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Coherent propagation of a short polarised radiation pulse in a one-dimensional resonance Bragg grating

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***Abstract*. The propagation of an optical ultrashort pulse in a resonance Bragg grating is considered taking into account the polarisation of electromagnetic radiation. It is assumed that the grating is formed by thin films containing two-level atoms with the triply degenerate upper energy level. The system of equations is derived for the envelopes of electromagnetic pulses counterpropagating in such a grating. In the long-wavelength (continual) approximation, the system of equations generalising the known system for scalar waves is obtained. The solutions corresponding to elliptically (in particular, linearly and circularly) polarised stationary pulses are found. An arbitrary degree of ellipticity is possible only in a medium with a preliminary prepared stage of resonance atoms.**

Keywords: *Bragg grating, polarised radiation, coherent pulse propagation.*

1. Introduction

One of the objects of nonlinear optics attracting attention for a long time is a localised electromagnetic wave in a nonlinear medium with the linear refractive index periodically changing along the propagation direction of the wave. If the wave vectors of counterpropagating waves are related by the Bragg condition, such localised waves are called Bragg solitons and media themselves are called Bragg gratings or one-dimensional photonic crystals [1 – 3]. A Bragg grating can be obtained not only by varying periodically the linear refractive index. In particular, alternating layers of linear and nonlinear materials give another example of the Bragg grating. In [4, 5], a homogeneous linear dielectric medium consisting of parallel thin films spaced by the step a and doped with resonance impurities was considered. Such a medium was called a resonance Bragg grating (RBG). The term 'thin film' means here that the film thickness l_f is smaller than the wavelength of light propagating through this medium. By using the model of two-level atoms interacting with an optical ultrashort pulse (USP), it was shown [5 – 11] that a stationary pulse, which is similar to the 2π self-induced transparency pulse, can propagate in a RBG. It was found [9] that, aside from bright solitons, dark Bragg solitons can also propagate in the RBG. The dipole – dipole interaction between resonance atoms was considered in [12]. The results of many studies in this field are presented in review [13]. Recent numerical simulations [14] showed that a nonstationary pulse can exist in the RBG, which looks like the 2π pulse with periodically changing propagation velocity (the so-called optical buzzeron). In [14], the approximate analytic expression describing this pulse with good accuracy was obtained.

It was assumed in all the above-mentioned studies that the electromagnetic field is always linearly polarised. This assumption is valid if the states of a two-level atom are singlet states. It remains valid if one or both states are doublets but radiation is circularly polarised. The polarisation properties of Bragg solitons in the RBG can be complicated if the energy levels of atoms are degenerate over the projections of angular momenta [15 – 17].

In this paper, we consider the simplest model of impurity atoms in a thin film, which allows us to analyse the possibility of propagation of a stationary electromagnetic-field pulse, i.e. the vector Bragg soliton (VBS). We derive exact equations for the amplitudes of counterpropagating waves in the RBG. For USPs of duration shorter than all the relaxation times of the resonance medium, the system of Maxwell – Bloch equations can be obtained in the long-wavelength approximation. This system generalises the equations of the scalar theory of Bragg solitons in the RBG whose solutions describe different VBSs.

2. Coupling equations

Consider an optical pulse propagating along the x axis and intersecting successively thin isotropic films located in planes with coordinates $\ldots x_{n-1},\ x_n,\ x_{n+1}\ldots$ (Fig. 1). A medium between the layers has the dielectric constant ε. For definiteness, we will consider the TE wave. Let us assume that the field strength vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ lie in the plane of thin films. These vectors and the polarisation $\boldsymbol{P}$ of atoms inside a thin film are represented by the integrals

$$\boldsymbol{E}(x,y,z,t)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\boldsymbol{k}_\perp}{(2\pi)^2}\exp(-\mathrm{i}\omega t+\mathrm{i}\boldsymbol{r}\boldsymbol{k}_\perp)\boldsymbol{E}(x,\boldsymbol{k}_\perp,\omega),$$

$$\boldsymbol{H}(x,y,z,t)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\boldsymbol{k}_\perp}{(2\pi)^2}\exp(-\mathrm{i}\omega t+\mathrm{i}\boldsymbol{r}\boldsymbol{k}_\perp)\boldsymbol{H}(x,\boldsymbol{k}_\perp,\omega),$$

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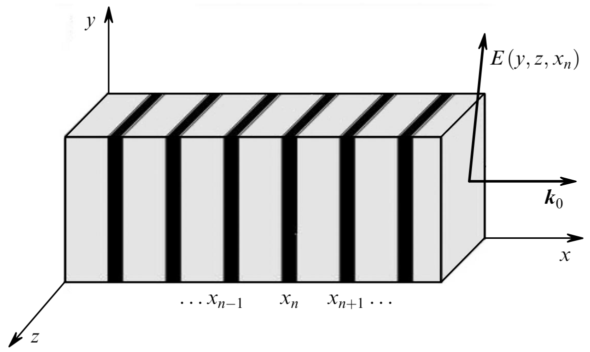

Figure 1. Schematic structure of a Bragg grating formed by the layers of resonance atoms.

$$\boldsymbol{P}(x_n, y, z, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}\boldsymbol{k}_\perp}{(2\pi)^2} \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{r}\boldsymbol{k}_\perp)\boldsymbol{P}(x, \boldsymbol{k}_\perp, \omega).$$

