PACS numbers: 42.25.Dd; 42.62.Be; 42.30.Wb; 87.64.Cc
DOI: 10.1070/QE2006v036n11ABEH013303

Stability of the Henyey – Greenstein phase function and fast path integration under conditions of multiple light scattering

V.M. Petnikova, E.V. Tret'yakov, V.V. Shuvalov

***Abstract*. It is shown that the stability of the Henyey – Greenstein phase function allows the calculation rate of light propagation through strongly scattering objects to be drastically increased by using the same *a priori* information on interaction processes as in the initial formulation of the problem. The increase in the calculation rate is accompanied by a gradual impairment of simulation accuracy from the accuracy of the Monte-Carlo method to that of the diffusion approximation. By using a standard assumption about the statistical independence of the photon mean free path and photon scattering angle, an exact analytic expression relating the effective number of scattering events with the optical path is obtained.**

Keywords: *Henyey – Greenstein phase function, multiple-scattering phase function, fast path integration.*

1. Introduction

The problem of light propagation under multiple-scattering conditions is usually solved numerically by the methods of radiation transfer theory [1, 2], Monte Carlo [3, 4] or path integration [5 – 7]. These calculations are based on the parameters that *a priori* describe the statistics: absorption and scattering coefficients $\mu_{a,s}$ and the phase function $P_s^{(1)}(\boldsymbol{\Theta})$ characterising the distribution of the probability density of single scattering within the two-dimensional angle $\boldsymbol{\Theta} = (\theta, \varphi)$, where θ and φ are the azimuthal and polar scattering angles [1 – 4]. Because the problem of multiple scattering cannot be solved analytically in such 'exact' formulation based only on *a priori* statistics [$\mu_{a,s}$ and $P_s^{(1)}(\boldsymbol{\Theta})$], it is usually simplified by introducing some approximations [2, 7 – 11]. Unfortunately, the verification of the results obtained within the framework of these approximations by the methods mentioned above for propagation over distances ~ 1000 scattering lengths becomes virtually impossible due to enormous calculation times. The matter is that more or less reliable calculation (with the relative error $\sim 1\,\%$) of only the distribution of the probability density for propagating photons from a source to a detector certainly requires simulations of more than 10^{13} realisations, which simply follows from the necessity to fill the data matrix (10^4 photons per cell) corresponding to this distribution.

The numerical calculation rate in any of the above-mentioned exact methods can be considerably increased by introducing the 2D distribution $P_s^{(k)}(\boldsymbol{\Theta})$ *a priori* describing k-fold scattering [11 – 14]. It was assumed in this method [13, 14] that the function

$$P_s^{(k)}(\boldsymbol{\Theta}) = \frac{1}{4\pi} \frac{1 - g_k^2}{(1 + g_k^2 - 2g_k \cos\theta)^{3/2}}, \quad k = 1, 2, ... \qquad (1)$$

is the Henyey – Greenstein phase function [15] with the anisotropy parameter $g_k = g_1^k = \langle \cos \boldsymbol{\Theta} \rangle$, which determines the mean cosine of the angle of k-fold scattering and varies from 0 (isotropic scattering) to 1 (forward scattering). It was assumed that the number k of scattering events on the path of length Δz transforms to a new effective constant $k_{\rm eff}$, which depends on Δz and is expressed in terms of the mean value $\langle k \rangle = \mu_s \Delta z$ [12 – 14]. The dependence $k_{\rm eff}(\Delta z)$ required for calculations was introduced semiempirically.

We will show below that, because the distribution $P_s^{(1)}(\boldsymbol{\Theta})$ belongs to the class of stable distributions [16] and the dependence $P_s^{(k)}(\boldsymbol{\Theta})$ reproduces the dependence $P_s^{(1)}(\boldsymbol{\Theta})$ [see (1)], there is no need to use semiempirical considerations and the dependence $k_{\rm eff}(\Delta z)$ can be found exactly. Therefore, within the framework of any of the above-mentioned exact methods, by using the same *a priori* information on a scattering medium as in the initial formulation of the problem, a principal advantage in the calculation rate can be achieved in the solution of the problem of multiple small-angle scattering.

