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# Peculiarities of the light propagation in semi-infinite fibre arrays

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***Abstract*. The spatial intensity distribution of radiation propagating in semi-infinite directional coupler arrays with different dependences of coupling constants on the fibre number is studied.**

***Keywords***: *directional coupler, semi-infinite fibre array, propagation constant, coupling constant.*

## 1. Introduction

Linear and nonlinear optical effects in directional couplers representing fibre arrays attract great interest because they can be used in fibreoptic communication lines and all-optical data processing systems. In addition, fibre arrays, in which each of the fibres is coupled with its nearest neighbours, are examples of discrete optical systems whose functional features have not been studied completely so far. Fibre arrays proved to be very useful in the development of semiconductor lasers [1 – 3]. The properties of light propagation in fibre arrays were studied theoretically in a number of papers [4 – 9]. It was shown in [10 – 12] that the inhomogeneous system of tunnel-coupled fibres is characterised by the total internal reflection of light. A light beam in such a system propagates along a wave-like spiral trajectory. The problem of light propagation in a system of tunnel-coupled fibres in which propagation constants change linearly was considered in [13 – 16]. Upon excitation of one fibre in such structures, the oscillations of the beam width are observed along the axis during light propagation. Upon variations of the input intensity and phase, the effects of switching, light localisation in several fibres, and control of light propagation can be observed in nonlinear fibre arrays [17 – 20]. It was shown in [21, 22] that soliton pulses can propagate in nonlinear fibre arrays.

A further study of the functional possibilities of fibre arrays is of current interest. In this connection we note that the properties of light propagation in couplers are different for infinite, semi-infinite, and finite arrays. Below, we present the results of our theoretical study of light propagation in semi-infinite linear fibre arrays by the method of coupled waves. This method is especially attractive because it reduces the problem to a one-dimensional system of coupled linear first-order differential equations. An important circumstance is that in the absence of absorption losses, the coupling between neighbouring fibres is symmetric, i.e. the coupling constants between the $i$th and $j$th fibres are equal: $\kappa_{ji} = \kappa_{ij}$.

A complicated system of fibre arrays prevents the obtaining of the general solution of the system of equations describing light propagation for arbitrary relations between the propagation constants of individual fibres and their coupling constants. As for model systems, exact analytic solutions can be obtained in some cases.

The infinite system of identical fibres was the first system for which the exact analytic solution of the system of equations for coupled waves was obtained [4, 23]. The number of model, exactly solvable systems can be considerably increased in the case of semi-infinite arrays by using the theory of orthogonal classical polynomials. Such solutions are of interest because complicated arrays can contain rather large parts in which coupling constants have a certain dependence on the fibre number in the array.

In this paper, we obtained the exact solutions of difference – differential equations describing the spatial distribution of the field amplitude in each fibre of the semi-infinite array of identical fibres. The arrays under study differ only in the dependence of coupling constants between adjacent fibres on the fibre number in the array. Although the approach used in the paper is simplified, nevertheless it predicted a number of qualitatively new effects having their own scientific and practical significance.

The difference – differential equations considered below are also used, in particular, for solving the Schrödinger equation for multilevel equidistant systems excited by resonance radiation [24 – 28] and studying the time evolution of electronic transitions between quantum dots [29, 30], and other fields of physics. Therefore, the solution of such equations for a broad class of systems is undoubtedly of interest.

## 2. Method of the problem solution

Consider a semi-infinite system of identical fibres characterised by the same propagation constant $\beta$. We will describe the stationary propagation of laser radiation in each of the fibres by the system of difference – differential equations [4 – 8, 23]

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$$\mathrm{i}\frac{\mathrm{d}E_n}{dz} + \beta E_n + \kappa_{n-1,n}E_{n-1} + \kappa_{n,n+1}E_{n+1} = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

where $n$ is the fibre number; $E_n$ is the field amplitude of a wave propagating in the $n$th fibre; $\kappa_{n-1,n}$ is the coupling constant between the $n-1$th and $n$th fibres; $z$ is the coordinate along the axis of each of the fibres measured from the end of the corresponding fibre. We assume that all the fibres are located in the same plane and the coupling constants are $\kappa_{n-1,n} = \kappa\kappa_n$ $(n = 0, 1, 2)$, where $\kappa_0 = 0$, $\kappa_1 = 1$, so that $\kappa_{0,1} \equiv \kappa$. Particular systems will differ in the dependence of the coupling constants $\kappa_n$ on the fibre number $n$. We will study below the dependence of the propagation of laser radiation on the shape of the function $\kappa_n$. By assuming in (1) that $E_n = f_n \exp(\mathrm{i}\beta z)$ and introducing the variable $x = \kappa z$, we obtain

$$\mathrm{i}\frac{\mathrm{d}f_n}{\mathrm{d}x} + \kappa_n f_{n-1} + \kappa_{n+1}f_{n+1} = 0, \qquad n = 0, 1, 2, \dots . \tag{2}$$