Here, the vector $\boldsymbol{k}_\perp$ lies in a plane normal to the propagation direction of the waves. The Fourier components $\boldsymbol{E}(x, \boldsymbol{k}_\perp, \omega)$ and $\boldsymbol{H}(x, \boldsymbol{k}_\perp, \omega)$ of the vectors outside the film are determined by Maxwell's equations, and for $x = x_n$ – by the continuity condition, so that for the case of transverse waves under study, we arrive to analysis of the system of equations

$$\frac{\mathrm{d}^2 E_j}{\mathrm{d}x^2} + k_0 \varepsilon E_j = 0, \quad j = z, y,$$
$$H_z = -\frac{\mathrm{i}}{k_0}\frac{\mathrm{d}E_y}{\mathrm{d}x}, \quad H_y = \frac{\mathrm{i}}{k_0}\frac{\mathrm{d}E_z}{\mathrm{d}x} \tag{1a}$$

with the boundary conditions

$$E_z(x_n - 0) = E_z(x_n + 0),$$
$$H_z(x_n + 0) - H_z(x_n - 0) = 4\pi \mathrm{i} k_0 P_y(\boldsymbol{k}_\perp, \omega),$$
$$E_y(x_n - 0) = E_y(x_n + 0), \tag{1b}$$
$$H_y(x_n + 0) - H_y(x_n - 0) = -4\pi \mathrm{i} k_0 P_z(\boldsymbol{k}_\perp, \omega).$$

Here, $k_0 = \omega/c$. The solution of Eqn (1a) in the region $x_n < x < x_{n+1}$ has the form

$$E_y(x, \boldsymbol{k}_\perp, \omega) = A_n(\boldsymbol{k}_\perp, \omega)\exp[\mathrm{i}q(x - x_n)] + B_n(\boldsymbol{k}_\perp, \omega)\exp[-\mathrm{i}q(x - x_n)],$$
$$E_z(x, \boldsymbol{k}_\perp, \omega) = C_n(\boldsymbol{k}_\perp, \omega)\exp[\mathrm{i}q(x - x_n)] + D_n(\boldsymbol{k}_\perp, \omega)\exp[-\mathrm{i}q(x - x_n)],$$
$$H_z(x, \boldsymbol{k}_\perp, \omega) = qk_0^{-1}\{A_n(\boldsymbol{k}_\perp, \omega)\exp[\mathrm{i}q(x - x_n)] - B_n(\boldsymbol{k}_\perp, \omega)\exp[-\mathrm{i}q(x - x_n)]\},$$
$$H_y(x, \boldsymbol{k}_\perp, \omega) = -qk_0^{-1}\{C_n(\boldsymbol{k}_\perp, \omega)\exp[\mathrm{i}q(x - x_n)] - D_n(\boldsymbol{k}_\perp, \omega)\exp[-\mathrm{i}q(x - x_n)]\},$$

where $q = k_0\sqrt{\varepsilon}$. Therefore, the amplitudes A_n, B_n, C_n, and D_n determine the electromagnetic field in the medium under study. Consider the point x_n. The electric field for $x = x_n - \delta x$ ($\delta x \ll a$) is determined by the amplitudes A_n^{L}, B_n^{L}, C_n^{L}, and D_n^{L}, and for $x = x_n + \delta x$ – by the amplitudes A_n^{R}, B_n^{R}, C_n^{R}, and D_n^{R}. The continuity conditions (1b) give the relation between these amplitudes:

$$A_n^{\mathrm{R}} + B_n^{\mathrm{R}} = A_n^{\mathrm{L}} + B_n^{\mathrm{L}},$$
$$A_n^{\mathrm{R}} - B_n^{\mathrm{R}} = A_n^{\mathrm{L}} - B_n^{\mathrm{L}} + 4\pi \mathrm{i} k_0^2 q^{-1} P_{yn},$$
$$C_n^{\mathrm{R}} + D_n^{\mathrm{R}} = C_n^{\mathrm{L}} + D_n^{\mathrm{L}}, \tag{2}$$
$$C_n^{\mathrm{R}} - D_n^{\mathrm{R}} = C_n^{\mathrm{L}} - D_n^{\mathrm{L}} + 4\pi \mathrm{i} k_0^2 q^{-1} P_{zn},$$

where $P_{jn} = P_j(A_n^{\mathrm{R}} + B_n^{\mathrm{R}}, C_n^{\mathrm{R}} + D_n^{\mathrm{R}})$ are the Cartesian components of the surface polarisation vector for a thin film at the point x_n induced by the electric field inside the film. We can find from Eqns (2) that

$$A_n^{\mathrm{R}} = A_n^{\mathrm{L}} + 2\pi \mathrm{i} k_0^2 q^{-1} P_{yn}, \quad B_n^{\mathrm{R}} = B_n^{\mathrm{L}} - 2\pi \mathrm{i} k_0^2 q^{-1} P_{yn},$$
$$C_n^{\mathrm{R}} = C_n^{\mathrm{L}} + 2\pi \mathrm{i} k_0^2 q^{-1} P_{zn}, \quad D_n^{\mathrm{R}} = D_n^{\mathrm{L}} - 2\pi \mathrm{i} k_0^2 q^{-1} P_{zn}. \tag{3}$$

Taking into account the dependence of the electric field strength on x in the region between films, we obtain

$$A_{n+1}^{\mathrm{L}} = A_n^{\mathrm{R}}\exp(\mathrm{i}qa), \quad B_{n+1}^{\mathrm{L}} = B_n^{\mathrm{R}}\exp(-\mathrm{i}qa),$$
$$C_{n+1}^{\mathrm{L}} = C_n^{\mathrm{R}}\exp(\mathrm{i}qa), \quad D_{n+1}^{\mathrm{L}} = D_n^{\mathrm{R}}\exp(-\mathrm{i}qa). \tag{4}$$