2. Multiple-scattering phase function

We will assume that a photon $k = 0, 1, ...$ times changes its propagation direction by the angle $\boldsymbol{\Theta}_k = (\theta_k, \varphi_k)$ on an interval of length Δz parallel to the z axis. By considering only the final change in the photon propagation direction ($\boldsymbol{\Theta} = \sum_{k=0}^{\infty} \boldsymbol{\Theta}$) and assuming that all scattering events are independent, we introduce the effective multiple-scattering phase function in the form

$$P_s(\boldsymbol{\Theta}, \Delta z) = \sum_{k=0}^{\infty} P^{(k)}(\Delta z) P_s^{(k)}(\boldsymbol{\Theta}). \qquad (2)$$

V.M. Petnikova, E.V. Tret'yakov, V.V. Shuvalov International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119992 Moscow, Russia; e-mail: vsh@vsh.phys.msu.su

Received 19 June 2006; revision received 10 July 2006
Kvantovaya Elektronika **36** (11) 1039 – 1042 (2006)
Translated by M.N. Sapozhnikov

Here, $P^{(k)}(\Delta z)$ is the probability of k-fold scattering on the interval Δz, which will also take into account below the probability of the absence of absorption (path integration method);

$$P_{\rm s}^{(k)}(\boldsymbol{\Theta}) = \int\!\!\int {\rm d}\boldsymbol{\Theta}' P_{\rm s}^{(k-1)}(\boldsymbol{\Theta}') P_{\rm s}^{(1)}(\boldsymbol{\Theta} - \boldsymbol{\Theta}'), \quad k = 1, 2, ... \qquad (3)$$

is the k-fold scattering phase function; and $P_{\rm s}^{(0)}(\boldsymbol{\Theta}) \equiv \delta(\boldsymbol{\Theta})$ is the delta function over the angle $\boldsymbol{\Theta} = (\theta, \varphi)$.

By using the mathematical apparatus of characteristic functions, we can easily show that for any phase function $P_{\rm s}^{(1)}(\boldsymbol{\Theta})$ it follows already from the condition of the independence of single-scattering events for any integer $n > 1$ that

$$\left\langle \cos \sum_{k=1}^{n} \boldsymbol{\Theta}_k \right\rangle = \langle \cos \boldsymbol{\Theta}_1 \rangle^n. \qquad (4)$$

Therefore, if the single-scattering phase function were determined by distribution (1) and this distribution were stable [16], we would obtain a rather simple and convenient relation $g_k = g_1^k$ [14].

3. Stability of the Henyey – Greenstein phase function

To illustrate the stability of the Henyey – Greenstein phase function (1) we present the result of numerical integration of (3) for $P_{\rm s}^{(1)}(\boldsymbol{\Theta})$ given by expression (1) and $0.99 > g_1 > 0.15$ and $k = 1, 2, ... 50$. Figure 1 shows the dependences $F^{(k)}(\theta) = \int_0^{2\pi} {\rm d}\varphi P_{\rm s}^{(k)}(\boldsymbol{\Theta})$ on the plane θ, k calculated by this method (points) and dependences

$$F^{(k)}(\theta) = \frac{1}{2} \frac{1 - g_1^{2k}}{(1 + g_1^{2k} - 2g_1^k \cos\theta)^{3/2}}, \qquad (5)$$

calculated by using expression (1) and the condition $g_k = g_1^k$ for $g_1 = 0.95$ (solid curves). It is easy to verify that for $g_k = 0.99 - 0.07$ (almost isotropic scattering), the deviation of the results of numerical integration of (3) from those obtained by expression (5) does not exceed 10^{-3}. Taking into account the accuracy of numerical calculations, this confirms that distribution (1) is indeed stable [16]. Note that the same result can be also obtained analytically [11, 14].