The boundary condition for system (2) in the case when only one $j$th fibre is pumped to the end can be written in the form

$$f_n(x = 0) = \delta_{nj}, \tag{3}$$

where $\delta_{nj}$ is the Kronecker delta. The normalised radiation intensity $\mathscr{P}_n$ in the $n$th fibre is described by the expression

$$\mathscr{P}_n(x) = |f_n(x)|^2. \tag{4}$$

The solution of the system of equations (2) for arbitrary $\kappa_n$ cannot be obtained. However, we can obtain the solution of the problem for a broad class of functions $\kappa_n$. We will seek the solution of equation (2) under condition (3) in the form

$$f_n(x) = \int_a^b \sigma(y)\frac{P_j(y)}{d_j}\frac{P_n(y)}{d_n}\exp(\mathrm{i}rxy)\mathrm{d}y, \tag{5}$$

where $P_n(x)$ is the classical orthogonal polynomial with the standardisation $P_0(x) = 1$ and the norm $d_n$ defined in the interval $[a, b]$ with the weight $\sigma(x)$, i.e.

$$\int_a^b \sigma(x)P_n(x)P_m(x)\mathrm{d}x = d_n^2\delta_{nm}. \tag{6}$$

By substituting (5) into (2), we arrive at the recurrence relation

$$\kappa_{n+1}\frac{P_{n+1}(x)}{d_{n+1}} + \kappa_n\frac{P_{n-1}(x)}{d_{n-1}} = rx\frac{P_n(x)}{d_n}, \tag{7}$$

to which orthonormalised polynomial $P_n(x)/d_n$ satisfy. By selecting different polynomials $P_n(x)$, and thereby specifying their weight $\sigma(x)$ and the interval $[a, b]$, we obtain the constant $r$ and the functional dependence of the coupling constant $\kappa_n$ from the known expressions [31–34]

$$r = \frac{k_1}{k_0}\frac{d_0}{d_1}, \qquad \kappa_n = \frac{k_{n-1}}{k_n}\frac{d_n}{d_{n-1}}r, \tag{8}$$

where $k_n$ is the coefficient at the term $x^n$ of the polynomial. Thus, we determine a particular system to which solution (5) corresponds. By integrating (5), we obtain the explicit solution $f_n(x)$ for the selected fibre system. Expression (5) for $x = 0$ satisfies boundary condition (3) due to the orthonormality of classical polynomials with the weight $\sigma(y)$:

$$\int_a^b \sigma(y)\frac{P_j(y)}{d_j}\frac{P_n(y)}{d_n}\mathrm{d}y = \delta_{n,j}. \tag{9}$$

Consider now particular models of semi-infinite arrays which will be referred to by the name of a classical polynomial giving the solution of the problem.

## 3. Chebyshev array of the I kind

Chebyshev polynomials of the I kind $T_n(x)$, $n = 0, 1, 2, ...$, are orthogonal in the interval $-1 \leqslant x \leqslant 1$ with the weight $\sigma(x) = 1/\sqrt{1-x^2}$ and satisfy the recurrence relation [31–36]

$$T_{n+1}(x) + T_{n-1}(x) - \frac{2x}{1+\delta_{n,0}}T_n(x) = 0, \quad n = 0, 1, 2, \dots . \tag{10}$$

Here, $T_0(x) = 1$. The norms of these polynomials are $d_0 = \sqrt{\pi}$, $d_n = \sqrt{\pi/2}$, $n = 1, 2, 3, ...$, and the coefficient $k_n$ at $x^n$ is $k_0 = 1$, $k_n = 2^{n-1}$, $n = 1, 2, 3\dots$ . Therefore,

$$r = \frac{k_1}{k_0}\frac{d_0}{d_1} = \sqrt{2},$$

$$\kappa_n = \frac{k_{n-1}}{k_n}\frac{d_n}{d_{n-1}}r = \begin{cases} 1, & n = 1, \\ 1/\sqrt{2}, & n = 2, 3, \dots . \end{cases} \tag{11}$$

By calculating integral (5) for the determined values of $r$ and $\kappa_n$ and assuming that the fibre with the number $n = j = 0$ is end-pumped (the boundary condition $f_{n|x=0} = \delta_{n,0}$), we find the solution of the system of equations (2) in the form

$$f_n(x) = \begin{cases} J_0(\sqrt{2}x), & n = 0, \\ \mathrm{i}^n\sqrt{2}\,J_n(\sqrt{2}x), & n = 1, 2, 3, \dots , \end{cases} \tag{12}$$

where $J_n(x)$ is the Bessel function of the $n$th order [32–37].

