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On the bifurcation of the circular polarisation of the fifth and seventh pump-field harmonics generated in the plasma produced by the ionisation of a gas of excited hydrogen-like atoms

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***Abstract*.** **Within the framework of the Bethe ionisation model we considered theoretically the dependences of the degree of circular polarisation of the fifth and seventh pump-field harmonics, which are generated due to bremsstrahlung, on the electric intensity of the pump field, the degree of its circular polarisation, and the principal quantum number of the excited states of hydrogen-like atoms of a gas ionised by the pump field. A bifurcation of the circular polarisation of these harmonics was discovered, which confirms our previous hypothesis that this effect is common for harmonics generated due to the bremsstrahlung in the pump field when the plasma electrons oscillate in this field. We determined how the relationships under consideration are scaled with $V_E n/V_Z$, the product of electron oscillation velocity and the principal quantum number of the excited electron divided by the Coulomb velocity.**

Keywords: *bifurcation, degree of circular polarisation, harmonics, excited states of a hydrogen-like atom.*

1. The generation of harmonics of pump radiation field in plasmas has been studied for many years. A bremsstrahlung mechanism was proposed in Ref. [1] for the interpretation of this effect, and in Ref. [2] the influence exerted on this effect by the nonlinear properties of transfer in plasma was determined. There is good reason to believe that the list of possible mechanisms of harmonic generation is not limited to the bremsstrahlung mechanism. While speaking on the experimental properties of the radiation generation by a pump field, we emphasise that the radiation intensity rises by several orders of magnitude when the plasma is photoionised from a gas with preliminarily excited atoms (in comparison with the case of harmonic generation in a plasma photoionised from a gas with unexcited atoms). This effect was theoretically discovered in the model of bremsstrahlung in the pump field [3] and experimentally discovered in Ref. [4].

In the present work, on the basis of the model of bremsstrahlung harmonic generation in the pump field and the Bethe ionisation model (see, for instance, Ref. [5]) we concentrate our attention upon the degree of circular polarisation of the harmonic radiation field. We restrict our consideration to the properties of only the fifth and seventh harmonics. We will determine the nonlinear dependences of the fields of these harmonics on the polarisation and electric intensity of the pump field. Of greatest interest is the effect of bifurcation of complete circular polarisation of the harmonics, which manifests itself in this case. The bifurcation effect for the third harmonic was discovered in our earlier works [5, 6], and in doing this we framed a hypothesis that this effect is common for the bremsstrahlung harmonic generation. This hypothesis is borne out by the present paper.

We briefly consider the question of what the effect of bifurcation of complete circular harmonics polarisation consists. Recall that, when the degree A of circular polarisation of the pump field is equal to $+1$ or -1, the electric field intensity E does not vary with time and there occurs no harmonic generation in the dipole approximation. However, even for a small deviation of A from $+1$ or -1 the generation of harmonics becomes possible, and their degree of circular polarisation is little different from $+1$ or -1. In this sense the values ± 1 are those limiting values which define the limits of harmonic generation; while these harmonics tend to zero in intensity, they simultaneously tend to the prefect circular polarisation.

For low intensities of the pump field, under the variation of the degree of circular pump polarisation there occurs a monotonic change of the degree of circular harmonic polarisation, which nevertheless does not become complete. However, this takes place only for low intensities of the pump field. The picture becomes qualitatively different when the pump field intensity increases to exceed some threshold value $E_{\text{th pol}}$ defined by the nonlinear nature of the pump field – plasma interaction. In particular, in Refs [5, 6] it was shown that the generated third harmonic may possess perfect circular polarisation for certain values of the degree of circular polarisation $(-1 < A < +1)$ when the electric intensity of the pump field exceeds some threshold value. In this case, we hypothesised that this effect, which we termed the bifurcation of complete circular polarisation, was common for the generation of other harmonics. The following expression was derived in Refs [5, 6] for the threshold field of the bifurcation in the third harmonic generation:

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$$E^{(3)}_{\text{th pol}} = 1.8445 \frac{m\omega}{|e|n} V_Z, \tag{1}$$

where e and m are the electron charge and mass; n is the principal quantum number of the excited state of a hydrogen-like atom of the gas whose ionisation produces the plasma;

$$V_Z = \frac{Ze^2}{\hbar} \tag{2}$$

is the Coulomb velocity unit [7]; and Z is the nuclear charge number. The corresponding thresholds for the fifth and seventh harmonics are derived below.

2. We consider a completely ionised plasma in the pump field with an electromagnetic field intensity $\boldsymbol{E} = (E_x, E_y, 0)$:

$$E_x = e_x E \cos(\omega t - kz + \varphi),$$
$$E_y = -e_y E \sin(\omega t - kz + \varphi). \tag{3}$$

Here, E is the real amplitude of the electric intensity of the pump field; φ is the phase of this field; ω and k are its frequency and wave vector related by the equation $\omega^2 = \omega^2_{\text{Le}} + c^2k^2$, where $\omega_{\text{Le}} = (4\pi e^2 N_{\text{e}}/m)^{1/2}$ is the electron Langmuir frequency; and N_{e} is the electron plasma density.