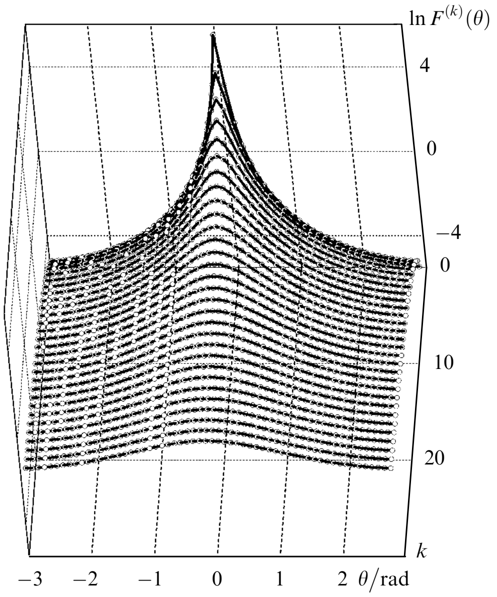

Figure 1. Dependence $F^{(k)}(\theta)$ as a function of k for $g_1 = 0.95$. Points are the results of numerical integration of expression (3), solid curves correspond to expression (5).

4. Statistical moments in k-fold scattering

Note at once that due to the different lengths of paths with a different number of scattering events, it cannot be assumed that the function $P^{(k)}(\Delta z)$ in (2) is given by the standard Poisson distribution (see, for example, [13, 14]). Therefore, we will calculate the statistical moments of the distribution $P^{(k)}(\Delta z)$ by using the following simple considerations.

Let us assume that the path of any photon upon k-fold scattering is a broken line consisting of $k + 1$ linear segments $\Delta \boldsymbol{l}_i$ ($i = 0, 1, ..., k$) at the ends of which a photon is scattered by the 2D angle $\boldsymbol{\Theta}_i$ (Fig. 2). We assume that the lengths Δl_i of these segments are distributed exponentially with the first- and second-order moments $\langle \Delta l_i \rangle = \langle \Delta l \rangle = \mu_{\rm s}^{-1}$ and $\langle \Delta l_i^2 \rangle = \langle \Delta l^2 \rangle = 2\mu_{\rm s}^{-2}$, respectively. In this case, the mean length $\langle \Delta l^{(k)} \rangle = (k+1)\mu_{\rm s}^{-1}$ of the paths under study depends on the total number k of scattering events, and $\langle [\Delta l^{(k)}]^2 \rangle = 2(k+1)\mu_{\rm s}^{-2}$. By projecting now all scattering points to the segment Δz, which is the continuation of Δl_0, we construct on it $k + 1$ successive segments with different lengths $\Delta z_i = \Delta l_i \cos(\sum_{m=0}^{i} \boldsymbol{\Theta}_m)$, where $\boldsymbol{\Theta}_0 \equiv 0$ because $\boldsymbol{\Theta}_0$ is the entrance angle of a photon to the path under study. By averaging Δz_i over Δl_i and $\boldsymbol{\Theta}_i$ assuming that the mean free paths Δl_i and single-scattering angles are statistically independent (the approximation of point scattering centres), we obtain

$$\langle \Delta z^{(k)} \rangle = \sum_{i=0}^{k} \langle \Delta z_i \rangle = \mu_{\rm s}^{-1} \sum_{i=0}^{k} g_1^i = \mu_{\rm s}^{-1} \frac{1 - g_1^{k+1}}{1 - g_1}, \qquad (6)$$

where $\langle \Delta z^{(k)} \rangle$ is the mean displacement length of a photon along the z axis after k-fold scattering. It follows from expression (6) that

$$k = \frac{\ln\left[1 - \mu_{\rm s} \langle \Delta z^{(k)} \rangle (1 - g_1)\right]}{\ln g_1} - 1, \qquad (7)$$

and the total mean path for such scattering multiplicity is

$$\langle \Delta l^{(k)} \rangle = \mu_{\rm s}^{-1} \frac{\ln[1 - \mu_{\rm s} \langle \Delta z^{(k)} \rangle (1 - g_1)]}{\ln g_1}. \qquad (8)$$

It is easy to verify that, although $\langle \Delta l^{(k)} \rangle \to \infty$ for $k \to \infty$, the mean displacement $\langle \Delta z^{(k)} \rangle$ of the photon along