Expressions (3) and (4) allow us to write the recurrent relations

$$A_{n+1}^{\mathrm{L}} = A_n^{\mathrm{L}}\exp(\mathrm{i}qa) + 2\pi \mathrm{i} k_0^2 q^{-1} P_{yn}\exp(\mathrm{i}qa),$$
$$B_{n+1}^{\mathrm{L}} = B_n^{\mathrm{L}}\exp(-\mathrm{i}qa) - 2\pi \mathrm{i} k_0^2 q^{-1} P_{yn}\exp(-\mathrm{i}qa),$$
$$C_{n+1}^{\mathrm{L}} = C_n^{\mathrm{L}}\exp(\mathrm{i}qa) + 2\pi \mathrm{i} k_0^2 q^{-1} P_{zn}\exp(\mathrm{i}qa), \tag{5}$$
$$D_{n+1}^{\mathrm{L}} = D_n^{\mathrm{L}}\exp(-\mathrm{i}qa) - 2\pi \mathrm{i} k_0^2 q^{-1} P_{zn}\exp(-\mathrm{i}qa).$$

Below, we will omit the superscript at amplitudes. Equations (5) are exact equations, which are similar to the differential equations of the method of coupled waves, but here no assumptions were used about the rate of changing electromagnetic field amplitudes or resonance conditions. By assuming that the grating spacing a depends on the number of the site in which a thin film is located, we obtain the model of an inhomogeneous grating.

3. Coupling equation in the continual limit

Let us assume that the inequality $a\Delta q = aq - 2\pi \ll 1$ corresponding to the Bragg resonance condition is fulfilled [4, 5]. In this case, the amplitudes of the forward and backward wave change slowly at the scale of the order of a few interplane spacing, and we can pass from discrete equations (5) to differential equations. Such approximate equations correspond to the continual (or long-wavelength limit) in the initial problem. By introducing the notation $na = x$ (where n is an integer), we rewrite Eqns (5) as the system of differential equations, which can be obtained in a standard way:

$$\frac{\partial A}{\partial x} = \mathrm{i}\Delta q A + \mathrm{i}KP_y, \quad \frac{\partial B}{\partial x} = -\mathrm{i}\Delta q B - \mathrm{i}KP_y,$$
$$\frac{\partial C}{\partial x} = \mathrm{i}\Delta q C + \mathrm{i}KP_z, \quad \frac{\partial D}{\partial x} = -\mathrm{i}\Delta q B - \mathrm{i}KP_z, \tag{6}$$

where $K = 2\pi k_0^2 q^{-1}$ and $\Delta q = q - 2\pi/a$. The quantities A, B, C, and D in these equations are the Fourier components of the amplitudes of counterpropagating electromagnetic fields rapidly varying in time. If quasi-harmonic waves are considered, we can use the relation between the Fourier components of slowly varying pulse envelopes $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ and the quantities A, B, C, and D [15]:

$$A(\omega_0 + \omega) = \tilde{A}(\omega), \quad B(\omega_0 + \omega) = \tilde{B}(\omega),$$
$$C(\omega_0 + \omega) = \tilde{C}(\omega), \quad D(\omega_0 + \omega) = \tilde{D}(\omega).$$

Then, we obtain from (6) the system of equations

$$\frac{\partial \tilde{A}}{\partial x} = \mathrm{i}\Delta q(\omega_0 + \omega)\tilde{A} + \mathrm{i}K(\omega_0 + \omega)\tilde{P}_y,$$
$$\frac{\partial \tilde{B}}{\partial x} = -\mathrm{i}\Delta q(\omega_0 + \omega)\tilde{B} - \mathrm{i}K(\omega_0 + \omega)\tilde{P}_y,$$
$$\frac{\partial \tilde{C}}{\partial x} = \mathrm{i}\Delta q(\omega_0 + \omega)\tilde{C} + \mathrm{i}K(\omega_0 + \omega)\tilde{P}_z, \tag{7}$$
$$\frac{\partial \tilde{D}}{\partial x} = -\mathrm{i}\Delta q(\omega_0 + \omega)\tilde{D} - \mathrm{i}K(\omega_0 + \omega)\tilde{P}_z.$$

Because the envelopes of pulses change slowly, the corresponding Fourier components are nonzero only for small values of arguments, i.e. for $\omega \ll \omega_0$. Coefficients in (7) are described by the approximate expressions

$$\Delta q(\omega_0 + \omega) \approx q_0 - \frac{2\pi}{a} + q_1\omega + \frac{q_2\omega^2}{2},$$
$$K(\omega_0 + \omega) \approx K_0. \tag{8}$$