Let $e_\alpha(\alpha = x, y)$ be the polarisation vector components which satisfy the relation

$$e_x^2 + e_y^2 = 1. \tag{4}$$

Since the polarisation tensor of the field (3) is of the form [8]

$$R_{\alpha\beta} = \begin{vmatrix} e_x^2 & \mathrm{i}e_x e_y \\ -\mathrm{i}e_x e_y & e_y^2 \end{vmatrix} \equiv \frac{1}{2}\begin{pmatrix} 1+\xi_3 & \xi_1 - \mathrm{i}\xi_2 \\ \xi_1 + \mathrm{i}\xi_2 & 1-\xi_3 \end{pmatrix}, \tag{5}$$

for the Stokes parameters we have $\xi_1 = 0$, $\xi_2 = -2e_x e_y$, $\xi_3 = e_x^2 - e_y^2$. In what follows our interest is the dependences of harmonic-characterising quantities on the degree $A = -2e_x e_y$ of circular polarisation of the field of the fundamental harmonic (3), i.e. the pump field. In the general case, the degree A of circular polarisation of the $(2N+1)$th harmonic is related to its Stokes parameter as

$$\xi_2^{(2N+1)} = A(2N+1). \tag{6}$$

Here, $2N+1$ is the number of an odd harmonic generated by the electron bremsstrahlung in the plasma under the action of the pump field in the nonrelativistic (dipole) approximation.

Under the assumption that the effective collision frequencies are low in comparison with the pump field frequency (see below), from Eqn (9) of Ref. [5] for the electromagnetic field of the $(2N+1)$th harmonic one can obtain

$$E_x^{(2N+1)} = e_x E \frac{4\pi\sigma_{xx}^{(2N+1)}(2N+1)\omega}{\omega^2_{\text{Le}} - (2N+1)^2(\omega^2 - c^2k^2)}$$
$$\times \sin[(2N+1)(\omega t - kz)], \tag{7}$$

$$E_y^{(2N+1)} = e_y E \frac{4\pi\sigma_{yy}^{(2N+1)}(2N+1)\omega}{\omega^2_{\text{Le}} - (2N+1)^2(\omega^2 - c^2k^2)}$$
$$\times \cos[(2N+1)(\omega t - kz)], \tag{8}$$

where the following notation was used for the nonlinear complex conductivities:

$$\sigma_{xx}^{(2N+1)} = \frac{e^2 N_{\text{e}} \nu_{xx}^{(2N+1)}(n, E, \rho)}{m\omega^2},$$
$$\sigma_{yy}^{(2N+1)} = \frac{e^2 N_{\text{e}} \nu_{yy}^{(2N+1)}(n, E, \rho)}{m\omega^2}. \tag{9}$$

In this case, the effective nonlinear collision frequencies are defined by the formulas

$$\nu_{xx}^{(2N+1)} = \frac{16e^4 Z N_{\text{e}} \Lambda}{\rho^3 m^2 V_E^3} D\alpha^{(+)}(2N+1, \alpha, \rho)\Big|_{b=1}, \tag{10}$$

$$\nu_{yy}^{(2N+1)} = \frac{16e^4 Z N_{\text{e}} \Lambda}{\rho^3 m^2 V_E^3} D\alpha^{(-)}(2N+1, \alpha, \rho)\Big|_{b=1}. \tag{11}$$

Here, Λ is the Coulomb logarithm; $\rho = (1 - A^2)^{1/4}$ is the highest degree of linear polarisation of the pump field; and $V_E = |eE|/m\omega$ is the electron velocity oscillation amplitude in the pump field. Taking into account the l-degeneracy of the excited state of a hydrogen-like atom, for the operator D we obtain the following expression:

$$D = 1 - \frac{\mathrm{d}}{\mathrm{d}b} + \frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}b^2}. \tag{12}$$

We apply this operator to the functions $\alpha^{(+)}$ and $\alpha^{(-)}$ in expressions (10) and (11) and assume that $b = 1$. The explicit expressions of these functions, $\alpha^{(+)}(2N+1, \alpha, \rho)$ and $\alpha^{(-)}(2N+ 1, \alpha, \rho)$, are given in Appendix 1 for the fifth $(N = 2)$ and seventh $(N = 3)$ harmonics. In this case, $\alpha = V_Z b/(V_E n)$.

