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LASER APPLICATIONS AND OTHER TOPICS IN QUANTUM ELECTRONICS

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Application of the Wigner function and matrix optics to describe variations in the shape of ultrashort laser pulses propagating through linear optical systems

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Abstract. **The transformation of the shape of ultrashort laser pulses (USPs) in time can be described similarly to the process of image formation in space. It is shown that the wave description of imaging is simplified by using the Wigner function, this description in the quadratic approximation being identical to the use of the *ABCD* matrices. The transformation of USPs propagating through linear optical systems was described and these systems were classified by the methods of matrix optics.**

Keywords: *ultrashort pulses, Wigner function, matrix optics.*

1. Introduction

As the advent of point light sources allowed the understanding of the 'spatial' process of image formation in optical systems, the development of ultrashort-pulse lasers permits the insight into the 'temporal' process of image formation in these systems. Moreover, there exists a formal analogy between the so-called spatial and temporal optical systems [1]: the cross-section width of a narrow wave beam propagating through optically homogeneous media and lenses is transformed identically to the duration of a narrowband pulse propagating through a dispersive medium and phase modulators. This analogy permits one to treat phase modulators as temporal lenses, which can be used for the development of temporal analogues of conventional spatial optical systems such as microscopes, telescopes, Fourier converters, etc. The processes of image formation in such temporal systems are described by means of the mathematical apparatus that is traditionally used for the description of image formation in spatial optical systems.

At present the Wigner distribution function (WDF) is finding increasing use for describing the image formation in space. This function allows one to consider the wave and geometrical optics from the unitary point of view [2, 3] by preserving the apparatus of these theories based on duality [4, 5]. When the WDF is used, the 'modulator – filters' duality in wave optics [6] naturally transforms in the quadratic approximation to the 'refraction matrix – transfer matrix' duality in geometrical optics [7].

The aim of this paper is the use of the WDF and its matrix approximation to describe the transformation of the USP shape in optical systems with temporal lenses, i.e., for the description of the temporal analogues of conventional optical systems.

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2. Wigner function and its properties

A scalar signal can be described both by the dependence of the complex amplitude U on time t and the dependence of the complex spectrum $\tilde{U}$ on frequency ω. These descriptions are equivalent, i.e.,

$$\tilde{U}(\omega) = \mathrm{F}_{t\to\omega}\{U(t)\} \quad \text{and} \quad U(t) = \mathrm{F}^{-1}_{\omega\to t}\{\tilde{U}(\omega)\}, \qquad (1)$$

because they are related by the Fourier transforms

$$\mathrm{F}_{t\to\omega}\{...\} \equiv \frac{1}{\sqrt{2\pi \mathrm{i}}}\int_{-\infty}^{\infty}\{...\}\exp(-\mathrm{i}\omega t)\mathrm{d}t$$

and

$$\mathrm{F}^{-1}_{\omega\to t}\{...\} \equiv -\frac{1}{\sqrt{2\pi \mathrm{i}}}\int_{-\infty}^{\infty}\{...\}\exp(\mathrm{i}\omega t)\mathrm{d}\omega.$$

A 'quasi-point' pulse can be conveniently described in the temporal representation $U(t)$, while a 'quasi-monochromatic' oscillations – in the spectral representation $\tilde{U}(\omega)$. A signal of the general form is reasonable to describe by a real function specified on a two-dimensional plane with coordinates t and ω. For this reason, the USP parameters are often described and estimated by such functions as the FROG [8], 'spectrogram' [9] or WDF [10 – 12]. Note that the WDF allows one to describe laser pulses before and after their transformation in optical systems, thereby describing these systems.

