PACS numbers: 42.55.Px; 42.60.Mi; 42.65.Sf
DOI: 10.1070/QE2007v037n02ABEH013213

Harmonic modulation of radiation of an external-feedback semiconductor laser

A.G. Sukharev, A.P. Napartovich

***Abstract*. The appearance of the harmonic modulation regime at the Hopf bifurcation point is described analytically for a delayed-feedback semiconductor laser. The second-order delay differential equation with complex coefficients is derived. The frequency of oscillations appearing at the Hopf bifurcation point is determined by the solution of two relatively simple transcendental equations, from which the bifurcation point itself is found. These equations contain dependences on all the control parameters of the problem. The exact upper and lower limits of the oscillation frequency are found. A comparison with numerical results shows that the modulation frequency is preserved almost constant in a broad range of feedback phases. A procedure is proposed for determining the parameters of the laser providing the presence of bifurcations with a passage to oscillations with the specified frequency. The results obtained in the paper are of interest for WDM communication systems.**

Keywords: *semiconductor laser, harmonic modulation of radiation, external feedback*.

1. Introduction

A delayed-feedback (DF) semiconductor laser is characterised by a variety of dynamic regimes [1], some of them being used in optical communication systems [2]. The complexity of these dynamic regimes is caused by the presence of two coupled resonators, of which one is formed by the reflecting end facets of a semiconductor crystal, and the other – by a combination of a highly reflecting end facet of the crystal and an external mirror. Lang and Kobayashi [3] derived dynamic equations for a laser diode with an additional external mirror by neglecting the reflection of incident radiation from the diode facet. The Lang – Kobayashi (LK) equations form the basis of modern theories describing the dynamics of DF laser diodes. The system of LK equations contains six control parameters, which results in a variety of dynamic regimes, of which chaotic regimes are of special interest. The fundamentals of the theory of external-feedback lasers and the results of recent studies (till the year 2000) are presented in [1].

The system of LK equations has stationary particular solutions in which the radiation intensity is independent of time and the rate of phase variation is determined by the frequency Ω found from the system itself. The number of stationary solutions depends on the control parameters and can be large. Many of these states prove to be unstable. The complication of the dynamics of a DF semiconductor laser with increasing number of stationary solutions was studied in our paper [4]. One of the results of this paper was the discovery of the regime of quasi-harmonic intensity self-oscillations. It was found that the frequency of these oscillations exceeds the relaxation oscillation frequency and can be quite high (it lies in the gigahertz frequency region).

Such regimes with the harmonic modulation of the radiation intensity can be used in WDM optical communication systems [2]. The power of a signal at the input to a channel is limited by nonlinear effects produced in the optical fibre. As a result, the length of an optical fibre between two neighbouring amplifiers is limited by 50 – 100 km. The harmonic modulation of the radiation power, which expands the spectrum of a signal, allows the use of input signals of higher powers, thereby increasing the distance between the amplifiers. Thus, semiconductor lasers operating in regimes of intensity self-oscillations in the gigahertz range are of interest for optical communication systems.

2. Basic equations

The LK equations describing an external-feedback laser diode in the dimensionless variables have the form [5, 6]

$$\frac{\partial E}{\partial t} = (1 - \mathrm{i}\alpha)NE(t) + ME(t-\tau)\exp[\mathrm{i}(\kappa + \pi/2)],$$
$$T\frac{\partial N}{\partial t} = P - N - (1 + 2N)|E|^2. \qquad (1)$$

The equations were derived by using the frequency dependence of the field in the form $\exp(-\mathrm{i}\omega_0 t)$; $\kappa + \pi/2 = \omega_0\tau(\mathrm{mod}\,2\pi)$. Here, α is the linewidth enhancement factor (antiwaveguide parameter) and τ is the delay time in the feedback loop. All the quantities having the dimensionality of time are divided by the photon lifetime τ_{ph} in the diode resonator of length L; $\tau_{\mathrm{ph}}^{-1} = (c/n)[\alpha_{\mathrm{int}} + (2L)^{-1}\ln r^{-1}]$; c/n is