The above formulas permit representing the degrees of circular polarisation of the fifth $[A(5, x, A)]$ and seventh $[A(7, x, A)]$ harmonics discussed below in the following form:

$$A(5, x, A) = 2 \operatorname{sign} A \frac{G(5, x, \rho) H(5, x, \rho)}{G^2(5, x, \rho) + H^2(5, x, \rho)}, \tag{13}$$

$$A(7, x, A) = 2 \operatorname{sign} A \frac{G(7, x, \rho) H(7, x, \rho)}{G^2(7, x, \rho) + H^2(7, x, \rho)}, \tag{14}$$

where

$$H(2N+1, x, \rho) = \left(\frac{1+\rho^2}{2}\right)^{1/2}$$
$$\times D\alpha^{(+)}(2N+1, \alpha, \rho)\frac{1}{\rho^3 x^3}; \tag{15}$$

$$G(2N+1, x, \rho) = \left(\frac{1-\rho^2}{2}\right)^{1/2}$$
$$\times D\alpha^{(-)}(2N+1, \alpha, \rho)\frac{1}{\rho^3 x^3}; \tag{16}$$

and $x = nV_E/V_Z$ is the dimensionless electric intensity of the pump field, which follows from the definition of V_E.

The analytic expressions for formulas (13), (14) are cumbersome and are therefore illustrated in a graphical form below.

3. As an illustration of the qualitative dependences, Fig. 1 shows the three-dimensional projection of the function $A(5, x, A) \equiv A(5)$. One can see that in the range of small values of the argument x, the function $A(5)$ varies smoothly throughout the range of the argument A. By contrast, when the x-argument values are not small, the function $A(5)$ in the range of small values of the argument A changes its sign abruptly approximately from its minimal value $A(5) \approx -1$ to its maximal value $A(5) \approx +1$. Here, we do not give the corresponding three-dimensional projection of the $A(7, x, A)$ function similar to that plotted in Fig. 1. Instead, given in Fig. 2 is the three-dimensional projection $A(7, x, A)$ *complementary* to Fig. 1, because it exists for $A(5, x, A)$ as well.

The degrees of circular polarisation of the fifth and seventh harmonics as functions of the degree A of circular polarisation of the electric intensity of the pump field and of its dimensionless amplitude x are shown in Figs 3 and 4. Curve (*1*) in Fig. 3 corresponds to $x = 1$, when the function $A(5, x, A)$ monotonically and smoothly increases from it minimal value -1 to its maximal value $+1$; curve (*2*) corresponds to the threshold intensity of the dimensionless electric field $x(5)_{\text{th pol}} = 2.25$, i.e., to the threshold [the curve $A(5, 2.25, A)$], which separates the nonbifurcation and bifurcation domains of the curves – the domains with monotonic and nonmonotonic A-dependences $A(5, x, A)$ in the range from -1 to $+1$. In this case, three curves are given in the domains of nonmonotonic dependence: curves (*3*) (which corresponds to $x = 3.5$), (*4*) ($x = 5$) and (*5*) ($x = 10$). All of them have the common property that the function $A(5, x, A)$ assumes the -1 value not only for $A = -1$, but also for a higher negative value which depends on the electric intensity of the pump field. In accordance with these three curves, the unit value of the $A(5, x, A)$ function is reached not only for $A = +1$, but for a smaller positive value as well, which also depends on the electric intensity of the pump field. We note that the two new values of the degree of circular pump polarisation (positive and negative) at which the fifth harmonic for a given x exhibits complete circular polarisation are equal in modulus.

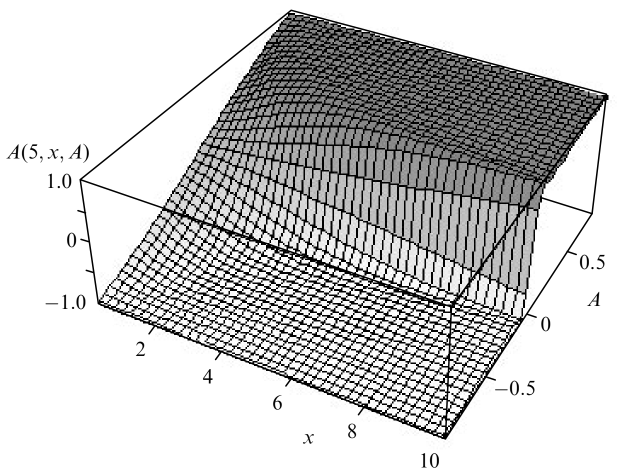

Figure 1. Three-dimensional representation of the degree of circular polarisation of the fifth harmonic as a function of the variables x and A.

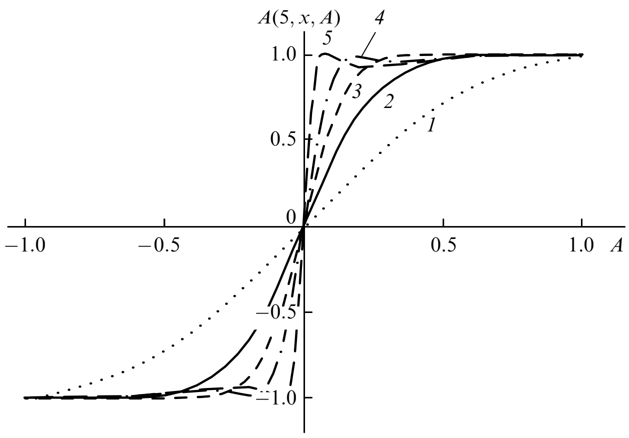

Figure 3. Degree $A(5, x, A)$ of circular polarisation of the fifth harmonic as a function of degree A of circular polarisation of the pump field for $x = 1$ (*1*), 2.25 (*2*), 3.5 (*3*), 5 (*4*), and 10 (*5*).