The WDF W is formally defined dually [2, 3, 10 – 12] both in terms of the temporal representation of the signal $U(t)$,

$$W(t,\omega) \equiv \mathrm{F}_{\Delta t\to\omega}\left\{U\left(t+\frac{\Delta t}{2}\right)U^*\left(t-\frac{\Delta t}{2}\right)\right\}, \qquad (2)$$

and in terms of the frequency representation of the signal $\tilde{U}(\omega)$,

$$W(t,\omega) \equiv \mathrm{F}^{-1}_{\Delta\omega\to t}\left\{\tilde{U}\left(\omega+\frac{\Delta\omega}{2}\right)\tilde{U}^*\left(\omega-\frac{\Delta\omega}{2}\right)\right\}. \quad (3)$$

The function W has many useful properties. By integrating it over time t and frequency ω, we obtain the signal energy

$$\mathscr{E} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t,\omega)\mathrm{d}t\mathrm{d}\omega,$$

by integrating it over the frequency [the projection of W on the coordinate axis t (Fig. 1b)], we obtain the intensity

$$I(t) = |U(t)|^2 = \int_{-\infty}^{\infty} W(t,\omega)\mathrm{d}\omega,$$

and by integration over time [the projection of W on the coordinate axis ω (Fig. 1c)], we obtain the intensity spectrum

$$\tilde{I}(\omega) = |\tilde{U}(\omega)|^2 = \int_{-\infty}^{\infty} \tilde{W}(t,\omega)\mathrm{d}t.$$

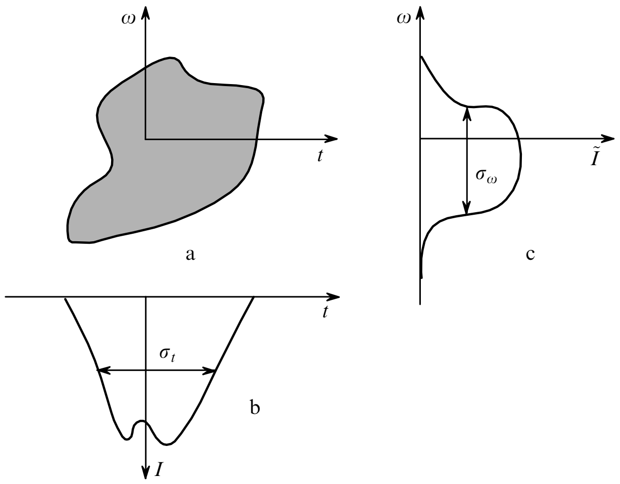

Figure 1. Wigner function (a) and its projections on the coordinate axes t (b) and ω (c).

Consider the matrices of the first and second moments of the function W [3]

$$E \equiv \begin{pmatrix} \bar{t} \\ \bar{\omega} \end{pmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} t \\ \omega \end{pmatrix} W(t,\omega)\mathrm{d}t\mathrm{d}\omega\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t,\omega)\mathrm{d}t\mathrm{d}\omega\right]^{-1},$$

$$D \equiv \begin{pmatrix} \sigma_t^2 & m_{t\omega} \\ m_{\omega t} & \sigma_\omega^2 \end{pmatrix} \quad (4)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} (t-\bar{t})(t-\bar{t})^* & (t-\bar{t})(\omega-\bar{\omega})^* \\ (\omega-\bar{\omega})(t-\bar{t})^* & (\omega-\bar{\omega})(\omega-\bar{\omega})^* \end{bmatrix} W(t,\omega)\mathrm{d}t\mathrm{d}\omega \times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t,\omega)\mathrm{d}t\mathrm{d}\omega\right]^{-1}.$$

Note that the elements of the principal diagonal of the matrix of second moments are the squares of the pulse duration σ_t and the spectral width σ_ω of the pulse. A pulse and its WDF are called transform limited [13, 14] if the matrix of second moments of the WDF is diagonal:

$$D = \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\omega^2 \end{pmatrix} \equiv \mathrm{diag}(\sigma_t^2, \sigma_\omega^2).$$