A.G. Sukharev, A.P. Napartovich State Research Center of Russian Federation 'Troitsk Institute for Innovation and Fusion Research', 142190 Troitsk, Moscow region, Russia; e-mail: sure@triniti.ru, napart@mail.ru

Received 4 May 2006
Kvantovaya Elektronika **37** (2) 149 – 153 (2007)
Translated by M.N. Sapozhnikov

the group speed of light in a medium; $\alpha_{\rm int}$ are internal losses; r is the reflection coefficient of the crystal facet; and $T = \tau_s/\tau_{\rm ph}$ is the lifetime of carriers. The gain in the active medium slightly above the lasing threshold $G_{\rm th}$ can be approximately described by a linear function of the carrier density N_c as $G(N_c)/G_{\rm th} = 1 + g\tau_{\rm ph}(N_c - N_{\rm th})$, where g is the differential amplification in the medium. The rest of the dimensionless variables are introduced as follows: the field amplitude is $E = (\frac{1}{2}g\tau_s)^{1/2}E_{\rm ph}$ ($|E_{\rm ph}|^2$ is the photon density); the inverse population measured from the threshold is $N = \frac{1}{2}g\tau_{\rm ph}(N_c - N_{\rm th})$; the normalised pump intensity is $P = \frac{1}{2}g\tau_{\rm ph}(j\tau_s - N_{\rm th})$, where $j = J/ed$ is the rate of injection of carriers into the active layer of thickness d (j is measured in cm^{-3} s^{-1}) for the current density J; the coupling constant is $M = (1-r)(R/r)^{1/2}(c\tau_{\rm ph}/2nL)$, where R is the reflection coefficient of the external mirror or the far end of the fibre taking into account splice, absorption, and diffraction losses.

The stationary solution is obtained from the conditions $\partial N/\partial t = 0$, and $\partial E/\partial t = {\rm i}E\Omega$, where Ω is the detuning of the radiation frequency from ω_0. For this solution, $N/M = \sin(\kappa - \Omega\tau)$ and $P - N = (1+2N)|E|^2$, and the detuning Ω is obtained from the equation

$$\Omega/M(1+\alpha^2)^{1/2} = \sin[\Omega\tau - \kappa + \arctan(1/\alpha)]. \quad (2)$$

3. Regime of harmonic oscillations

Because initial equations (1) have a strict 2π periodicity as a function of the parameter κ, we will study the behaviour of solutions by varying κ within the interval [0, 2π]. Previous investigations [1, 4] have shown that the number of stationary solutions increases with increasing the effective feedback strength $s = M\tau(1+\alpha^2)^{1/2}$. The analysis of the stability of these solutions is considerably complicated by the presence of six parameters of the problem (α, M, τ, κ, P, T). The two parameters α and T depend on the material and design of a semiconductor laser, and we used their typical values $\alpha = 3$ and $T = 1000$. Although stationary solutions depend formally on time, the analysis of their stability with respect to small perturbations exponentially depending on time as $\exp(\lambda t)$ is reduced to the system of three linear equations for variations of the field amplitude and phase and population inversion [1]. Some elements of the matrix A of this system contain the dependence of the type $1 - \exp(-\lambda\tau)$ on the eigenvalue λ. The value of λ is obtained from the transcendental characteristic equation $\det(A - \lambda E) = 0$.

The number of its roots in the case of a small effective feedback strength is equal to three and increases with increasing s. The scheme proposed in [4] allows us to determine the number of λ roots falling to the right half of the complex plane, i.e. the number n of unstable solutions. As parameters are changed, the roots move in the complex plane. The instant of intersection of the imaginary axis by the exponent λ corresponds to the bifurcation point. For the set of parameters presented in Fig. 1, the stationary solution can be either stable ($n = 0$) or unstable ($n > 0$), depending on the combination of Ω and κ.