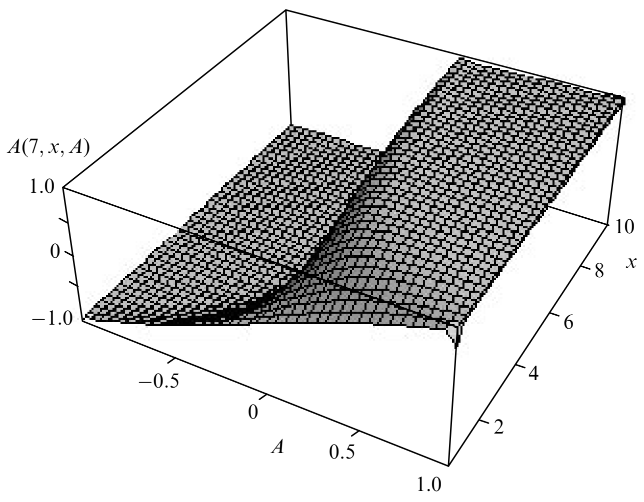

Figure 2. Three-dimensional representation of the degree of circular polarisation of the seventh harmonic as a function of the variables x and A.

It is pertinent to note here that the threshold curve in Fig. 3, like the nonmonotonic curves in a wide range of the degrees of circular pump polarisation, describe a dependence which is close to unity for $A > 0$ and to minus unity for $A < 0$. In other words, the circular polarisation of the fifth harmonic in these wide A-value ranges is close to the perfect circular polarisation. With increase of the electric intensity of the pump field there emerge A-value ranges, accordingly for positive and negative values of the degree of circular pump polarisation, in which there appears, against the background of almost complete circular polarisation, a departure from the perfect circular one. When the dimensionless electric field intensity is high enough, this departure becomes quite significant.

Lastly, as is evident from Fig. 3, the range of the degrees of circular pump polarisation in which the degree of circular polarisation of the fifth harmonic changes its sign becomes narrower when the pump field intensity is high enough. A sharp change of circular polarisation is also evident in Fig. 1.

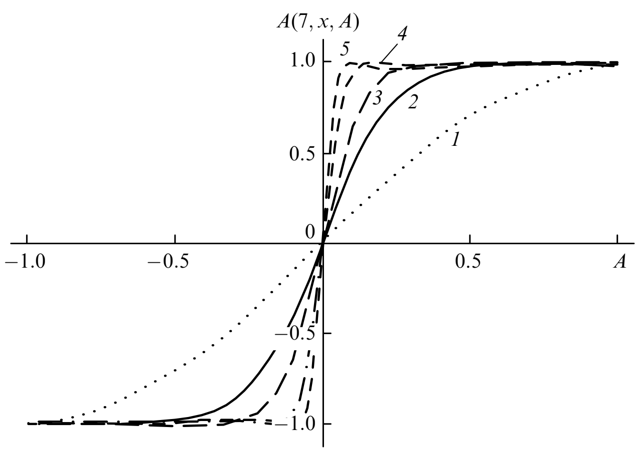

Figure 4. Degree $A(7,\ x,\ A)$ of circular polarisation of the seventh harmonic as a function of degree A of circular polarisation of the pump field for $x = 1$ (*1*), 2.57 (*2*), 4 (*3*), 7 (*4*), and 10 (*5*).

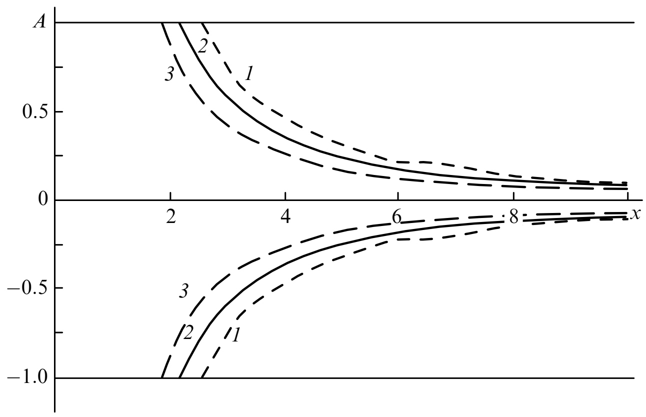

Figure 5. Curves which define the values of A and x in the case of perfect polarisation for the three harmonics. $A(2N+1, x, A) = 1$ for the curves in the upper half-plane and $A(2N+1, x, A) = -1$ in the lower half-plane. Curves (*1*) correspond to the seventh harmonic, (*2*) to the fifth harmonic, and (*3*) to the third one (taken from Ref. [5]).