In the general case, the duration and spectral width of a pulse are related by the Heisenberg inequality

$$\sigma_t\sigma_\omega \geqslant \sqrt{\det D}.$$

In particular, a signal with the rectangular WDF is transform limited [15, 16] (Fig. 2a):

$$W_0(t-t_0,\omega-\omega_0) = \mathrm{rect}\left(\frac{t-t_0}{\sigma_t}\right)\mathrm{rect}\left(\frac{\omega-\omega_0}{\sigma_\omega}\right),$$

where t_0 is the central instant of time; ω_0 is the carrier frequency of the pulse;

$$\mathrm{rect}(t) \equiv \begin{cases} 1 & \text{for } |t|<1/2, \\ 1/2 & \text{for } |t|=1/2, \\ 0 & \text{for } |t|>1/2 \end{cases}$$

is a rectangular function.

3. Transformation of the Wigner function in linear systems

It was shown in [6] that there exist two mutually complimentary (dual) classes of linear systems – modulators and filters.

3.1 Modulators

A modulator, i.e., a linear system invariant in the frequency space is described in the temporal region by the expression [6]

$$U_{\mathrm{M}}(t) = U(t)g(t), \quad (5)$$

where $g(t)$ is a modulation function. If

$$g(t) = \exp[-\mathrm{i}\psi(t)], \quad (6)$$

expression (5) describes a phase modulator.

By substituting expression (5) for the phase modulator into (2), we obtain

$$W_{\mathrm{M}}(t-t_0,\omega) \equiv \mathrm{F}_{(t-t_0)\to\Delta\omega}\left\{U_{\mathrm{M}}\left(t-t_0+\frac{\Delta t}{2}\right)U_{\mathrm{M}}^*\left(t-t_0-\frac{\Delta\omega}{2}\right)\right\}$$

$$= \mathrm{F}_{(t-t_0)\to\Delta\omega}\left\{U\left(t-t_0+\frac{\Delta t}{2}\right)U^*\left(t-t_0-\frac{\Delta t}{2}\right) \times\exp\left[-\mathrm{i}\psi\left(t-t_0+\frac{\Delta t}{2}\right)+\mathrm{i}\psi\left(t-t_0-\frac{\Delta t}{2}\right)\right]\right\}. \quad (7)$$

In the linear approximation, we have

$$\begin{aligned} W_M&\left\{\begin{pmatrix} t-t_0 \\ \omega \end{pmatrix}\right\} \\ &= F_{(t-t_0)\to\Delta\omega}\left\{\left\langle U\left(t-t_0+\frac{\Delta t}{2}\right)U^*\left(t-t_0-\frac{\Delta t}{2}\right)\right\rangle\right. \\ &\times \exp\left[-i\psi\left(t-t_0+\frac{\Delta t}{2}\right)+i\psi\left(t-t_0-\frac{\Delta t}{2}\right)\right]\Bigg\} \\ &\approx F_{(t-t_0)\to\Delta\omega}\left\{\left\langle U\left(t-t_0+\frac{\Delta t}{2}\right)U^*\left(t-t_0-\frac{\Delta t}{2}\right)\right\rangle\right. \\ &\times \exp\left(-i\frac{d\psi}{dt}\Delta t\right)\Bigg\}, \end{aligned} \tag{8}$$

and by using the shift theorem [6], we obtain

$$\begin{aligned} W_M&(t-t_0,\omega) \\ &\approx F_{(t-t_0)\to\Delta\omega}\left\{U\left(t-t_0+\frac{\Delta t}{2}\right)U^*\left(t-t_0-\frac{\Delta t}{2}\right)\right. \\ &\times \exp\left(-i\frac{d\psi}{dt}\Delta t\right)\Bigg\} = W\left(t-t_0,\omega+\frac{d\psi}{dt}\right). \end{aligned} \tag{9}$$

Note that expression (9) is much simpler than the initial relation

$$U_M(t) = U(t)\exp[-i\psi(t)],$$

because it is reduced to the frequency shift of W by the time derivative of the phase shift ψ. Thus, the modulator does not change the duration of the initial transform limited rectangular pulse but increases its frequency band (Fig. 2c), by producing a frequency chirped pulse.