The normalised radiation intensity in each of the fibres is

$$\mathscr{P}_n(x) = \begin{cases} J_0^2(\sqrt{2}x), & n = 0, \\ 2J_n^2(\sqrt{2}x), & n = 1, 2, 3, \dots . \end{cases} \tag{13}$$

If the $j$th fibre is pumped ($f_{n|x=0} = \delta_{n,j}$), the solution for the function $f_n(x)$ can be also easily found, and we obtain the expressions

$$\mathscr{P}_0(x) = J_0^2(\sqrt{2}x), \quad j = 0, n = 0,$$

$$\mathscr{P}_n(x) = 2J_n^2(\sqrt{2}x), \quad j = 0, n = 1, 2, 3, ...,$$

$$\mathscr{P}_0(x) = 2J_j^2(\sqrt{2}x), \quad j \neq 0, n = 0, \tag{14}$$

$$\mathscr{P}_n(x) = J_{n+j}^2(\sqrt{2}x) + J_{n-j}^2(\sqrt{2}x) + 2(-1)^j\times$$

$$\times J_{n+j}(\sqrt{2}x)J_{n-j}(\sqrt{2}x), \quad j \neq 0, \; n \geqslant 1$$

for the radiation intensity in an arbitrary fibre.

Figure 1 shows the spatial distributions of radiation intensity in the fibre array as functions of the normalised coordinate $x = \kappa z$ (hereafter, the radiation intensity in a fibre pumped to the end for $x = 0$ is unity). One can see that the radiation intensity in the pumped fibre ($n = 0$) is an oscillating and monotonically decreasing function of the distance $x$ (Fig. 1a). The radiation intensity in the pumped fibre ($n = 0$) in nonzero at a large distance from the fibre end. One can see from Fig. 1a that for $x = 2.8$ and 11.7, the radiation intensity is $\mathscr{P}_0 \approx 0.15$ and 0.05, respectively. For $x \gg 1$, the field intensity tends to zero proportionally to $x^{-1}$. The transfer of radiation to other fibres with $n > 0$ is also observed. In this case, the higher is the fibre number $n$, i.e. the farther the fibre is located from the end-pumped fibre, the farther the first maximum of the field intensity in the $n$th fibre appears from the fibre end. The radiation intensity maxima for $x \gg 1$ decrease proportionally to $x^{-1}$.

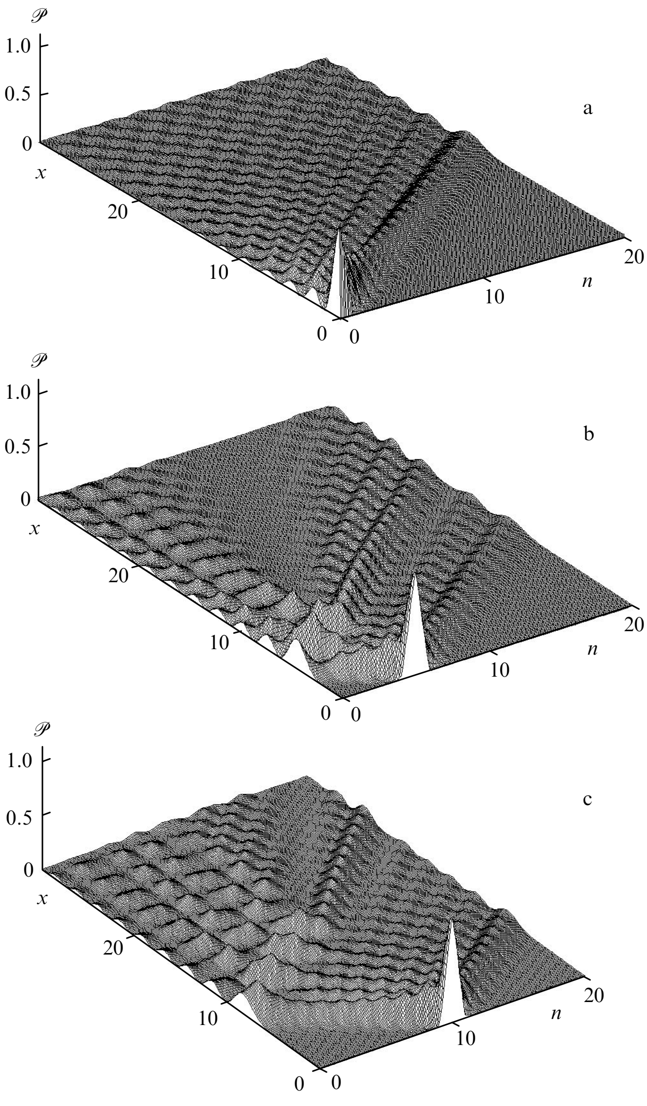


**Figure 1.** Spatial distributions of the radiation intensity $\mathscr{P}$ in fibres in the semi-infinite Chebyshev array of the I kind upon pumping the zero (a), fifth (b), and tenth (c) fibres.