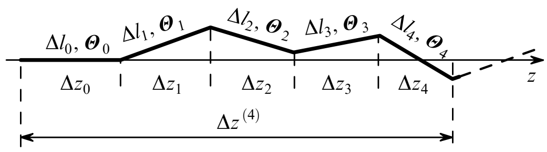

Figure 2. Schematic representation of the photon path upon four-fold scattering (see the text).

the z axis cannot exceed $\Delta z^{(\infty)} = (\mu'_s)^{-1}$, where $\mu'_s = \mu_s(1-g_1)$ is the transport scattering coefficient. This well-known result [see expression (28) in [17]] is explained by the fact that $\langle \Delta z^{(k)} \rangle$ increases only due to the regular component (the mean projection on the z axis is nonzero) of the photon velocity directed strictly along the z axis. It is the distance $\Delta z^{(\infty)}$ that determines the possibility of passing to the diffusion approximation in which the further propagation of photons is already described in terms of the second-order moment taking also into account the irregular component (the mean projection on the z axis is zero) of the total displacement.

To take into account this component of the photon displacement, we calculate the quantity

$$\langle [\Delta z^{(k)}]^2 \rangle = \Big\langle \Big(\sum_{i=0}^{k} \Delta z_i \Big)^2 \Big\rangle = \sum_{i=0}^{k} \langle \Delta z_i^2 \rangle + 2 \sum_{j>i=0}^{k} \langle \Delta z_i \Delta z_j \rangle. \quad (9)$$

Note at once that the main problem of the calculation is that Δz_i and Δz_j in the second term in the right-hand side of expression (9) are statistically dependent. Indeed, the direction of photon propagation after the jth scattering event described by the angle $\sum_{m=0}^{j} \boldsymbol{\Theta}_m$ depends on all the previous events because it also includes the sum $\sum_{m=0}^{i} \boldsymbol{\Theta}_m$ characterising the direction of photon propagation after the ith single-scattering event. However, taking into account the stability of the Henyey – Greenstein function (see above) and statistical independence of free mean paths Δl_i and single-scattering angles $\boldsymbol{\Theta}_i$, it is possible to perform exact averaging in (9) because

$$\sum_{i=0}^{k} \langle \Delta z_i^2 \rangle = \langle \Delta l^2 \rangle \sum_{i=0}^{k} \Big\langle \cos^2 \Big(\sum_{m=0}^{i} \boldsymbol{\Theta}_m \Big) \Big\rangle = \langle \Delta l^2 \rangle \sum_{i=0}^{k} \frac{1 + 2g_1^{2i}}{3}$$

$$= \frac{1}{3} \langle \Delta l^2 \rangle \left(k + 3 + 2g_1^2 \frac{1 - g_1^{2k}}{1 - g_1^2} \right), \quad (10)$$

$$2 \sum_{j>i=0}^{k} \langle \Delta z_i \Delta z_j \rangle = 2 \langle \Delta l \rangle^2 \sum_{j>i=0}^{k} \Big\langle \cos \Big(\sum_{m=0}^{j} \boldsymbol{\Theta}_m \Big) \cos \Big(\sum_{m=0}^{i} \boldsymbol{\Theta}_m \Big) \Big\rangle$$

$$= 2 \langle \Delta l \rangle^2 \left(\sum_{j=1}^{k} g_1^j + \sum_{j>i=0}^{k} \frac{g_1^{j-i} + 2g_1^{j+i}}{3} \right)$$

$$= 2 \langle \Delta l \rangle^2 \left[g_1 \frac{1 - g_1^k}{1 - g_1} + \frac{1}{3} \frac{g_1}{1 - g_1} \left(k - \frac{1 - g_1^k}{1 - g_1} \right) \right.$$

$$\left. + \frac{2}{3} \frac{g_1^2}{1 - g_1} \left(g_1 \frac{1 - g_1^{2k}}{1 - g_1^2} - g_1^k \frac{1 - g_1^k}{1 - g_1} \right) \right]. \quad (11)$$