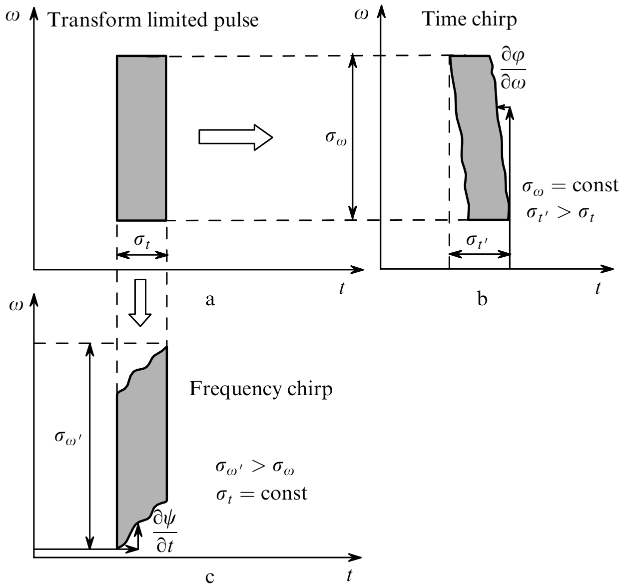

Figure 2. Transformation of a rectangular signal (a) by a filter (b) and a modulator (c).

Assuming that the pulse duration is small, we expand the phase function $\psi(t-t_0)$ in the vicinity of the central instant of time t_0 into a Taylor series:

$$\psi(t-t_0) = \psi_0 - \omega_0(t-t_0) + \frac{1}{2}P(t-t_0)^2 + \ldots,$$

where

$$\omega_0 \equiv \left.\frac{d\psi}{dt}\right|_{t=t_0}; \quad P \equiv \left.\frac{d^2\psi}{dt^2}\right|_{t=t_0}.$$

By restricting our consideration to the second-order terms, we obtain in this quadratic approximation from (9) (Fig. 3c)

$$W_M(t-t_0,\omega) \approx W[t-t_0,\omega-\omega_0+(t-t_0)P]. \tag{10}$$

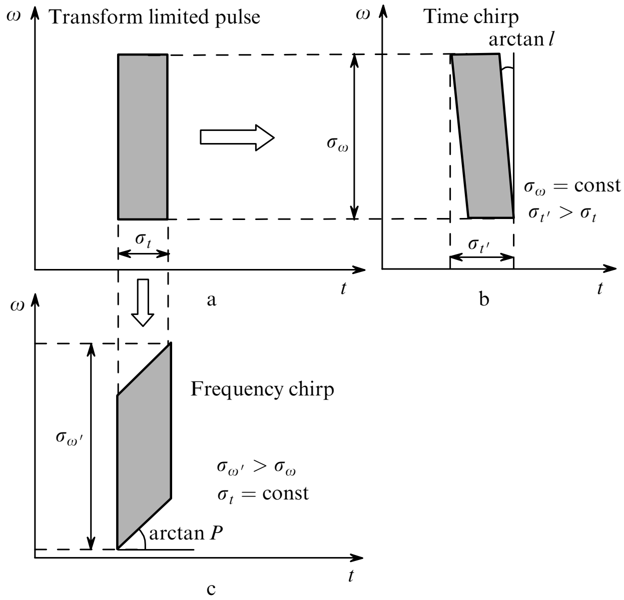

Figure 3. Transformation of a rectangular signal (a) by a quadratic filter (b) and a modulator (c).

3.2 Filters

A filter, i.e., a linear system invariant in time, is described in the frequency region by the expression [6]

$$\tilde{U}_F(\omega) = \tilde{U}(\omega)\tilde{f}(\omega), \tag{11}$$

where $\tilde{f}(\omega)$ is the transfer function. If

$$\tilde{f}(\omega) = \exp[-i\varphi(\omega)], \tag{12}$$

expression (11) describes a phase filter.