The radiation transfer between fibres due to interaction between them can be interpreted as the diffusion of radiation into the inner regions of the array in the direction perpendicular to the fibre axis. That is why the spatial distribution of the light intensity in the fibre array has generally a wave-like shape, with the amplitude decreasing with distance, the crests of several first waves being located deeper and deeper along the $x$ axis in the array with increasing the fibre number $n$. The shape of this structure is determined both by the effect of light propagation along the fibre axis and the diffusion of light in the transverse direction. Because the zero ($n = 0$) fibre is coupled only with the first one, whereas all other fibres interact with two nearest neighbours, the wave as if reflected from the array interface (zero fibre) due to diffusion and, by propagating along the $x$ axis, diffuses in the transverse direction. Near the end of the array, an unperturbed region appears which expands with increasing $n$.

Figures 1b, c show the spatial distributions of radiation intensity in the array upon end pumping the fifth and tenth fibres, respectively. One can see that the joint action of light propagation and diffusion produces a complicated interference structure of the spatial distribution with the amplified higher-order maxima. For example, upon end pumping the fifth fibre, a rapid transfer of radiation to the zero fibre occurs and its intensity at the maximum is nonzero even at the distance $x \simeq 25 - 30$. In addition, upon pumping the fifth fibre, the first two maxima in the 20th fibre are lower than the next three maxima.

Upon end pumping the tenth fibre, the first emission maximum in the zero fibre is observed at $x \approx 8$. Then, the intensity of maxima in the zero fibre monotonically decreases with increasing $x$, but noticeably differs from zero even at the distance $x \approx 30$, these maxima being more distinct than those observed upon pumping the fifth and zero fibres. One can also see that at the array interface in the vicinity of the zero fibre for $x > 0$, a rather complicated spatial structure of the intensity maxima and minima is formed, which is produced by a wave reflected from the array interface due to diffusion. In addition, radiation in the pumped (tenth) fibre first rapidly decreases with distance, completely vanishes at $x \simeq 9$, and then again appears, and the intensity of maxima in this fibre at large $x$ ($\sim 20$ and above) proves to be considerable. Note also that upon pumping into the tenth fibre, the amplitude of the sixth maximum in the 20th fibre is considerably greater than the amplitudes of nearest maxima. In this case, an unperturbed region is also observed on both sides of the end of the pumped fibre, which expands with distance from the fibre.

## 4. Chebyshev array of the II kind

Chebyshev polynomials of the II kind $U_n(x)$, $n = 0, 1, 2, \ldots$, are orthogonal in the interval $-1 \leqslant x \leqslant 1$ with the weight $\sigma(x) = \sqrt{1 - x^2}$ and satisfy the recurrence relation $U_{n+1}(x) + U_{n-1}(x) - 2xU_n(x) = 0$. The norm of these polynomials is $d_n = \sqrt{\pi/2}$, the coefficient $k_n$ at $x^n$ is $2^n$ [31–36]. According to (8), we obtain $r = 2$ and $\kappa_n = 1$. Thus, this array is characterised by the coupling constant which is the same for all fibres, i.e. independent of $n$. According to (5), we can obtain the solution for $f_n$ and then for the light intensity in the $n$th fibre. By assuming that for $x = 0$ only the fibre with $n = 0$ is pumped, i.e. the boundary conditions are

$f_{n|x=0} = \delta_{n,0}$, the solution of the system of equations (2) has, according to (5), the form

$$f_n(x) = \mathrm{i}^n \frac{n+1}{x} J_{n+1}(2x). \quad (15)$$

This means that the field amplitude in the $n$th fibre is an oscillating monotonically decreasing function of distance. For $x \to \infty$, the field amplitude $f_n(x)$ tends to zero. The field intensity $\mathscr{P}_n(x)$ at an arbitrary point is described by the expression

$$\mathscr{P}_n(x) = \frac{(n+1)^2}{x^2} J_{n+1}^2(2x). \quad (16)$$

By assuming that pumping is performed to the end of the $j$th fibre ($f_{n|x=0} = \delta_{n,j}$), we obtain