After some simple transformations taking the relation $\langle \Delta l^2 \rangle = 2 \langle \Delta l \rangle^2$ into account, we obtain from this the exact analytic expression

$$\left\langle \left[\Delta z^{(k)} \right]^2 \right\rangle = \frac{2}{3} \frac{\langle \Delta l \rangle^2}{1 - g_1} \left[k + 3(1 - g_1) + g_1(2 - g_1) \frac{1 - g_1^k}{1 - g_1} \right.$$

$$\left. - 2g_1^3 \frac{1 - g_1^{2k}}{1 - g_1^2} \right]. \quad (12)$$

Note that for $k = 0 - 3$, relation (12) can be transformed to expressions obtained earlier in [17]. At the same time, the result of exact averaging that we obtained differs from expression (13) presented in [18]. This is explained by the fact that expression (13) from [18] is not valid for small values of k because it was derived by assuming that the three spatial projections of the second displacement moment were initially equivalent.

The expressions for second-order moments $\langle [\Delta x^{(k)}]^2 \rangle = \langle [\Delta y^{(k)}]^2 \rangle$ of the photon displacement along the two x and y, which are orthogonal to each other and the z axis, upon k-fold scattering can be also easily written taking into account the exact analytic relation

$$\left\langle \left[\Delta x^{(k)} \right]^2 \right\rangle + \left\langle \left[\Delta y^{(k)} \right]^2 \right\rangle + \left\langle \left[\Delta z^{(k)} \right]^2 \right\rangle$$

$$= 2 \frac{\langle \Delta l \rangle^2}{1 - g_1} \left(k - \frac{1 - g_1^k}{1 - g_1} \right), \quad (13)$$

which corresponds to expression (25) in [19].

5. Fast solution of the light propagation problem by the path-integration method

Taking into account the exact analytic expressions presented above, the effective phase function (2) describing the probability distribution of photon propagation over the path of length Δz with a change in the propagation direction by the 2D angle $\boldsymbol{\Theta} = (\theta, \varphi)$ takes now the form

$$P_s(\boldsymbol{\Theta}, \Delta z) = \exp \left\{ - \left[k_{\text{eff}}(\Delta z) + 1 \right] \frac{\mu_a}{\mu_s} \right\}$$

$$\times \frac{1}{4\pi} \frac{1 - g_1^{2k_{\text{eff}}(\Delta z)}}{\left[1 + g_1^{2k_{\text{eff}}(\Delta z)} - 2g_1^{k_{\text{eff}}(\Delta z)} \cos \theta \right]^{3/2}}, \quad (14)$$

where $k_{\text{eff}}(\Delta z)$ is determined by the solution of transcendental equation (12). Figure 3 illustrates the typical dependence of k_{eff} on Δz normalised to the transport scattering length $(\mu'_s)^{-1}$ for $g_1 = 0.95$. Here, the solid curve shows in double logarithmic coordinates the exact dependence of $\Delta z(k_{\text{eff}})$ calculated from analytic expression (12), while the dashed curve presents this dependence calculated by expression (13) from [18]. It is easy to verify that for $\Delta z < (2 \div 3)(\mu'_s)^{-1}$, the difference between these two

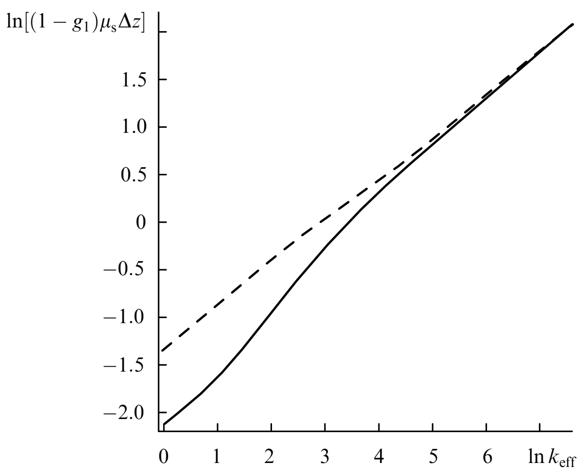

Figure 3. Dependences of $(1 - g_1)\mu_s \Delta z$ on k_{eff} for $g_1 = 0.95$. The solid curve is the dependence $\Delta z(k_{\text{eff}})$ calculated from (12), the dashed curve is this dependence calculated from expression (13) in [18].

calculations is considerable, and they coincide only in the diffusion limit $\Delta z \gg (\mu_s')^{-1}$.