Figure 4, like Fig. 3, shows five curves, this time for the seventh harmonic: curve (*1*) corresponds to $x = 1$, curve (*2*) corresponds to $x = 2.57$ and limits the domain of bifurcation behaviour of the dependence $A(7, x, A)$, curve (*3*) corresponds to $x = 4$, curve (*4*) to $x = 7$, and curve (*5*) to $x = 10$. Like Fig. 3, Fig. 4 describes the bifurcation of the degree of circular polarisation of a harmonic generated in the electron bremsstrahlung in the pump field. Figure 4 differs from Fig. 3 primarily in that curve (*2*), which delimits the bifurcation domains, corresponds to a dimensionless electric field intensity of 2.57 rather than 2.25. Furthermore, a comparison of Figs 3 and 4 shows that the sign of circular polarisation of the seventh harmonic changes more abruptly than that of the fifth harmonic with the change of sign of the degree of circular pump polarisation.

4. One can infer from Figs 1 and 2 as well as from Figs 3 and 4 that, first, $x(5)_{\text{th pol}} = 2.25$ and $x(7)_{\text{th pol}} = 2.57$, which correspond to the fifth and seventh harmonic bifurcation thresholds, are realised for $A = \pm 1$ and, second, $A(5, x, A) = \pm 1$ and $A(7, x, A) = \pm 1$, which correspond to perfect circular polarisation of the harmonics, are realised for different values of dimensionless electric pump field and of the degree of circular polarisation of the pump field. These values are linked by the relationships plotted in Fig. 5. To complete the picture, also plotted in Fig. 5 are curves (*3*), which correspond to the third harmonic and were obtained in Ref. [5]. In this case, the two parallel straight lines which represent $A = \pm 1$ correspond to the limiting values of the degree of polarisation, whereby the intensities of all harmonics tend to zero. Curves (*1*) correspond to the seventh harmonic and curves (*2*) to the fifth one. The points in all curves in Fig. 5 correspond to the unit values of the degree of circular harmonics polarisation multiplied by sign$A$. One can see that with increase in dimensionless electric field the corresponding degree of circular pump polarisation in the vicinity of the threshold values decreases relatively fast in magnitude, while on further increase in x this decrease becomes quite slow. To gain an impression of the intensities of the harmonics generated above the bifurcation threshold which correspond to complete circular polarisation of the harmonics for $A < 1$, we consider their production efficiency $\eta^{(2N+1)}$. By a harmonic generation efficiency is meant the ratio between the period-averaged squared electric intensity of a harmonic field and the period-averaged squared intensity of the pump field. From Eqns (7)–(11) and (15), (16) it follows that

$$\eta^{(2N+1)} = \frac{(2N+1)^2}{N^2(N+1)^2}\left(\frac{4e^4 Z N_{\mathrm{e}} \Lambda n^3}{m^2 V_Z^3 \omega}\right)^2$$

$$\times\{H^2[2N+1, x,\ (1-A^2)^{1/4}]$$

$$+ G^2[2N+1, x, (1-A^2)^{1/4}]\}. \quad (17)$$

Collected in Tables 1 and 2 for $A(5, x, A) = 1$ and $A(7, x, A) = 1$ are the values of the functions $\Psi(5, x, A)$ and $\Psi(7, x, A)$, which are, in accord with expression (17), defined by the formulas

Table 1.

$A(5,x,A)$	A	x	$\Psi(5,x,A)\times 10^6$
1	1	2.25	0
1	0.975	2.27	0.012
1	0.97	2.28	0.018
1	0.96	2.29	0.031
1	0.95	2.3	0.049
1	0.94	2.31	0.070
1	0.91	2.35	0.153
1	0.875	2.4	0.284
1	0.82	2.5	0.539
1	0.72	2.7	1.099
1	0.60	3	1.740
1	0.53	3.25	1.923
1	0.45	3.5	2.182
1	0.40	3.75	2.104
1	0.358	4	1.959
1	0.24	5	1.258
1	0.137	7	0.422
1	0.110	8	0.252
1	0.092	9	0.156
1	0.075	10	0.099

Table 2.

$A(7,x,A)$	A	x	$\Psi(7,x,A)\times10^8$
1	1	2.57	0
1	0.991	2.58	5.77×10^{-5}
1	0.96	2.6	5.20×10^{-3}
1	0.945	2.63	13.31×10^{-3}
1	0.915	2.7	4.67×10^{-2}
1	0.825	2.85	0.394
1	0.752	3	1.047
1	0.69	3.15	1.914
1	0.65	3.25	2.633
1	0.565	3.5	4.492
1	0.47	4	5.839
1	0.3	5	7.837
1	0.22	6	6.020
1	0.19	7	3.816
1	0.13	8	2.853
1	0.113	9	1.890
1	0.095	10	1.290

$$\Psi(5,x,A)=\frac{1}{36}\Big\{H^2[5,x,(1-A^2)^{1/4}]+G^2[5,x,(1-A^2)^{1/4}]\Big\},\tag{18}$$

$$\Psi(7,x,A)=\frac{1}{144}\Big\{H^2[7,x,(1-A^2)^{1/4}]+G^2[7,x,(1-A^2)^{1/4}]\Big\}.\tag{19}$$

