**, 197 (1995)].
117. Aleshin D.A., Kravtsov N.V., Lariontsev E.G., Chekina S.N. *Kvantovaya Elektron.*, **35**, 7 (2005) [*Quantum Electron.*, **35**, 7 (2005)].
118. Kravtsov N.V., Lariontsev E.G. *Kvantovaya Elektron.*, **34**, 487 (2004) [*Quantum. Electron.*, **34**, 487 (2004)].