$$\mathscr{P}_n(x) = J_{n+j+2}^2(2x) + J_{n-j}^2(2x)$$

$$+ 2(-1)^j J_{n+j+2}(2x) J_{n-j}(2x). \quad (17)$$

From this, we find for $n = j \neq 0$

$$\mathscr{P}_n(x) = J_0^2(2x) + J_{2(n+1)}^2(2x)$$

$$+ 2(-1)^j J_0(2x) J_{2(n+1)}(2x). \quad (18)$$

Figure 2 presents the spatial distributions of the light intensity in the Chebyshev array of the II kind upon end pumping the zero, fifth, and tenth fibres, respectively. Unlike the previous case, here radiation pumped to the zero fibre is transferred very rapidly to neighbouring fibres, so that the second maximum in the zero fibre for $x \simeq 2$ contains only 3 % of the pumped intensity (Fig. 2a). This is explained by the fact that for $x \gg 1$, the radiation intensity decreases with increasing $x$ much faster [$\mathscr{P}_n(x) \sim x^{-3}$] than in the Chebyshev array of the I kind, where $\mathscr{P}_n(x) \sim x^{-1}$. One can see from Fig. 2a that the stationary spatial distribution of the light intensity exhibits several wave crests with amplitudes rapidly decreasing with increasing coordinate $x$. Radiation is rapidly transferred from the pumped zero fibre to fibres with greater values of $n$, as if reflecting from the array wall (i.e. from the zero fibre). The spatial field structure is even more complicated upon pumping to the end of the fifth (Fig. 2b) and tenth (Fig. 2c) fibres, which is explained by the propagation and diffusion of radiation. Upon pumping the fifth fibre, a small trough is observed for large $n$ after the first two crests of the waves in the spatial intensity distribution. The extension of this trough increases upon pumping the tenth fibre, and a distinct crest is formed behind it, which is followed by a short trough and several crests with monotonically decreasing amplitudes. Note also that upon pumping the fifth fibre, the field distribution in the zero fibre drastically differs from that observed upon pumping the zero fibre itself. In this case, the joint action of radiation propagation and diffusion to the left with reflection from the system wall gives rise to a complicated interference structure of the spatial field distribution with maxima and minima located at large distances from the front end.

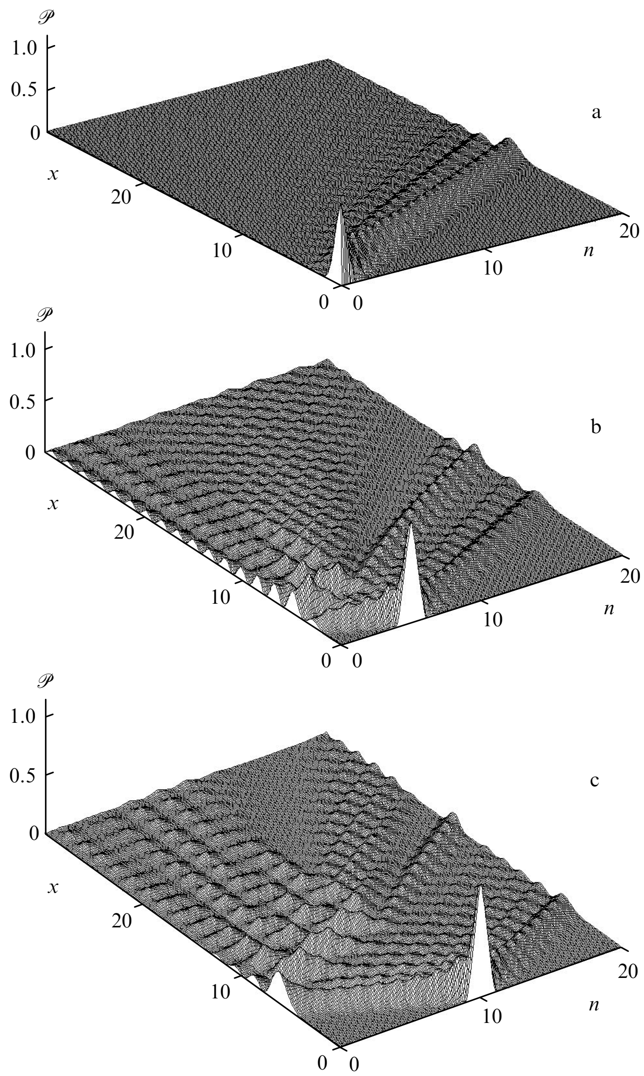


**Figure 2.** Spatial distributions of the radiation intensity $\mathscr{P}$ in fibres in the semi-infinite Chebyshev array of the II kind upon end pumping the zero (a), fifth (b), and tenth (c) fibres.