Here, $q_m = \mathrm{d}^m q(\omega)/\mathrm{d}\omega^m$ (where $m = 0, 1, 2$) for $\omega = \omega_0$. In particular, q_0 is the wave number and $q_1^{-1} = v_{\mathrm{g}}$ is the group velocity. By returning to the dynamic variables

$$\mathscr{A}(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(\omega, x)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega,$$
$$\mathscr{B}(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{B}(\omega, x)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega, \text{ etc.,}$$

and taking into account expansions (8), we can obtain the equations describing the evolution of slowly varying envelopes:

$$\mathrm{i}\left(\frac{\partial}{\partial x} + \frac{1}{v_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{A} - \frac{q_2}{2}\frac{\partial^2\mathscr{A}}{\partial t^2} + \Delta q_0\mathscr{A} = -K_0\mathscr{P}_y,$$
$$\mathrm{i}\left(\frac{\partial}{\partial x} - \frac{1}{v_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{B} + \frac{q_2}{2}\frac{\partial^2\mathscr{B}}{\partial t^2} - \Delta q_0\mathscr{B} = +K_0\mathscr{P}_y,$$
$$\mathrm{i}\left(\frac{\partial}{\partial x} + \frac{1}{v_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{C} - \frac{q_2}{2}\frac{\partial^2\mathscr{C}}{\partial t^2} + \Delta q_0\mathscr{C} = -K_0\mathscr{P}_z, \tag{9}$$
$$\mathrm{i}\left(\frac{\partial}{\partial x} - \frac{1}{v_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{D} + \frac{q_2}{2}\frac{\partial^2\mathscr{D}}{\partial t^2} - \Delta q_0\mathscr{D} = +K_0\mathscr{P}_z,$$

where polarisation slowly varying in time is defined in terms of the parameters of two-levels atoms in thin films; and $\Delta q_0 = q_0 - 2\pi/a$.

The electromagnetic field of a pulse in Eqns (9) is represented by the projections of vectors in Cartesian coordinates. Effects caused by a change in the polarisation vector of an electromagnetic wave are often discussed by using spherical coordinates instead of Cartesian. The electric field strength in spherical coordinates with the right-hand $[E^{(+)}]$ and left-hand $[E^{(-)}]$ circular polarisations is determined by the expression $E^{(\pm)} = E_y \pm \mathrm{i}E_z$. The spherical coordinates of the polarisation vector of atoms in the film are determined similarly as $P^{(\pm)} = P_y \pm \mathrm{i}P_z$. The corresponding complex envelopes of pulses can be written as $\mathscr{A}^{(\pm)} = \mathscr{A} \pm \mathrm{i}\mathscr{C}$, $\mathscr{B}^{(\pm)} = \mathscr{B} \pm \mathrm{i}\mathscr{D}$, and $\mathscr{P}^{(\pm)} = \mathscr{P}_y \pm \mathrm{i}\mathscr{P}_z$. Equations (9) are transformed to the system of equations

$$\mathrm{i}\left(\frac{\partial}{\partial x} + \frac{1}{_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{A}^{(\pm)} - \frac{q_2}{2}\frac{\partial^2\mathscr{A}^{(\pm)}}{\partial t^2} + \Delta q_0\mathscr{A}^{(\pm)} = -K_0\mathscr{P}^{(\pm)}, \tag{10a}$$

$$\mathrm{i}\left(\frac{\partial}{\partial x} - \frac{1}{_{\mathrm{g}}}\frac{\partial}{\partial t}\right)\mathscr{B}^{(\pm)} + \frac{q_2}{2}\frac{\partial^2\mathscr{B}^{(\pm)}}{\partial t^2} - \Delta q_0\mathscr{B}^{(\pm)} = +K_0\mathscr{P}^{(\pm)}. \tag{10b}$$

This system of equations describes the propagation of a polarised radiation pulse in a RBG formed by thin films of a polarisable substance (for example, resonance atoms) taking into account the second-order group velocity dispersion. The forward and backward waves are described by Eqns (10a) and (10b), respectively.

4. Resonance system

The system of equations (10) should be supplemented with equations determining the evolution of the states of atoms forming thin films or immersed in them. Let us assume that these atoms are two-level atoms with quantum transitions between levels degenerate in the projections j_{a} and j_{b} of angular momenta [16, 17] and consider the case $j_{\mathrm{a}} = 1 \to j_{\mathrm{b}} = 0$. It is convenient to represent the elements of the density matrix $\hat{\rho}$ describing transitions between the states $|\mathrm{a}, m\rangle = |j_{\mathrm{a}} = 1, m = \pm 1\rangle$ and $|\mathrm{b}\rangle = |j_{\mathrm{b}} = 0, m = 0\rangle$ in the form

$$\rho_{12} = \langle \mathrm{a}, -1|\hat{\rho}|\mathrm{a}, +1\rangle, \quad \rho_{13} = \langle \mathrm{a}, -1|\hat{\rho}|\mathrm{b}\rangle, \quad \rho_{23} = \langle \mathrm{a}, +1|\hat{\rho}|\mathrm{b}\rangle,$$

$$\rho_{11} = \langle \mathrm{a}, -1|\hat{\rho}|\mathrm{a}, -1\rangle, \quad \rho_{22} = \langle \mathrm{a}, +1|\hat{\rho}|\mathrm{a}, +1\rangle, \quad \rho_{33} = \langle \mathrm{b}|\hat{\rho}|\mathrm{b}\rangle,$$

$$\rho_{kl} = \rho_{lk}^*, \quad l, k = 1, 2, 3. \tag{11}$$

It is assumed that the pulse duration is much shorter than all the relaxation times of the resonance system. The generalised Bloch equations can be written in this case in the form