Because analytic relation (12) used to obtain (14) is expressed in terms of the same parameters that describe information on scattering (μ_s and g_1) and absorption (μ_a) processes in the initial formulation of the problem and is exact, the possibility of a fast and exact (in the above described sense) solution of the problem of multiple small-angle scattering by the path-integration method can be considered proved [5, 6, 12, 13]. This is illustrated in Fig. 4a which shows the central cross sections of the probability distribution $P_\Sigma(r)$ of propagation of photons through different points of a model object represented by a strongly scattering and weakly absorbing ($\mu_a = 0.01$ mm^{-1} and $\mu_s = 14$ mm^{-1}, $g_1 = 0.95$) medium in a cylindrical vessel of diameter $2R = 35$ mm with absorbing walls (r is the distance to the cylinder axis). A detector is located on the side surface of the cylinder at an angle of 90° to a radiation source (Fig. 4b). Distributions $P_\Sigma(r)$ were calculated by the Monte-Carlo and path-integration methods by using the procedure described in [13] for $\langle\Delta z\rangle = 8\mu_s^{-1}$ and employing the Henyey–Greenstein phase function. The value of $k_{\rm eff}(\Delta z)$ was expressed in terms of $\langle k\rangle(\Delta z)$ and calculated from (12). The angular aperture of the radiation source in the Monte-Carlo method was 10° and the receiving area was 1 mm^2.

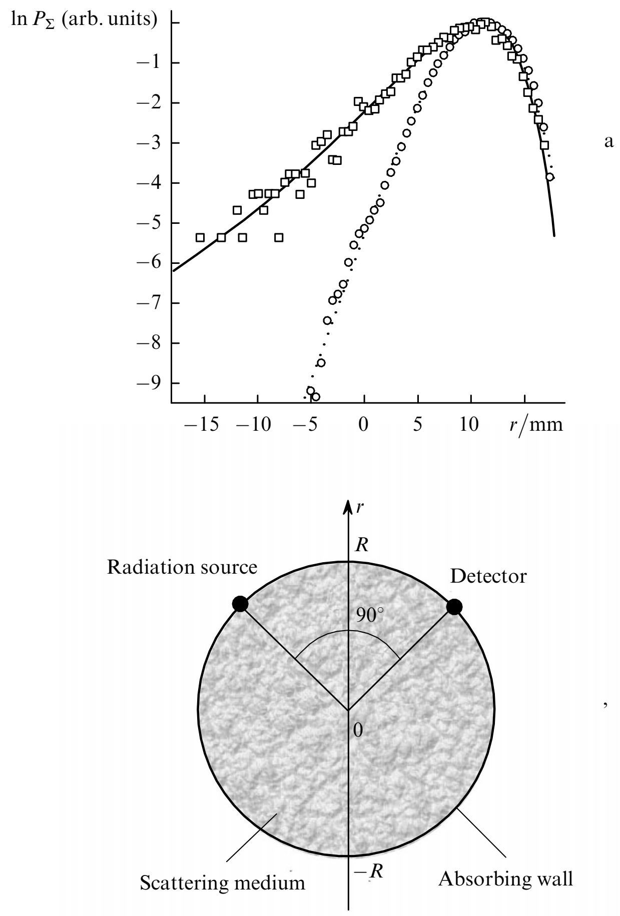

Figure 4. Central cross sections of the probability distribution $P_\Sigma(r)$ for the propagation of photons (a) and geometry of the experiment (b). Calculations were performed by the Monte-Carlo (□) and path-integration methods for $\langle\Delta z\rangle = 8\mu_s^{-1}$ and $k_{\rm eff}(\Delta z)$ expressed in terms of $\langle k\rangle(\Delta z)$ (dashed curve) and from expression (12) (solid curve).