## 5. Hermitean array

Hermitean polynomials $H_n(x)$ are orthogonal in the interval $(-\infty; +\infty)$ with the weight $\sigma(x) = \exp(-x^2)$. The norm and leading coefficient of polynomials are $d_n = \sqrt{\sqrt{\pi} 2^n n!}$, $k_n = 2^n$, respectively [31–36]. We find from (8) that $r = \sqrt{2}$, and $\kappa_n = \sqrt{n}$, and, according to (5), construct the solution under the condition that the system is pumped to the zero fibre ($f_{n|x=0} = \delta_{n,0}$):

$$f_n(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} \int_{-\infty}^{\infty} \exp(-y^2) H_n(y) \exp(\mathrm{i}\sqrt{2} xy) \mathrm{d}y. \quad (19)$$

The integral can be easily calculated to obtain

$$f_n(x) = \frac{\mathrm{i}^n x^n}{\sqrt{n!}} \exp\left(-\frac{x^2}{2}\right). \quad (20)$$

Then, the light intensity in the $n$th fibre is described by the expression

$$\mathscr{P}_n(x) = \frac{x^{2n}}{n!}\exp(-x^2). \tag{21}$$

For the given $n$, the intensity maximum is located at the point $x = \sqrt{n}$ and the maximum intensity in the $n$th fibre of the array is

$$\mathscr{P}_{n\,\max} = \frac{n^n \exp(-n)}{n!}.$$

When the system is pumped through the $i$th fibre, the solution has the form

$$f_n(x) = \mathrm{i}^{n-j}\sqrt{\frac{j!}{n!}}\,x^{n-j}\exp\left(-\frac{x^2}{2}\right)L_j^{n-j}(x^2), \quad n \geqslant j, \tag{22}$$

$$f_n(x) = \mathrm{i}^{j-n}\sqrt{\frac{n!}{j!}}\,x^{j-n}\exp\left(-\frac{x^2}{2}\right)L_n^{j-n}(x^2), \quad n \leqslant j, \tag{23}$$

where $L_n^m(x)$ is the Laguerre polynomial [31–36]. By using (4), (22), and (23), we can easily obtain the expression for the radiation intensity $\mathscr{P}_n(x)$. Figure 3a presents the spatial distribution of the radiation intensity $\mathscr{P}_n(x)$ according to (21) upon pumping the zero fibre. One can see that radiation is rapidly transferred to fibres with greater numbers $n$ with increasing $x$. The radiation propagation and diffusion processes produce the spatial intensity distribution in the form of a solitary wave-like profile. The maximum of this profile from the fibre end increases with increasing the fibre number $n$ in the array.

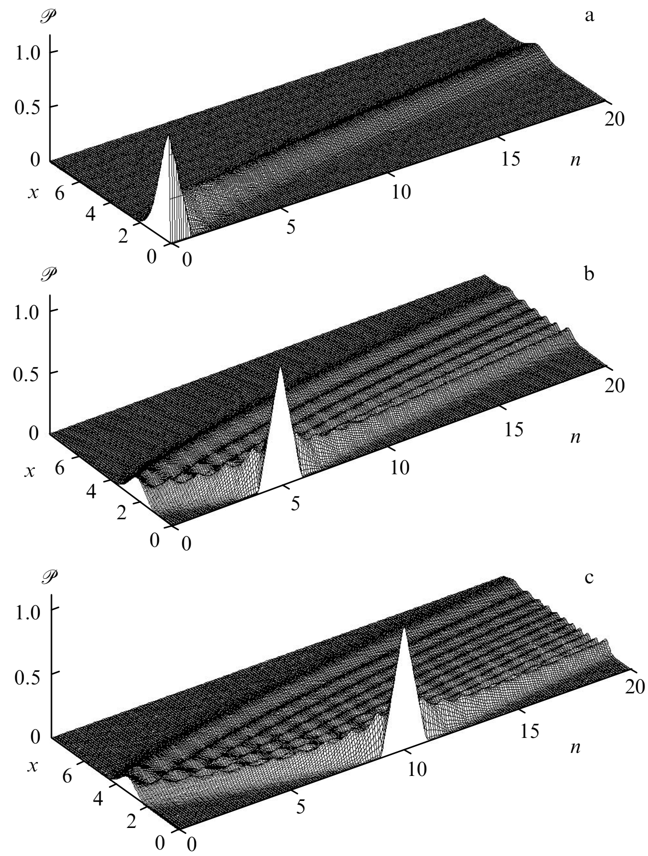


**Figure 3.** Spatial distributions of the radiation intensity $\mathscr{P}$ in fibres in the semi-infinite Hermitean array upon end pumping the zero (a), fifth (b), and tenth (c) fibres.