Let us put the eigenvalues in order of Reλ and study the phase portrait in the plane of a pair of eigenvectors senior in Reλ. The unstable stationary solutions correspond to the singularities of three types: the saddle point ($n = 1$), unstable focus ($n = 2$), and unstable (three-dimensional) knot–focus

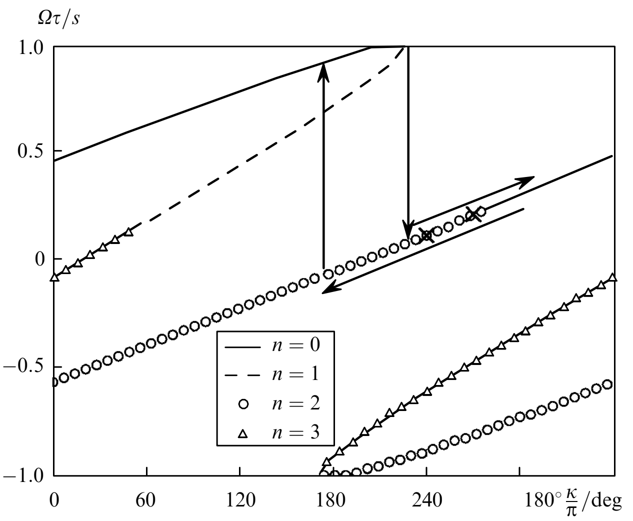

Figure 1. Stability diagram of stationary states for $P = 0.21$, $M = 0.02$, and $\tau = 80$; n is the number of roots of the characteristic equation with Re$\lambda > 0$ (crosses correspond to curves in Fig. 2).

($n = 3$). The bifurcation ($n = 0 \to 2$) is characterised by the simultaneous creation of the unstable focus and stable limit cycle. The winding of the phase trajectory on the limit cycle results in periodic intensity oscillations. The creation of the limit cycle is called the Hopf bifurcation of the system; in this case, the two complex conjugate roots simultaneously intersect the imaginary axis. As κ continuously decreases after the bifurcation point, the amplitude of oscillations increases if the calculation is started from the solution obtained for the previous value of κ. However, oscillations disappear near the point $\kappa = \pi$, which is indicated in Fig. 1 by the arrow directed upward, and a jump to the stationary solution occurs, which is preserved as κ further changes.

We are interested in the situation when oscillations of the field appear during the passage through the Hopf bifurcation. As a whole, the scenario of the development of self-oscillations is complicated, and under some conditions the multistability and hysteresis are observed (the arrows in Fig. 1 demonstrate the hysteresis found numerically for many-valued Ω by varying continuously κ). The solution has often the form of pulses or pulse packets, whose shape is far from harmonic. In [4], the conditions were found under which self-oscillations are close to harmonic oscillations at large enough delay times ($\tau = 80$).

For the parameters presented above, the Hopf bifurcation occurs in the vicinity of $\kappa = 3\pi/2$. Numerical calculations performed for $P = 0.2$, $M = 0.02$, and $\tau = 80$ show that the regime of harmonic oscillations of the field intensity with respect to the stationary level takes place in a broad range of feedback phases $1.05\pi < \kappa < 3\pi/2$ (Fig. 2). In the interval $\pi < \kappa < 1.28\pi$, both periodic and stationary solutions can be realised. The change of regimes occurs at the interval boundaries and is determined by the bypass direction (hysteresis). The change in κ after the Hopf bifurcation point almost does not affect the frequency of intensity oscillations, while the modulation remains harmonic up to the amplitude modulation ~ 1.

Near the bifurcation point ($\kappa < 3\pi/2$), equations (1) can be linearized in small deviations of solutions from stationary solution (2). Let us represent the field in the form $E = E_{\rm st}(1 + \Psi)$, where $E_{\rm st} = [P + M\sin(\Omega\tau - \kappa)]^{1/2}\exp({\rm i}\Omega t)$,