6. Conclusions

Thus, the problem of light propagation through strongly scattering objects can be solved considerably faster by introducing the multiple-scattering phase function (2). In the case of independent single-scattering events and stable distributions $P_s^{(1)}(\boldsymbol{\Theta})$, the same *a priori* information on the object is used [$\mu_{a,s}$ and $P_s^{(1)}(\boldsymbol{\Theta})$]. In the case of small-angle scattering, the calculation rate can be increased by $\sim 10^4$ times ($g_1 = 0.95$) and more when Δz varies from $\Delta z < \mu_s^{-1}$ to $\Delta z \sim (\mu_s')^{-1}$, although this is accompanied by a gradual decrease in the calculation accuracy from the accuracy of the Monte-Carlo method to that of the diffusion approximation [8]. This allows one to optimise (in the calculation rate and accuracy) the solution of the problem of multiple scattering and verify fast approximate algorithms proposed earlier for the diffusion optical tomography of objects of size of the order of 1000 scattering lengths [19].

Note also that the approach described in the paper can be easily extended to the known more complicated models of single-scattering processes, in which the function $P_s^{(1)}(\boldsymbol{\Theta})$ is described by a linear superposition of two or more Henyey–Greenstein phase functions [8, 9].

References

1. Duderstadt J.J., Martin W.R. *Transport Theory* (New York: John Wiley & Sons, 1979).
2. Ishimaru A. *Wave Propagation and Scattering in Random Media* (New York: Academic Press, 1978; Moscow: Mir, 1981) Vols 1, 2.
3. Metropolis N., Ulam S. *J. Am. Statistical Association*, **44**, 335 (1949).
4. Sobol' I.M. *Metod Monte-Karlo* (Monte-Carlo Method) (Moscow: Nauka, 1985).
5. Feynman R.P., Hibbs A.R. *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill Higher Education, 1965).
6. Kleinert H. *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (Singapore: World Scientific, 1995).
7. Perelman L.T. et al. *Phys. Rev. Lett.*, **72**, 1341 (1994).
8. Van de Hulst H.C. *Multiple Light Scattering* (New York: Acad. Press, 1980).
9. Zege E.P. et al. *Image Transfer Through a Scattering Medium* (Berlin: Springer, 1991).
10. Kim A., Ishimaru A. *Appl. Opt.*, **37**, 5313 (1998).
11. Kokhanovsky A.A. *J. Phys. D*, **30**, 2837 (1997); *Meas. Sci. Technol.*, **13**, 233 (2002).
12. Premoze S. et al. *Proc. Eurographics 14th Symp. Rendering' 2003* (Leuven, Belgium, 2003) pp 1–12.
13. Voronov A.V. et al. *Kvantovaya Elektron.*, **34**, 547 (2004) [*Quantum Electron.*, **34**, 547 (2004)].
14. Turcu I. *J. Opt. A: Pure Appl. Opt.*, **6**, 537 (2004); *Appl. Opt.*, **45**, 639 (2006).
15. Henyey L.G., Greenstein J.L. *Astrophys. J.*, **93**, 70 (1941); Jacques S.L. et al. *Lasers Life Sci.*, **1**, 309 (1987).
16. Uchaikin V.V., Zolotarev V.M. *Chance and Stability. Stable Distributions and their Applications* (The Netherlands, Utrecht: VSP, 1999).
17. Zaccanti G. et al. *Pure Appl. Opt.*, **3**, 897 (1994).
18. Gandjbakhche A.H. et al. *J. Statistical Physics*, **69** (1/2), 35 (1992).
19. Chursin D.A. et al. *Kvantovaya Elektron.*, **29**, 83 (1999) [*Quantum Electron.*, **29**, 921 (1999)]; Tret'yakov E.V. et al. *Kvantovaya Elektron.*, **31**, 1095 (2001) [*Quantum Electron.*, **31**, 1095 (2001)].