To estimate the order of magnitude of the resultant expressions, we write the relation

$$\left(\frac{4e^4ZN_{\rm e}\Lambda n^3}{m^2V_Z^3\omega}\right)^2=\left(\frac{\Lambda n^3N_{\rm e}}{Z^2\omega}\right)^2\left(\frac{4\hbar^3}{m^2e^2}\right)^2=$$

$$=\frac{n^6}{Z^4}\left(\frac{2\times10^{15}\,{\rm c}^{-1}}{\omega}\right)^2\left(\frac{N_e}{10^{17}\,{\rm cm}^{-3}}\right)^2\Lambda^2(115\times10^{-8})^2.\tag{20}$$

For $\omega=2\times10^{15}$ s^{-1}, $N_{\rm e}=10^{17}$ cm^{-3}, $n=5$, $Z=1$, and $\Lambda=6$ expression (20) is approximately equal to 0.73×10^{-6}. We are reminded that the treatment of the problem undertaken in the present work corresponds to the Bethe ionisation model, whereby the pump intensity should, according to Refs [4, 5] satisfy the inequality

$$q>q_{\rm B}=\frac{Z^6}{n^8}\,1.37\times10^{14}\ {\rm W\ cm^{-2}}.\tag{21}$$

In particular, for $Z=1$ and $n=5$ we have $q_{\rm B}=3.5\times10^8$ W cm^{-2}. Here, we also give the expressions for the threshold values of the pump radiation intensity which correspond to the emergence of bifurcation for the third, fifth, and seventh harmonic:

$$q^{(3)}_{\rm th\,pol}\approx\frac{Z^2}{n^2}\left(\frac{\hbar\omega}{1\ {\rm eV}}\right)^2 1.7\times10^{14}\ {\rm W\ cm^{-2}},\tag{22}$$

$$q^{(5)}_{\rm th\,pol}\approx\frac{Z^2}{n^2}\left(\frac{\hbar\omega}{1\ {\rm eV}}\right)^2 2.4\times10^{14}\ {\rm W\ cm^{-2}},\tag{23}$$

$$q^{(7)}_{\rm th\,pol}\approx\frac{Z^2}{n^2}\left(\frac{\hbar\omega}{1\ {\rm eV}}\right)^2 3.2\times10^{14}\ {\rm W\ cm^{-2}}.\tag{24}$$

For $\hbar\omega=1$ eV, $Z=1$, and $n=5$ we have $q^{(5)}_{\rm th\,pol}\approx10^{13}$ W cm^{-2} and $q^{(7)}_{\rm th\,pol}\approx1.3\times10^{13}$ W cm^{-2}. It is evident that these values exceed $q_{\rm B}$, i.e., satisfy the Bethe inequality (21).

We now give a numerical estimate of the fifth harmonic intensity. The function $\Psi(5,x,A)$ peaks when the pump intensity exceeds the bifurcation threshold by about a factor of 2.4, which is equal to $\sim2.4\times10^{13}$ W cm^{-2}. By using the above estimate of expression (20) and taking the value $\Psi_{\max}(5,x,A)\approx0.2\times10^{-5}$, we obtain a value of $\sim97.5\times10^{-11}$ for the efficiency of fifth harmonic generation and a value ~950 W cm^{-2} for the intensity of the fifth harmonic.

For comparison we give the corresponding estimate for the third harmonic. According to Appendix 2, the function $\Psi(3,x,A)$ peaks when the intensity exceeds the bifurcation threshold by about a factor of 1.55, and therefore the pump intensity corresponding to this peak is equal to $\sim4.4\times10^{12}$ W cm^{-2} for the parameters employed in the foregoing. According to Table 3 (see Appendix 2), $\Psi_{\max}(3,x,A)\sim3.4\times10^{-4}$. And so, in view of the above-estimated expression (20) (0.73×10^{-6}), we arrive at a value of 20×10^{-9} for the efficiency of third harmonic generation and a value of $\sim9\times10^3$ W cm^{-2} for the intensity of the third harmonic.

5. To summarise the above discussion, we indicate first of all that the bifurcation effect of perfect circular polarisation was theoretically determined for the fifth and seventh harmonics. This bears out the hypothesis of Refs [5, 6] that this effect is common for the harmonics generated due to the bremsstrahlung in plasma. We demonstrated how the harmonic polarisation scales with the dimensionless field $x=nV_E/V_Z$. This scaling defines how the degree of circular harmonic polarisation depends on the principal quantum number of the electrons of the hydrogen-like gas atoms whose ionisation gives rise to the plasma. In this case, owing to electron collisions, the n-dependence ceases to manifest itself for pulses longer than $\tau_n\sim2(N_{\rm e}/10^{17}\ {\rm cm}^{-3})^{-1}\times(Z^2n^{-3})\times10^8$ s, i.e. for $Z=1$, $n=5$, and $N_{\rm e}=10^{17}$ cm^{-3} we have $\tau_n\sim1.6\times10^{-11}$ s. In conclusion we emphasise the following fact established in our work: beginning with the near-threshold values of the pump intensity and for its higher values, the circular polarisation of the harmonics is close to the perfect one outside of the domain of very small values of the degree of circular pump polarisation. The last-named domain becomes narrower with increase in pump intensity. This property is quite characteristic for the bremsstrahlung mechanism of harmonic generation.