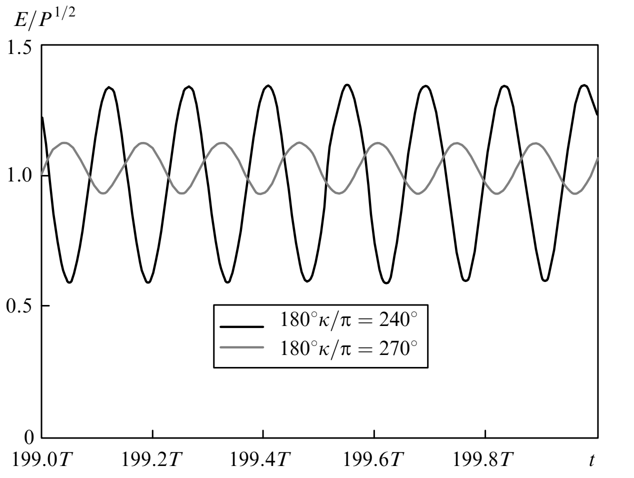

Figure 2. Field amplitude for $180°\kappa/\pi = 240°$ and $270°$ for $P = 0.2$, $M = 0.02$, and $\tau = 80$.

$|\Psi| \ll 1$ and $|N| \ll 1$. For a small perturbation Ψ, the delay differential equation can be derived:

$$-\ddot{\Psi} = M(\dot{\Psi} - \dot{\Psi}_\tau)\exp[-\mathrm{i}\Omega\tau + \mathrm{i}(\kappa + \pi/2)] \tag{3}$$

$$+ (1 - \mathrm{i}\alpha)\omega_\mathrm{r}^2 \mathrm{Re}\Psi,$$

where

$$\Psi_\tau = \Psi(t - \tau); \quad \omega_\mathrm{r}^2 = \frac{2}{T}\left[P + M\sin(\Omega\tau - \kappa)\right]. \tag{4}$$

Equation (3) was derived taking into account that the relaxation oscillation frequency in diode lasers is usually $\omega_\mathrm{r} \gg 1/T$. Equation (3) belongs to the class of differential delay equations. In addition, the appearance of $\mathrm{Re}\Psi$ in its right-hand side leads to the additional coupling between $\mathrm{Re}\Psi$ and $\mathrm{Im}\Psi$ (similar situation appears in the problem of a small-scale focusing [7]). In accordance with the definition of the Hopf bifurcation, the perturbation Ψ at the bifurcation point itself has the form of harmonic oscillations. The oscillation frequency ω should be found from the condition of existence of nontrivial solutions (3). A direct substitution of solutions in the form of exponentials with constant complex coefficients gives the system of transcendental equations of a rather complex structure.

To analyse this system, we study the group properties (see [8]) of the operator $L_\tau\psi = \mathrm{d}/\mathrm{d}t[\psi(t) - \psi(t - \tau)]$, entering the right-hand side of equation (3). It is easy to verify that the action of the operator on the function $\psi = \cos(\omega t)$ is reduced to the multiplication by the parameter $D = -2\omega\sin(\omega\tau/2)$ and the backward displacement of the argument of the initial function by $\tau/2$:

$$L_\tau\psi = -2\omega\sin(\omega\tau/2)\cos\omega(t - \tau/2) = D\psi_{\tau/2}.$$

This circumstance suggests that it is reasonable to seek the solution of (3) in the form of the linear combination

$$\Psi = c_1\cos(\omega t) + c_2\cos\omega(t - \tau/2). \tag{5}$$

The action of the operator L_τ on the second term in (5) produces a function of an argument delayed by τ. To express this function in terms of the functions introduced above, note the following. The action of the operator $L_{-\tau}$ changing the sign of the shift τ produces the function with the argument shifted forward by $\tau/2$: $L_{-\tau}\psi = -D\psi_{-\tau/2}$. The action of the difference of operators $(L_\tau - L_{-\tau})$ retains the argument of the harmonic function:

$$(L_\tau - L_{-\tau})\psi(t) = -2\omega\sin(\omega\tau)\psi(t) = B\psi(t). \tag{6}$$

By using this property, we can express the result of the action of the operator L_τ on $\cos\omega(t - \tau/2)$ in terms of functions introduced in (5): $L_\tau\psi_{\tau/2} = [L_{-\tau} + (L_\tau - L_{-\tau})] \times \psi_{\tau/2} = -D\psi + B\psi_{\tau/2}$.