$$\begin{aligned}
&\mathrm{i}\hbar \frac{\partial \rho_{13}}{\partial t} = \hbar\Delta\omega\rho_{13} - d_{13}(\rho_{33} - \rho_{11})A_1^{\mathrm{in}} + d_{23}\rho_{12}A_2^{\mathrm{in}},\\
&\mathrm{i}\hbar \frac{\partial \rho_{23}}{\partial t} = \hbar\Delta\omega\rho_{23} - d_{23}(\rho_{33} - \rho_{22})A_2^{\mathrm{in}} + d_{13}\rho_{21}A_1^{\mathrm{in}},\\
&\mathrm{i}\hbar \frac{\partial \rho_{12}}{\partial t} = -d_{13}\rho_{32}A_1^{\mathrm{in}} + d_{32}\rho_{13}A_2^{\mathrm{in}*},\\
&\mathrm{i}\hbar \frac{\partial}{\partial t}(\rho_{33} - \rho_{11}) = 2(d_{13}\rho_{31}A_1^{\mathrm{in}} - d_{31}\rho_{13}A_1^{\mathrm{in}*})\\
&\qquad + (d_{23}\rho_{32}A_2^{\mathrm{in}} - d_{32}\rho_{23}A_2^{\mathrm{in}*}),\\
&\mathrm{i}\hbar \frac{\partial}{\partial t}(\rho_{33} - \rho_{22}) = (d_{13}\rho_{31}A_1^{\mathrm{in}} - d_{31}\rho_{13}A_1^{\mathrm{in}*})\\
&\qquad + 2(d_{23}\rho_{32}A_2^{\mathrm{in}} - d_{32}\rho_{23}A_2^{\mathrm{in}*}).
\end{aligned} \tag{12}$$

Here, d_{kl} are the matrix elements of the dipole moment operator; $A_{1,2}^{\mathrm{in}}$ are the slowly varying envelopes of electromagnetic pulses acting on resonance atoms. In the problem under study, they are described by the expressions $A_1^{\mathrm{in}} = \mathscr{A}^{(+)} + \mathscr{B}^{(+)}$ and $A_2^{\mathrm{in}} = \mathscr{A}^{(-)} + \mathscr{B}^{(-)}$ [15, 18, 19].

The boundary conditions (for $t \to -\infty$) for the non-diagonal elements of the density matrix $\hat{\rho}$ have the form $\rho_{12} = \rho_{13} = \rho_{23} = 0$. For diagonal elements, we can set $\rho_{33} = 1$ and $\rho_{22} = \rho_{11} = 0$, which corresponds to the atoms in the ground state before the pulse arrival. The interaction with resonance atoms is determined in terms of the elements of the density matrix by the expressions

$$K_0\mathscr{P}^{(+)} = \frac{2\pi\omega_0 n_{\mathrm{at}} d_{13}}{c\tilde{n}(\omega_0)} \langle\rho_{13}\rangle,$$

$$K_0\mathscr{P}^{(-)} = \frac{2\pi\omega_0 n_{\mathrm{at}} d_{13}}{c\tilde{n}(\omega_0)} \langle\rho_{23}\rangle.$$

Here, the angle brackets denote summation over all the atoms detuned by the frequency $\Delta\omega$ from the centre of the inhomogeneous absorption band; $\tilde{n}(\omega_0)$ is the refractive index of a dielectric medium into which films are immersed; n_{at} is the effective density of atoms in a thin film, which is described by the expression $n_{\mathrm{at}} = N_{\mathrm{at}}(l_{\mathrm{f}}/a)$, where N_{at} is the volume density of atoms, l_{f} is the film thickness, and a is the grating spacing.

It is convenient to introduce the dimensionless variables

$$e_1^{(\pm)} = \frac{t_0 d_{13}\mathscr{A}^{(\pm)}}{\hbar}, \; e_2^{(\pm)} = \frac{t_0 d_{13}\mathscr{B}^{(\pm)}}{\hbar}, \; \zeta = \frac{z}{v_{\mathrm{g}} t_0}, \; \tau = \frac{t}{t_0}, \tag{13}$$

where t_0 is the time interval determining the characteristic scale, for example, it can be the initial pulse duration. By introducing the notation $\delta = v_{\mathrm{g}} t_0 \Delta q_0$ and neglecting the second-order group velocity dispersion, we obtain the system of equations in the form

$$\begin{aligned}
&\mathrm{i}\left(\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\tau}\right)e_1^{(\pm)} + \delta e_1^{(\pm)} = -\kappa\langle\sigma^{(\pm)}\rangle,\\
&\mathrm{i}\left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\tau}\right)e_2^{(\pm)} - \delta e_2^{(\pm)} = +\kappa\langle\sigma^{(\pm)}\rangle,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\mathrm{i}\,\frac{\partial \rho_{13}}{\partial\tau} = \varDelta\rho_{13} - n_1 e_1 + \rho_{12} e_2,\\
&\mathrm{i}\,\frac{\partial \rho_{23}}{\partial\tau} = \varDelta\rho_{23} - n_2 e_2 + \rho_{12}^* e_1,\\
&\mathrm{i}\,\frac{\partial \rho_{12}}{\partial\tau} = -\rho_{23}^* e_1 + \rho_{13} e_2^*,\\
&\frac{\partial n_1}{\partial\tau} = -4\,\mathrm{Im}(\rho_{13} e_1^*) - 2\,\mathrm{Im}(\rho_{23} e_2^*),\\
&\frac{\partial n_2}{\partial\tau} = -2\,\mathrm{Im}(\rho_{13} e_1^*) - 4\,\mathrm{Im}(\rho_{23} e_2^*),
\end{aligned} \tag{15}$$

where $\kappa = v_{\mathrm{g}} t_0/L_{\mathrm{a}}$; $L_{\mathrm{a}} = cn(\omega_0)\hbar/(2\pi\omega_0 t_0 n_{\mathrm{at}}|d_{13}|^2)$ is the resonance absorption length; $\varDelta = \Delta\omega t_0$ is the normalised frequency detuning; $n_1 = \rho_{33} - \rho_{11}$ and $n_2 = \rho_{33} - \rho_{22}$ are the population differences; and $e_1 = e_1^{(+)} + e_2^{(+)}$ and $e_2 = e_1^{(-)} + e_2^{(-)}$. In equations (14) the notation $\sigma^{(+)} = \rho_{13}$ and $\sigma^{(-)} = \rho_{23}$ is used.