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Appendix 1

Here, we give the expressions for the functions $\alpha^{(+)}(5,\alpha,A)$, $\alpha^{(-)}(5,\alpha,A)$ and $\alpha^{(+)}(7,\alpha,A)$, $\alpha^{(-)}(7,\alpha,A)$, which were derived in Ref. [9]:

$$\alpha^{(+)}\left(5,\frac{b}{x},\rho\right)=\frac{4b}{x\rho}\Big[-\frac{2}{5}+\frac{8}{5\rho^2}+\frac{16b^2}{15x^2\rho^2}+$$

$$+\Big(-\frac{2}{15}-\frac{8}{5\rho^2}-\frac{16b^2}{15x^2\rho^2}\Big)\Big(\frac{2b^2+x^2-x^2\rho^2}{2b^2+x^2+x^2\rho^2}\Big)^{1/2}\Big]$$

$$+\frac{1}{(1+\rho^2)^{1/2}}2^{3/2}\rho\Big[\Big(\frac{3}{10}-\frac{16}{15\rho^4}-\frac{2}{3\rho^2}\Big)$$

$$\times E\Big(\arctan\Big[\frac{x^2(1+\rho^2)}{2b^2}\Big]^{1/2},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)+\Big(-\frac{1}{6}$$

$$+\frac{16}{15\rho^4}-\frac{2}{5\rho^2}\Big)F\Big(\arctan\Big[\frac{x^2(1+\rho^2)}{2b^2}\Big]^{1/2},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big],$$

$$\alpha^{(-)}\Big(5,\frac{b}{x},\rho\Big)=\frac{1}{15\rho^3(1-\rho^2)}\Big\{\frac{8b}{x}(1-\rho^2)$$

$$\times\Big(12+3\rho^2+\frac{8b^2}{x^2}\Big)+\Big[\frac{6b}{x}\Big(\frac{2b^2+x^2+x^2\rho^2}{2b^2+x^2-x^2\rho^2}\Big)^{1/2}$$

$$\times\Big(4-2\rho^2-\rho^4-\frac{4b^2\rho^2}{x^2}+\frac{16b^2}{x^2}+\frac{16b^4}{x^4}\Big)$$

$$-\frac{2b}{x}\Big(\frac{2b^2+x2-x^2\rho^2}{2b^2+x^2+x^2\rho^2}\Big)^{1/2}\Big(60-2\rho^2-19\rho^4+\frac{80b^2}{x^2}$$

$$+\frac{4b^2\rho^2}{x^2}+\frac{48b^4}{x^4}\Big)\Big]+\sqrt{2}(1+\rho^2)^{1/2}(-32+20\rho^2+9\rho^4)$$

$$\times E\Big(\arctan\Big[\frac{x^2(1+\rho^2)}{2b^2}\Big]^{1/2},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)$$

$$-\frac{\sqrt{2}(1-\rho^2)}{(1+\rho^2)^{1/2}}(-32-12\rho^2+5\rho^4)$$

$$\times F\Big(\arctan\Big[\frac{x^2(1+\rho^2)}{2b^2}\Big]^{1/2},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big\},$$

$$\alpha^{(+)}\Big(7,\frac{b}{x},\rho\Big)=\frac{1}{1+\rho^2}\Big\{-\frac{2b\rho}{7x}\Big\{1-\frac{4}{\rho^2}-\frac{4}{\rho^4}+\frac{8}{\rho^6}$$

$$+\frac{b^2}{x^2\rho^2}\Big(-8-\frac{16}{\rho^2}+\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(-16+\frac{96}{\rho^2}\Big)$$

$$+64\frac{b^6}{x^6\rho^6}+\Big(\frac{x^2+2b^2-x^2\rho^2}{x2+2b^2+x^2\rho^2}\Big)^{1/2}\Big[1+\frac{4}{\rho^2}-\frac{4}{\rho^4}$$

$$-\frac{8}{\rho^6}+\frac{b^2}{x^2\rho^2}\Big(8-\frac{16}{\rho^2}-\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(-16-\frac{96}{\rho^2}\Big)$$

$$-64\frac{b^6}{x^6\rho^6}\Big]\Big\}+\Big\{2\rho\Big[\Big(\frac{64b^7}{7\rho^6x^7}+\frac{96b^5}{5\rho^6x^5}+\frac{16b^5}{5\rho^4x^5}$$

$$+\frac{48b^3}{\rho^6x^3}+\frac{16b^3}{3\rho^4x^3}-\frac{8b^3}{3\rho^2x^3}+\frac{8b}{\rho^6x}+\frac{4b}{\rho^4x}-\frac{4b}{\rho^2x}-\frac{b}{x}\Big)$$