Thus, operators in Eqn (3) acting on linear combination (5) retain the expansion form. Generally speaking, the coefficients c_1 and c_2 are complex numbers for which Eqn (3) is reduced to the homogeneous system of linear equations. Without loss of generality, the coefficient c_2 can be chosen purely imaginary ($c_2 = \mathrm{i}y$). By substituting the solution in form (5) into Eqn (3) and using the properties of the operator L_τ described above and the linear independence of functions $\cos(\omega t)$ and $\cos\omega(t - \tau/2)$, we obtain the homogeneous linear system of equations for the coefficients c_1 and y:

$$\omega^2 c_1 = M\exp(-\mathrm{i}\Omega\tau + \mathrm{i}\kappa)Dy + (1 - \mathrm{i}\alpha)\omega_\mathrm{r}^2\mathrm{Re}c_1, \tag{7}$$

$$\exp(\mathrm{i}\Omega\tau - \mathrm{i}\kappa)\omega^2 y = MDc_1 + \mathrm{i}MBy. \tag{8}$$

By equating real parts in Eqn (8), we find the relation between y and $\mathrm{Re}c_1$:

$$\omega^2 y\cos(\Omega\tau - \kappa) = MD\mathrm{Re}c_1. \tag{9}$$

Then, by substituting this expression for y into Eqn (7) and equating real parts, we find one of the conditions for the existence of the nontrivial solution of (3):

$$f_1(\omega) \equiv -\omega^2 + \omega_\mathrm{r}^2 + M^2D^2\omega^{-2}$$

$$= -\omega^2 + \omega_\mathrm{r}^2 + 4M^2\sin^2(\omega\tau/2) = 0. \tag{10}$$

The second condition, which follows from Eqns (7) and (8), contains the explicit dependence on α:

$$f_2(\omega) \equiv \omega^2 + [M\omega\sin(\omega\tau)\sin^{-1}(\Omega\tau - \kappa)]$$

$$-\frac{1}{2}\omega_\mathrm{r}^2[1 - \alpha\tan^{-1}(\Omega\tau - \kappa)] = 0. \tag{11}$$

Thus, according to (10), $\omega^2 = \omega_\mathrm{r}^2 + 4M^2\sin^2(\omega\tau/2)$. This relation gives exact upper and lower boundaries for the frequency of small harmonic oscillations ω: $\omega_\mathrm{r} \leqslant \omega \leqslant (\omega_\mathrm{r}^2 + 4M^2)^{1/2}$ for the specified value of ω_r.

Relations (10) and (11) determine the oscillation frequency and position of the Hopf bifurcation point. Functions f_1 and f_2 are constructed in Fig. 3 for $\kappa = 3\pi/2$, $P = 0.2$, $M = 0.02$ and $\tau = 80$. The functions f_1 and f_2 vanish at the bifurcation point; in this case, the frequency ω is 0.044, which is approximately equal to $2.2\omega_\mathrm{r}$. For the parameters selected and the photon lifetime in the diode resonator $\tau_\mathrm{ph} = 1$ ps, the oscillation frequency is 7 GHz. This coincides with the value obtained by the direct numerical integration of LK equations with the help of the program described in [4].

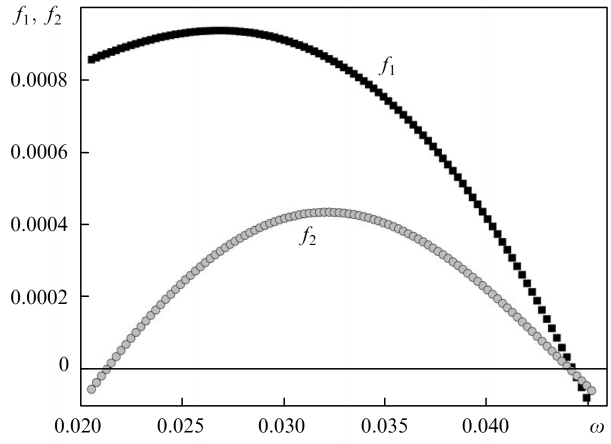

Figure 3. Plots of functions $f_1(\omega)$ and $f_2(\omega)$.