By substituting expression (11) for the phase filter into (3), we obtain

$$W_F(t,\omega-\omega_0)$$

$$\equiv F^{-1}_{\Delta\omega\to t}\left\{\tilde{U}_F\left(\omega-\omega_0+\frac{\Delta\omega}{2}\right)\tilde{U}^*_F\left(\omega-\omega_0-\frac{\Delta\omega}{2}\right)\right\} =$$

$$= \mathrm{F}_{\Delta\omega \to t} -1\left\{ \tilde{U}\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) \tilde{U}^*\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right) \right.$$

$$\left. \times \exp\left[-\mathrm{i}\varphi\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) + \mathrm{i}\varphi\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right)\right]\right\}. \quad (13)$$

In the linear approximation, we have [17]

$$W_{\mathrm{F}}(t, \omega - \omega_0)$$

$$= \mathrm{F}^{-1}_{\Delta\omega \to t}\left\{ \tilde{U}\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) \tilde{U}^*\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right) \right.$$

$$\left. \times \exp\left[-\mathrm{i}\varphi\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) + \mathrm{i}\varphi\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right)\right]\right\}$$

$$\approx \mathrm{F}^{-1}_{\Delta\omega \to t}\left\{ \tilde{U}\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) \tilde{U}^*\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right) \right.$$

$$\left. \times \exp\left(-\mathrm{i}\frac{\partial\varphi}{\partial\omega}\Delta\omega\right)\right\}, \quad (14)$$

and by using the shift theorem [6], we obtain

$$W_{\mathrm{F}}(t, \omega - \omega_0)$$

$$\approx \mathrm{F}^{-1}_{\Delta\omega \to t}\left\{ \tilde{U}\left(\omega - \omega_0 + \frac{\Delta\omega}{2}\right) \tilde{U}^*\left(\omega - \omega_0 - \frac{\Delta\omega}{2}\right) \right.$$

$$\left. \times \exp\left(-\mathrm{i}\frac{\partial\varphi}{\partial\omega}\Delta\omega\right)\right\} = W\left(t - \frac{\partial\varphi}{\partial\omega}, \omega - \omega_0\right). \quad (15)$$

Note that expression (15) is much simpler than the initial relation

$$\tilde{U}_{\mathrm{F}}(\omega) = \tilde{U}(\omega)\exp[-\mathrm{i}\varphi(\omega)],$$

because it is reduced to the temporal shift of W by the time derivative of the phase shift φ. Thus, the filter does not change the frequency band of the initial transform limited rectangular pulse but increases its duration (Fig. 2b) by producing a time chirped pulse.

Assuming that the frequency band of the pulse is narrow, we expand the phase function $\varphi(\omega - \omega_0)$ in a Taylor series in the vicinity of the carrier frequency ω_0:

$$\varphi(\omega - \omega_0) = \varphi_0 + t_0(\omega - \omega_0) + \frac{1}{2}l(\omega - \omega_0)^2 + \ldots,$$

where

$$t_0 \equiv \left.\frac{\mathrm{d}\varphi}{\mathrm{d}\omega}\right|_{\omega - \omega_0};$$

$$l \equiv \left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\omega^2}\right|_{\omega = \omega_0}$$

is the effective thickness of the filter, which can be both positive and negative. By restricting our consideration to the second-order terms, we obtain from (15) (Fig. 3b)

$$W_{\mathrm{F}}(t, \omega - \omega_0) \approx W[t - t_0 - (\omega - \omega_0)l, \omega - \omega_0]. \quad (16)$$

4. Matrix formalism of the theory of ideal optical systems

Engineering calculations and classification of the spatial optic systems are based on the theory of ideal (aberration-free) optical Maxwell systems [18] and the mathematical apparatus of matrix algebra. Therefore, it is very important to study the passage from the general wave description [1] to the matrix aberration-free description [19–22] in temporal optics as well.