After the change of variables

$$\begin{aligned}
&e_1^{(\pm)} + e_2^{(\pm)} = f_{\mathrm{s}}^{(\pm)}\exp(\mathrm{i}\delta\tau), \quad e_1^{(\pm)} - e_2^{(\pm)} = f_{\mathrm{a}}^{(\pm)}\exp(\mathrm{i}\delta\tau)\\
&\rho_{13} = r_{13}\exp(\mathrm{i}\delta\tau), \quad \rho_{23} = r_{23}\exp(\mathrm{i}\delta\tau), \quad \rho_{12} = r_{12},
\end{aligned} \tag{16}$$

the system of equations (14), (15) can be rewritten in the form

$$\frac{\partial f_{\mathrm{s}}^{(\pm)}}{\partial\zeta} + \frac{\partial f_{\mathrm{a}}^{(\pm)}}{\partial\tau} = 0\,, \tag{17a}$$

$$\frac{\partial f_{\mathrm{a}}^{(\pm)}}{\partial\zeta} + \frac{\partial f_{\mathrm{s}}^{(\pm)}}{\partial\tau} = 2\mathrm{i}\kappa\langle r^{(\pm)}\rangle\,, \tag{17b}$$

$$\begin{aligned}
&\mathrm{i}\,\frac{\partial r_{13}}{\partial\tau} = (\delta + \varDelta)r_{13} - n_1 f_{\mathrm{s}}^{(+)} + r_{12} f_{\mathrm{s}}^{(-)},\\
&\mathrm{i}\,\frac{\partial r_{23}}{\partial\tau} = (\delta + \varDelta)r_{23} - n_2 f_{\mathrm{s}}^{(-)} + r_{12}^* f_{\mathrm{s}}^{(+)},\\
&\mathrm{i}\,\frac{\partial r_{12}}{\partial\tau} = -r_{23}^* f_{\mathrm{s}}^{(+)} + r_{13} f_{\mathrm{s}}^{(-)*},\\
&\frac{\partial n_1}{\partial\tau} = -4\,\mathrm{Im}\big(r_{13} f_{\mathrm{s}}^{(+)*}\big) - 2\,\mathrm{Im}\big(r_{23} f_{\mathrm{s}}^{(-)*}\big),\\
&\frac{\partial n_2}{\partial\tau} = -2\,\mathrm{Im}\big(r_{13} f_{\mathrm{s}}^{(+)*}\big) - 4\,\mathrm{Im}\big(r_{23} f_{\mathrm{s}}^{(-)*}\big).
\end{aligned} \tag{18}$$

A further study can be performed by solving numerically the system of equations (17), (18). The analytic solutions of these equations can be found in the simplest way by neglecting the inhomogeneous broadening of the absorption line and assuming that the condition of the Bragg and optical resonances $\delta + \Delta = 0$ is fulfilled.

5. Stationary solutions describing polarised solitons

By assuming that all the variables in (17) and (18) are the functions of one variable $\xi = \tau + \alpha\zeta$, we can write the system of equations in partial derivatives in the form of the system of ordinary equations describing the stationary propagation of the forward and backward waves. It follows from (17a) that

$$\alpha \frac{\partial f_{\rm s}^{(\pm)}}{\partial \xi} + \frac{\partial f_{\rm a}^{(\pm)}}{\partial \xi} = 0.$$

By assuming that the fields vanish at some spatial point (in particular, at infinity), we obtain the relation $\alpha f_{\rm s}^{(\pm)} + f_{\rm a}^{(\pm)} = 0$ after integration of this equation. By substituting $f_{\rm a}^{(\pm)}$ into (17b), we obtain the equations

$$\frac{\partial f_{\rm s}^{(+)}}{\partial \xi} = \frac{2{\rm i}\kappa}{1-\alpha^2} r_{13}, \qquad \frac{\partial f_{\rm s}^{(-)}}{\partial \xi} = \frac{2{\rm i}\kappa}{1-\alpha^2} r_{23}. \tag{19}$$

The Bloch equations in this case take the form

$${\rm i}\frac{\partial r_{13}}{\partial \xi} = -n_1 f_{\rm s}^{(+)} + r_{12} f_{\rm s}^{(-)}, \qquad {\rm i}\frac{\partial r_{23}}{\partial \xi} = -n_2 f_{\rm s}^{(-)} + r_{12}^* f_{\rm s}^{(+)}, \tag{20a}$$

$${\rm i}\frac{\partial r_{12}}{\partial \xi} = -r_{23}^* f_{\rm s}^{(+)} + r_{13} f_{\rm s}^{(-)*}, \tag{20b}$$

$$\frac{\partial n_1}{\partial \xi} = -4\,{\rm Im}\big(r_{13} f_{\rm s}^{(+)*}\big) - 2\,{\rm Im}\big(r_{23} f_{\rm s}^{(-)*}\big), \tag{20c}$$

$$\frac{\partial n_2}{\partial \xi} = -2\,{\rm Im}\big(r_{13} f_{\rm s}^{(+)*}\big) - 4\,{\rm Im}\big(r_{23} f_{\rm s}^{(-)*}\big). \tag{20d}$$