Figures 3b, c presents field distributions upon pumping the fifth and tenth fibres. One can see that, upon pumping the fibre with the number $j > 0$, the field structure in fibres with numbers $n$ from zero to $j$ is rather complicated. The field distribution in the fibre with $n = 0$ is one pronounced maximum at the point $x = \sqrt{j}$ and $\mathscr{P}_{0\,\max} = j^j \exp(-j)/j!$. The amplitude of this maximum decreases with increasing $j$ as $\mathscr{P}_{0\,\max} \sim j^{-1/2}$. As the fibre number $n$ increases, the number of maxima in the field distribution gradually increases. This is caused by a complicated interference of the waves propagating from the $j$th fibre towards the zero fibre and the waves reflected from it. However, for $n > j$, the number of maxima of the spatial field distribution is $j + 1$, and, as follows from Figs 3b, c, is independent of $n$ (upon pumping the fifth and tenth fibres, 6 and 11 maxima of the field intensity are formed, respectively, which are shifted to greater $n$).

## 6. Legendre array

Legendre polynomials $P_n(x)$, $n = 0, 1, 2, \ldots$, are orthogonal in the interval $-1 \leqslant x \leqslant 1$ with the weight $\sigma(x) = 1$ and satisfy the recurrence relation $(n+1)P_{n+1}(x) + nP_{n-1}(x) - (2n+1)xP_n(x) = 0$. The norm of these polynomials is $d_n = \sqrt{2/(2n+1)}$ and the leading coefficient is $k_n = 2^n\Gamma(n+\frac{1}{2})/(\sqrt{\pi}n!)$. Then, $r = \sqrt{3}$ and $\kappa_n = \sqrt{3}n/\sqrt{4n^2-1}$. As $n$ increases, the coupling coefficient $\kappa_n$ tends asymptotically to $\sqrt{3}/2$. By assuming that only the zero fibre is pumped ($f_{n|x=0} = \delta_{n,0}$), we obtain the light intensity in an arbitrary fibre

$$\mathscr{P}_n(x) = \frac{(2n+1)\pi}{2\sqrt{3}x}J^2_{n+1/2}(\sqrt{3}x). \tag{24}$$

Figure 4 shows the spatial intensity distribution, which is quite similar to profiles presented in Fig. 2a. The radiation intensity in the zero fibre rapidly decreases (proportionally to $x^{-2}$) with increasing $x$, so that only the first two maxima are pronounced. As $x$ increases, radiation is rapidly transferred to fibres with large $n$.

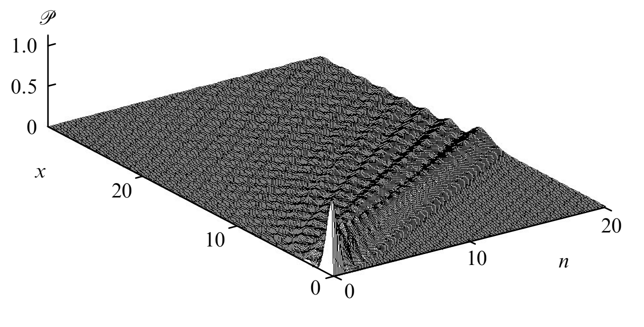


**Figure 4.** Spatial distribution of the radiation intensity $\mathscr{P}$ in fibres in the semi-infinite Legendre array upon end pumping the zero fibre.

## 7. Gegenbauer array

Ultraspherical Gegenbauer polynomials $C_n^\lambda(x)$ ($\lambda > -1/2$, $n = 0, 1, 2, \ldots$) are orthogonal in the interval $-1 \leqslant x \leqslant 1$ with the weight $\sigma(x) = (1-x^2)^{\lambda-1/2}$. The norm of these polynomials is

$$d_n = \left[\frac{\sqrt{\pi}(2\lambda)_n\Gamma(\lambda+0.5)}{(n+\lambda)n!\Gamma(\lambda)}\right]^{1/2},$$

the coefficient $k_n = 2^n(\lambda)_n/n!$, where $(\lambda)_n$ is the Pochhammer symbol [27] and $\Gamma$ is the gamma function. Then, according to (8), $r = \sqrt{2(\lambda+1)}$ and

$$\kappa_n = \left[\frac{n(n+2\lambda-1)(\lambda+1)}{2(n+\lambda-1)(n+\lambda)}\right]^{1/2}.$$