Note that the point (t, ω) in the phase space (see Fig. 2) can be characterised by the column matrix

$$\begin{pmatrix} t \\ \omega \end{pmatrix},$$

and, therefore, W can be represented as the function of the column matrix

$$W\left\{\begin{pmatrix} t \\ \omega \end{pmatrix}\right\} \equiv W(t, \omega). \quad (17)$$

By using this matrix representation, modulator (10) and filter (16) WDFs can be written in the form

$$W_{\mathrm{M}}\left\{\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\} = W\left\{\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix}\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\}$$

$$= W\left\{M\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\}, \quad (18)$$

where

$$M \equiv \begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix},$$

$$W_{\mathrm{F}}\left\{\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\} = W\left\{\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\}$$

$$= W\left\{F\begin{pmatrix} t - t_0 \\ \omega - \omega_0 \end{pmatrix}\right\}, \quad (19)$$

where

$$F \equiv \begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}.$$

The quadratic phase modulator is described by the matrix M producing a linear frequency shift from the transform limited rectangular pulse (Fig. 3a) to the pulse with a linear frequency chirp (Fig. 3c). The quadratic phase filter is described by the matrix F producing a liner temporal shift from the transform limited rectangular pulse to the pulse with a linear time chirp (Fig. 3b).

5. Modulators and filters in temporal optics

Phase filters used in the temporal optics of USPs represent a layer of a dispersive medium of thickness z (or, for example, a compressor – a pair of parallel diffraction gratings separated by the distance z [23] or a stretcher – a device containing elements with dispersions of opposite signs [24]). In the quadratic approximation, a dispersive medium layer is described by expression (19) [1], the effective thickness l

of the filter being proportional to the real thickness z of the dispersive medium.

An example of the phase modulator is an electro-optic modulator varying the phase of a transmitted laser beam by the harmonic law [1]

$$\psi(t) = \Phi_\mathrm{m}\cos(\omega_\mathrm{m} t + \theta),$$

where Φ_m, ω_m, and θ are the amplitude, frequency, and initial phase of modulation. If the phase θ is zero and the pulse duration is much shorter than the modulation period $T_\mathrm{m} = 2\pi/\omega_\mathrm{m}$, by expanding the phase shift into a Taylor series in the vicinity of point $t = 0$ and restricting our consideration to the second-order terms, we obtain

$$\psi(t) \approx \Phi_\mathrm{m}\left(1 - \frac{\omega_\mathrm{m}^2 t^2}{2}\right).$$

In this approximation, the electro-optic modulator is described by expression (18) if $P \equiv 1/f$, where $f = 1/(\Phi_\mathrm{m}\omega_\mathrm{m}^2)$ is the 'focal time' of the modulator (temporal lens) [1].

6. Modulators and filters in spatial optics

According to the dual sense of the concept of spatial frequencies [25], the spatial frequency u determines the propagation direction of a diffracted light beam. Therefore, by replacing time t by the distance x between the light beam and optical axis, and the frequency ω by the spatial frequency u, we obtain from the above-considered column matrix

$$\begin{pmatrix} t \\ \omega \end{pmatrix}$$

the column matrix

$$\begin{pmatrix} x \\ u \end{pmatrix}$$

for a light beam.

Note that the transformation of this beam in a lens with the lens power $P \equiv 1/f$ (in the phase modulator) and a layer of free space of thickness l (phase filter) are described by the same matrices (18) and (19), respectively.

7. *ABCD* matrices of optical systems

By combining modulators and filters, we can synthesise a variety of different systems obtained by multiplying matrices M and F [7]. Each of such systems is described by the $ABCD$ matrix T:

$$T \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} = M_1 F_2 M_3 M_4 M_5 F_6 \dots .$$

Because matrices M and F are noncommutative it is important not to change their multiplication order. Because

$$\det\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix} = 1 \quad \text{and} \quad \det\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix} = 1,$$

we have

$$\det T \equiv \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC = 1.$$

Therefore, the matrix T describes a linear transformation of the WDF carrier, and this transformation does not change its volume.