From (19), we obtain

$$r_{13} = -\frac{\rm i}{a_0}\frac{\partial f_{\rm s}^{(+)}}{\partial \xi}, \quad r_{23} = -\frac{\rm i}{a_0}\frac{\partial f_{\rm s}^{(-)}}{\partial \xi}, \tag{21}$$

where $a_0 = 2\kappa(1-\alpha^2)$. Taking (21) into account, Eqn (20b) can be written in the form

$${\rm i}\frac{\partial r_{12}}{\partial \xi} = -\frac{\rm i}{a_0}\frac{\partial}{\partial \xi}\big(f_{\rm s}^{(+)} f_{\rm s}^{(-)*}\big).$$

This gives

$$r_{12} = -\frac{1}{a_0}\big(f_{\rm s}^{(+)} f_{\rm s}^{(-)*}\big). \tag{22}$$

Therefore, Eqns (20a) take the form

$${\rm i}\frac{\partial r_{13}}{\partial \xi} = -n_1 f_{\rm s}^{(+)} - \frac{1}{a_0}\big|f_{\rm s}^{(-)}\big|^2 f_{\rm s}^{(+)}, \qquad {\rm i}\frac{\partial r_{23}}{\partial \xi} = -n_2 f_{\rm s}^{(-)} - \frac{1}{a_0}\big|f_{\rm s}^{(+)}\big|^2 f_{\rm s}^{(-)}. \tag{23}$$

Taking (23) and (19) into account, we can show that if the electric field strength is assumed a real quantity, the variables r_{13} and r_{23} will be imaginary, i.e. $f_{\rm s}^{(+)} = a_1$, $f_{\rm s}^{(-)} = a_2$, $r_{13} = {\rm i}u_1$ and $r_{23} = {\rm i}u_2$. Taking (21) into account, we can write equations for population differences:

$$\frac{\partial n_1}{\partial \xi} = \frac{1}{a_0}\frac{\partial}{\partial \xi}\big(2a_1^2 + a_2^2\big),$$

$$\frac{\partial n_2}{\partial \xi} = \frac{1}{a_0}\frac{\partial}{\partial \xi}\big(a_1^2 + 2a_2^2\big),$$

which gives

$$n_1 = n_{10} + \frac{1}{a_0}\big(2a_1^2 + a_2^2\big), \qquad n_1 = n_{20} + \frac{1}{a_0}\big(a_1^2 + 2a_2^2\big), \tag{24}$$

where n_{10} and n_{20} are the integration constants which are determined from the boundary conditions for $\xi \to \infty$. If the resonance system was in the inverted state before the arrival of pulses, the parameter a_0 should be negative, or $\alpha^2 > 1$. By using the definition of the variable ξ, we can obtain that the velocity $\upsilon_{\rm st}$ of the stationary pulse and the group velocity $\upsilon_{\rm g}$ are related by the expression $\alpha\upsilon_{\rm st} = \upsilon_{\rm g}$. Therefore, a stationary pulse propagates in a medium with non-inverted resonance states slower than linear waves. By substituting (24) into (23) and taking (21) into account, we obtain equations for the amplitudes of stationary electromagnetic pulses

$$\frac{{\rm d}^2 a_1}{{\rm d}\eta^2} + \big(a_1^2 + a_2^2\big)a_1 - \frac{1}{2}|a_0| n_{10} a_1 = 0, \qquad \frac{{\rm d}^2 a_2}{{\rm d}\eta^2} + \big(a_1^2 + a_2^2\big)a_2 - \frac{1}{2}|a_0| n_{20} a_2 = 0, \tag{25}$$

where $\eta = \sqrt{2}\xi$.

As a rule, the population of excited states corresponding to the different projections of the angular momentum is the same. If all the atoms in a medium are in the ground state in the absence of USPs, then $n_{10} = n_{20} = 1$. This situation corresponds to a preliminary nonpolarised resonance medium. Equations (25) become symmetrical with respect to the interchange of the subscripts of the real envelopes a_1 and a_2. This symmetry can be broken by transmitting, for example, a weakly circularly polarised radiation through the resonance medium. If this radiation is continuous, then the population difference established between the energy levels coupled by different transitions will depend on the intensity and polarisation of this radiation. For a high-power short pulse directed to the medium prepared in this way, this medium will be polarised, i.e. the populations of excited levels will be different: $n_{10} \neq n_{20}$.

The system of equations (25) was encountered earlier in the study of propagation of optical solitons in birefringent fibres [20, 21]. The analysis of propagation of polarised USPs in resonance or nonresonance nonlinear media is reduced in some cases to the solution of the same equations [22, 23]. The results of papers [20–23] can be used to write at once particular solutions of (25). According to [20, 21], we set $a_1 = g/f$ and $a_2 = h/f$ and rewrite (25) in the bilinear form

$$D_y^2(g\cdot f) = b_1^2 gf, \quad D_y^2(h\cdot f) = b_2^2 hf, \qquad D_y^2(f\cdot f) = g^2 + h^2 \tag{26}$$

by using the Hirota D operators [24]

$$D_y(\tilde a\cdot\tilde b) \equiv \lim_{x\to y}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\tilde a(x)\tilde b(y)$$

and the notation $2b_1^2 = |a_0|n_{10}$ and $2b_2^2 = |a_0|n_{20}$. To solve bilinear equations, the functions g, h, and f are written in the form of polynomials, for example,

$$g = \chi g_1 + \chi^3 g_3, \quad h = \chi h_1 + \chi^3 h_3, \quad f = 1 + \chi^2 f_2 + \chi^4 f_4. \tag{27}$$