$$+\frac{1}{105x^7}\Big\{b\Big(\frac{x^2+2b^2-x^2\rho^2}{x^2+2b^2+x^2\rho^2}\Big)^{1/2}\Big[-\frac{960b^6}{\rho^6}+x^6$$

$$\times\Big(143+\frac{92}{\rho^2}-\frac{876}{\rho^4}-\frac{840}{\rho^6}\Big)-48b^4x^2\Big(\frac{42}{\rho^6}+\frac{1}{\rho^4}\Big)$$

$$-8b^2x^4\Big(\frac{1}{\rho^2}+\frac{210}{\rho^6}+\frac{166}{\rho^4}\Big)\Big]\Big\}+\Big(63+\frac{208}{\rho^2}$$

$$-\frac{224}{\rho^4}-\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times E\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)$$

$$+\Big(25-\frac{144}{\rho^2}-\frac{160}{\rho^4}+\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times F\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big]\Big\}\Big\},$$

$$\alpha^{(-)}\Big(7,\frac{b}{x},\rho\Big)=\frac{1}{1-\rho^2}\Big\{-\frac{2b\rho}{7x}\Big\{-1-\frac{4}{\rho^2}+\frac{4}{\rho^4}+\frac{8}{\rho^6}$$

$$+\frac{b^2}{x^2\rho^2}\Big(-8+\frac{16}{\rho^2}+\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(16+\frac{96}{\rho^2}\Big)$$

$$+64\frac{b^6}{x^6\rho^6}-\Big(\frac{x^2+2b^2+x^2\rho^2}{x^2+2b^2-x^2\rho^2}\Big)^{1/2}\Big[1-\frac{4}{\rho^2}-\frac{4}{\rho^4}$$

$$+\frac{8}{\rho^6}+\frac{b^2}{x^2\rho^2}\Big(-8-\frac{16}{\rho^2}+\frac{48}{\rho^4}\Big)+\frac{b^4}{x^4\rho^4}\Big(-16+\frac{96}{\rho^2}\Big)$$

$$+64\frac{b^6}{x^6\rho^6}\Big]\Big\}+\Big\{2\rho\Big[\Big(\frac{64b^7}{7\rho^6x^7}+\frac{96b^5}{5\rho^6x^5}-\frac{16b^5}{5\rho^4x^5}$$

$$+\frac{16b^3}{\rho^6x^3}-\frac{16b^3}{3\rho^4x^3}-\frac{8b^3}{3\rho^2x^3}+\frac{8b}{\rho^6x}-\frac{4b}{\rho^4x}-\frac{4b}{\rho^2x}+\frac{b}{x}\Big)$$

$$+\frac{1}{105x^7}\Big\{b\Big(\frac{x^2+2b^2-x^2\rho^2}{x^2+2b^2+x^2\rho^2}\Big)^{1/2}\Big[-\frac{960b^6}{\rho^6}+x^6$$

$$\times\Big(17+\frac{484}{\rho^2}-\frac{36}{\rho^4}-\frac{840}{\rho^6}\Big)+576b^4x^2\Big(-\frac{7}{2\rho^6}-\frac{1}{4\rho^4}\Big)$$

$$-16b^2x^4\Big(-\frac{41}{2\rho^2}+\frac{105}{\rho^6}+\frac{13}{\rho^4}\Big)\Big]\Big\}+\Big(-63+\frac{208}{\rho^2}$$

$$+\frac{224}{\rho^4}-\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times E\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)$$

$$+\Big(-185+\frac{210\rho^2}{1+\rho^2}+\frac{304}{\rho^2}-\frac{608}{\rho^4}+\frac{384}{\rho^6}\Big)\frac{(1+\rho^2)^{1/2}}{105\sqrt{2}}$$

$$\times F\Big(\arctan\frac{x(1+\rho^2)^{1/2}}{\sqrt{2}b},\Big(\frac{2\rho^2}{1+\rho^2}\Big)^{1/2}\Big)\Big]\Big\}\Big\},$$

where $E(\varphi,k)$ and $F(\varphi,k)$ are elliptic functions defined according to Ref. [10].

Appendix 2

For comparison of the results obtained in the present work with our previous results for the third harmonic, below we give a Table of the values of the function $\Psi(3,x,A)$ (Table 3) obtained on the basis of the calculations of Ref. [5]. Here, similar to formulas (18) and (19),

$$\Psi(3,x,A) = \frac{1}{4}\{H^2[3,x,\ (1-A^2)^{1/4}] + G^2[3,x,(1-A^2)^{1/4}]\}.$$

Table 3.

$A(3,x,A)$	A	x	$\Psi(3,x,A)\times 10^4$
1	1	1.8445	0
1	0.97	1.89	0.553
1	0.945	1.9	1.011
1	0.905	1.95	1.620
1	0.865	2	2.141
1	0.67	2.3	3.427
1	0.57	2.5	3.413
1	0.42	3	2.404
1	0.26	4	0.990
1	0.17	5	0.420
1	0.13	6	0.197
1	0.08	8	0.053
1	0.05	10	0.018

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