We will classify temporal optical systems similarly to optical systems in geometrical optics [7], which are classified by equating different elements of the $ABCD$ matrix to zero. In this case, the sign rule is used: the effective length of the free space layer is assumed positive if the layer is located in front of the lens, and negative if the layer is located behind the lens. Consider this classification. Note that the primed quantities belong to the image space of the optical system.

(1) The case $A = 0$, i.e., $x' = Bu$ (Fig. 4a) in geometrical optics corresponds to the determination of the image focal plane and is realised in the cascade consisting of a modulator with the focal distance f and a filter of thickness l' located behind the modulator, when the condition

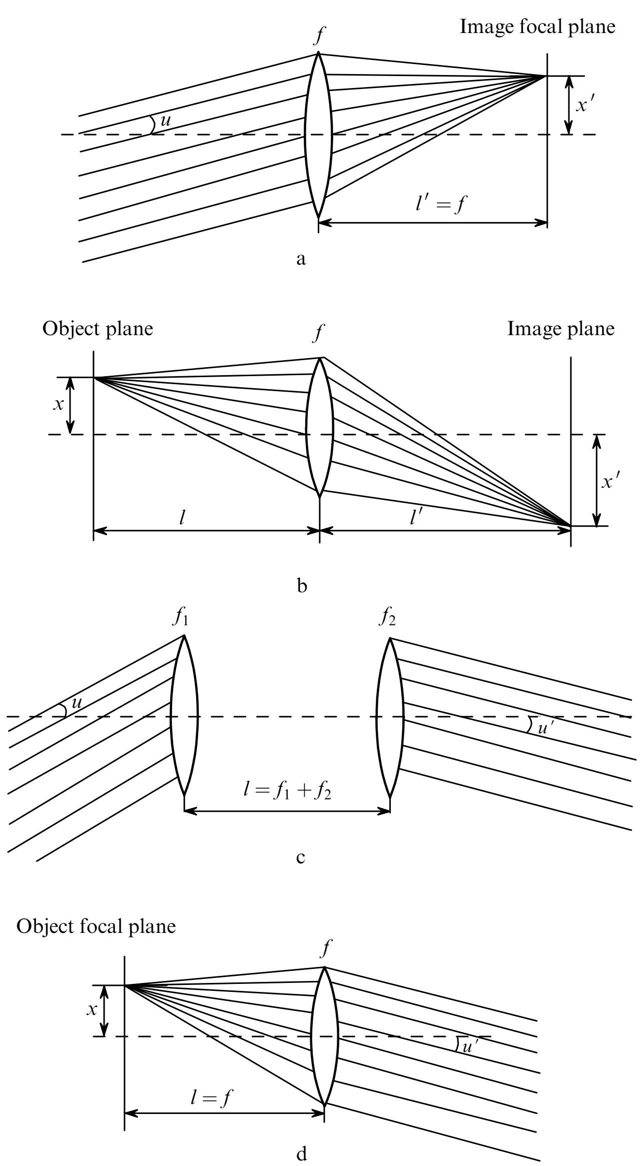

Figure 4. Classification of optical systems: determination of the image focal plane (a), optically conjugate planes (b), telescope (c), and object focal plane (d).

$$F'M = \begin{pmatrix} 1 & -l' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 - l'/f & -l' \\ 1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

is fulfilled. Therefore, $1 - l'/f = 0$, i.e., $l' = f$. If this condition is fulfilled, the initial matrix is simplified:

$$F'M = \begin{pmatrix} 0 & -f \\ 1/f & 1 \end{pmatrix}.$$

In temporal optics, such a system transforms the frequency modulation of a signal (wave form) to the temporal modulation [13, 14].