By substituting these expansions into (26) and equating the coefficients at the same powers of χ, we obtain the system of linked linear equations with variable coefficients [20–23]. By solving successively these equations, we can obtain the solution of system (26) for chosen polynomials (27), where χ is set equal to unity:

$$g = 2\sqrt{2}b_1\exp\theta_1[1+\exp(2\theta_2+b_{12})],$$

$$h = 2\sqrt{2}b_2\exp\theta_2[1-\exp(2\theta_1+b_{12})],$$

$$f = 1+\exp 2\theta_1+\exp 2\theta_2+\exp(2\theta_1+2\theta_2+2b_{12}),$$

where

$$\exp b_{12} = \frac{b_1-b_2}{b_1+b_2};$$

$\theta_{1,2} = b_{1,2}(y-y_{1,2})$, and $y_{1,2}$ are the integration constants; other integration constants are selected so that the obtained solution would correspond to a solitary wave. Now the solution of initial system of equations (25) can be written in the form

$$a_1(y) = \frac{2\sqrt{2}b_1\exp\theta_1[1+\exp(2\theta_2+b_{12})]}{1+\exp 2\theta_1+\exp 2\theta_2+\exp(2\theta_1+2\theta_2+2b_{12})},$$
$$a_2(y) = \frac{2\sqrt{2}b_2\exp\theta_2[1-\exp(2\theta_1+b_{12})]}{1+\exp 2\theta_1+\exp 2\theta_2+\exp(2\theta_1+2\theta_2+2b_{12})}. \tag{28}$$

This solution contains important limiting cases. Let us assume that the medium is prepared in the state with $n_{10} \neq 0$ and $n_{20} = 0$. In this case, $b_2 = 0$ and $\exp b_{12} = 1$. Then, we obtain from (28) the expressions $a_1(y) = \sqrt{2}b_1\times \mathrm{sech}[b_1(y-y_1)]$ and $a_2(y) = 0$ which describe a right-hand circularly polarised pulse. Similarly, we can obtain the expressions $a_1(y) = 0$ and $a_2(y) = \sqrt{2}b_2\mathrm{sech}[b_2(y-y_2)]$ for a left-hand polarised pulse if the medium is in the state with $n_{10} = 0$ and $n_{20} \neq 0$. The propagation velocity of such a pulse is independent of polarisation and coincides with that obtained for a scalar field in [4–7]. For a nonpolarised medium, we have $b_1 = b_2 = b_0$ and $\exp b_{12} = 0$. In this case, we obtain from (28)

$$a_1(y) = \frac{2\sqrt{2}b_0\exp\theta_1}{1+\exp 2\theta_1+\exp 2\theta_2},$$

$$a_2(y) = \frac{2\sqrt{2}b_0\exp\theta_2}{1+\exp 2\theta_1+\exp 2\theta_2}.$$

By introducing the parameter y_0 with the help of the equation

$$\exp(-2by_0) = \exp(-2by_1)+\exp(-2by_2),$$

we obtain the pulse envelope described by the vector with components

$$a_{1,2}(y) = \sqrt{2}bl_{1,2}\,\mathrm{sech}[b(y-y_0)], \tag{29}$$

where

$$l_{1,2} = \exp[b(y_0-y_{1,2})].$$

This solution corresponds to an elliptically polarised stationary radiation pulse propagating in a RBG.

6. Conclusions

We have studied the propagation of a polarised radiation USP in a one-dimensional RBG formed by the periodic layers of resonantly absorbing atoms. The equations for USPs of the forward and backward waves in the RBG have been derived in the slowly varying pulse envelope approximation. The resulting system of equations is the generalisation of the Maxwell–Bloch equations which were used to describe the formation and propagation of a resonance Bragg soliton of a scalar electromagnetic field.

We have considered in this paper thin films containing resonance atoms; however, films containing molecules, molecular aggregates*, quantum dots, metal nanoparticles, microcavities, Bose condensate drops, etc. can be also considered.

Our analysis of the possibility of propagation of a VBS has shown that the type of propagation of such a pulse depends on the state of the medium. Thus, if the population difference between the ground state and excited states with different projections of the angular momentum satisfies the condition $n_{10} = n_{20} = 1$, then a VBS in a one-dimensional RBG is an elliptically polarised radiation pulse. In particular, circularly and linearly polarised VBSs are trivial generalisations of stationary pulses known in the scalar theory of RBGs [4–13].

We have shown that a new solution of the system of Maxwell–Bloch equations appears when the resonance medium has been prepared preliminary in the state with the asymmetric population distribution of excited states with different projections of the angular momentum: $n_{10} \neq n_{20}$. In this case, the VBS ellipticity changes inside the pulse itself.

* Such RBGs were studied by A.A. Zabolotskii (Institute of Automation and Electrometry, Siberian Branch, RAS, Novosibirsk)

It does not follow from the obtained results that other solitary waves, similar to solitons in photonic crystals and RBGs, do not exist. Such waves, if they exist, are not stationary like breathers, buzzerons or multisoliton waves. It seems that their propagation can be studied only by numerical methods. The model developed in this paper can be used to study the reflection of polarised radiation pulses from gratings, the generation of harmonics in RBGs, and multidimensional VBSs.

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