The coupling constant $\kappa_n$ asymptotically tends to the value $[(\lambda+1)/2]^{1/2}$ with increasing $n$. For the boundary conditions $f_{n|x=0} = \delta_{n0}$, the solution of Eqn (2) has the form

$$f_n(x) = \mathrm{i}^n 2^\lambda\Gamma(\lambda)\left[\frac{\lambda(n+\lambda)}{n!}(2\lambda)_n\right]^{1/2}\frac{J_{n+\lambda}[\sqrt{2(\lambda+1)}x]}{[\sqrt{2(\lambda+1)}x]^\lambda}, \quad (25)$$

and we obtain the light intensity in the $n$th fibre

$$\mathscr{P}_n(x) = \frac{2^{2\lambda}[\Gamma(\lambda)]^2\lambda(n+\lambda)(2\lambda)_n}{n!}$$

$$\times\frac{1}{[\sqrt{2(\lambda+1)}x]^{2\lambda}}J^2_{n+\lambda}[\sqrt{2(\lambda+1)}x]. \quad (26)$$

The one-parametric family of solutions (25) and (26) with the parameter $\lambda$ for $\lambda = 0$, 1/2, 1, –1/2 and $\infty$ transforms to the known solutions. For $\lambda = 0$, the Gegenbauer polynomial transforms to the Chebyshev polynomial of the I kind $T_n(x) = (n/2+\delta_{n,0})\times C_n^0(x), n = 0\ 1,\ 2,\ ...$; correspondingly, the coupling constant is $\kappa_n = 1$ for $n = 1$ and $\kappa_n = 1/\sqrt{2}$ for $n$ = 2, 3,..., and solution (25) transforms to (12). For $\lambda = 1/2$, the Gegenbauer polynomials transform to the Legendre polynomials $P_n(x) = C_n^{1/2}(x)$, while solution (26) is reduced to expression (24). For $\lambda = 1$, ultraspherical polynomials transform to the Chebyshev polynomials of the II kind $C_n^1(x) = U_n(x)$. In this case, $\kappa_n = 1$ and solutions (25) and (26) transform to solutions (15) and (16), respectively.

Consider the limit $\lambda\to\infty$. It follows from the definition of $\kappa_n$ for the Gegenbauer array that $\lim_{\lambda\to\infty}\kappa_n = \sqrt{n}$. By using the asymptotics of the Bessel function $J_{n+\lambda}(\sqrt{2(\lambda+1)}\times x)$ for large values of the order and argument [31]

$$J_{n+\lambda}(\sqrt{2(\lambda+1)}x)_{\lambda\to\infty}\to\frac{[\sqrt{(\lambda+1)/2}x]^{n+\lambda}\exp(-x^2/2)}{\lambda^{n+1}\Gamma(\lambda)}$$

and substituting the latter expression into (25), we obtain (20).

For $\lambda = -1/2$, we obtain the known solution describing the propagation of light in a directional coupler consisting of two identical fibres [23]. In this case, $\kappa_n = [n(n-2)/(2n-3)(2n-1)]^{1/2}$, for which it follows that $\kappa_1 = 1$, and $\kappa_2 = 0$ in this system. Upon pumping the zero fibre ($n = 0$), light is transferred to the first fibre ($n = 1$), but coupling between the first and second fibres is absent because $\kappa_2 = 0$. Although the coupling constants $\kappa_n$ for $n > 2$ are nonzero, nevertheless because of successive excitation transfer from one fibre to the adjacent fibre, all fibres with $n\geqslant 2$ will not be excited and, therefore, the features of light propagation in such a system are identical to those for light propagation in a two-channel coupler. Indeed, by substituting $\lambda = -1/2$ into solution (25) and using relations $(-1)_n = \delta_{n,0}-\delta_{n,1}$, $[(-1/2)(n-1/2)(-1)_n]^{1/2} = (\delta_{n,0}+\delta_{n,1})/2$, and explicit expressions for the Bessel function of the half-integer order, we obtain $f_n(x) = \mathrm{i}^n[\delta_{n,0}\cos(x)+\delta_{n,1}\sin(x)]$, i.e. $f_0 = \cos(x)$, $f_1 = \mathrm{i}\sin(x)$, in agreement with known results [23].

## 8. Conclusions

We have obtained exact analytic solutions for an infinite system of difference – differential equations describing the stationary spatial distribution of the field amplitude and intensities in each of the fibres of a semi-infinite planar array of identical fibres with a certain dependence of coupling constants between nearest neighbouring fibres on their number in the array. The general property of all the arrays studied is that after end pumping one of the array fibres, radiation is rapidly transferred to fibres with greater numbers $n$. The features of the spatial distribution of the radiation field are explained by the radiation propagation and diffusion across the array due to coupling between adjacent fibres in the array.

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