(2) The case $B = 0$, i.e., $x' = Ax$ (Fig. 4b) in geometrical optics corresponds to the linear transformation of coordinates in the optically conjugate planes and is realised in the cascade consisting of a filter of thickness l, a modulator with the focal distance f behind the filter, and then a filter of thickness l', when the condition

$$F'MF = \begin{pmatrix} 1 & -l' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - l'/f & l - l' + ll'/f \\ 1/f & 1 - l/f \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

is fulfilled, i.e., $l - l' + ll'/f = 0$. From this, we obtain the formula for a thin lens:

$$\frac{1}{l} - \frac{1}{l'} = \frac{1}{f}.$$

When this condition is fulfilled, the initial matrix of the system is simplified:

$$F'MF = \begin{pmatrix} \mu & 0 \\ 1/f & 1/\mu \end{pmatrix},$$

where $\mu \equiv l'/l$ is the temporal magnification of the system. In temporal optics, such a matrix describes a temporal microscope, i.e., the optical system changing the temporal scale of a signal (wave form) [1].

(3) The case $C = 0$, i.e., $u' = Du$ (Fig. 4c) in geometrical optics corresponds to the linear transformation of the slopes of the input and output parallel beams and is realised in the cascade consisting of a modulator with the focal distance f_1 a filter of thickness l, and a modulator with the focal distance f_2, when the condition

$$M_2FM_1 = \begin{pmatrix} 1 & 0 \\ 1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - l/f_1 & -l \\ (-l + f_1 + f_2)/(f_1f_2) & 1 - l/f_2 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

is fulfilled. Therefore, $(-l + f_1 + f_2)/(f_1f_2) = 0$, i.e., $l = f_1 + f_2$. When this condition is fulfilled, the matrix of the system is simplified:

$$M_2FM_1 = \begin{pmatrix} -1/\eta & f_1 + f_2 \\ 0 & -\eta \end{pmatrix},$$

where $\eta \equiv f_1/f_2$ is the spectral magnification of the system. In temporal optics, such a matrix describes a temporal telescope, i.e., an optical system changing the frequency scale of a signal (wave form) [26].

(4) The case $D = 0$, i.e., $u' = Cx$ (Fig. 4d) in geometrical optics corresponds to the determination of the object focal plane and is realised in a cascade consisting of a filter of thickness l and a modulator with the focal distance f located behind the filter, when the condition

$$MF = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -l \\ 1/f & 1 - l/f \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is fulfilled. Therefore, $1 - l/f = 0$, i.e., $f = l$. When this condition is fulfilled, the initial matrix of the system is simplified:

$$MF = \begin{pmatrix} 1 & -f \\ 1/f & 0 \end{pmatrix}.$$

In temporal optics, such a system transforms the temporal modulation of a signal (wave form) to the frequency modulation.

Consider two special cases. In geometrical optics, the $2f$ system for $l = l' = f$ corresponds to the Fourier converter described by the expression

$$FMF' = \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f \\ 1/f & 0 \end{pmatrix} = \begin{pmatrix} 0 & -f \\ 1/f & 0 \end{pmatrix}.$$

In temporal optics, this matrix is realised in the cascade, consisting of a filter of thickness f, a modulator with the focal distance f located behind the filter, and then a filter of the same thickness f, and describes the '$t - \omega$ converter' [27].

In geometrical optics, the $4f$ system – the cascade of two identical optical $2f$ systems, is described by the matrix

$$\begin{pmatrix} 0 & -f \\ 1/f & 0 \end{pmatrix}\begin{pmatrix} 0 & -f \\ 1/f & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In temporal optics, the corresponding system inverts a train of pulses both in time (replacing the 'past' by the 'future') and frequency [27].

8. Conclusions

It has been shown that ultrashort laser pulses and their transformations in optical systems with temporal lenses can be simply described by using the Wigner functions. These functions can describe the pulse transformation in the most general form. Such a description in the quadratic approximation is equivalent to the use of $ABCD$ matrices from geometrical optics. The methods of matrix optics allow one to classify and determine the main operating parameters of temporal optical systems such as temporal lenses, microscopes, telescopes, Fourier converters, etc.

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