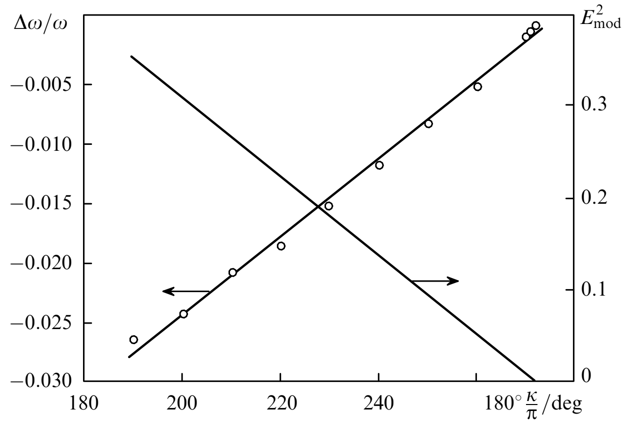

Figure 4. Plots of the functions of the frequency detuning $\Delta\omega/\omega$ and square of the field-modulation amplitude $E^2_{\rm mod}$ ($\kappa < 1.51\pi$).

As the pump intensity is decreased ($P < 0.2$), the functions f_1 and f_2 no longer vanish simultaneously at the point $\kappa = 3\pi/2$, i.e. the Hopf bifurcation is absent because the stationary solution $\Omega \approx 0.2M(1+\alpha^2)^{1/2}$ becomes stable. As the pump intensity P is increased, the region of instability with respect to the Hopf bifurcation (see the middle branch with $n = 2$ in Fig. 1) expands, and for $P = 0.2$ the boundary of this region intersects the point $\kappa = 3\pi/2$. The direct numerical integration of LK equations showed that the amplitude of intensity oscillations increased with further increasing the pump intensity, which is explained by the moving of the bifurcation point to the right (see Fig. 1). For $P = 0.2$, harmonic oscillations were preserved in a broad range of κ: $4\pi/3 < \kappa < 3\pi/2$ (see Fig. 2). The distortions of the harmonic modulation were small over the entire interval $1.05\pi < \kappa < 3\pi/2$ of the existence of oscillations.

The dependences of the frequency and square of the amplitude of field oscillations on the feedback phase obtained by the numerical integration of LK equations are presented in Fig. 4. One can see from Fig. 4 that the frequency decreases almost linearly with decreasing κ in the entire range of feedback phases, and its relative change does not exceed 2.7 %, whereas the square of the oscillation amplitude $E^2_{\rm mod}$ increases up to 0.36 [here, $E_{\rm mod} = (E_{\rm max} - E_{\rm min})/2$]. The dependences presented in Fig. 4 can be found analytically by taking into account nonlinear corrections to Eqn (3). Nonlinear corrections, as the anharmonicity of oscillations in classical mechanics, cause the shift of the oscillation frequency, which increases with increasing the oscillation amplitude. Taking into account the quadratic nonlinearity, the perturbation theory leads to the renormalisation of the relaxation frequency $\tilde{\omega}_{\rm r}^2 = \omega_{\rm r}^2(1-I)$, where $I = E^2_{\rm mod}$. In this case, the form of Eqn (3) is preserved, and only the substitution $\omega_{\rm r}^2 \to \tilde{\omega}_{\rm r}^2$ takes place. The same renormalisation $\omega_{\rm r}$ also occurs in Eqns (10) and (11) determining the oscillation eigenfrequency. For a small vicinity of the bifurcation point in Fig. 4, the expressions

$$\frac{1}{\omega}\frac{\Delta\omega}{\Delta I} \approx -\frac{\omega_{\rm r}^2}{2\omega^2}, \quad \frac{1}{\omega}\frac{\Delta\omega}{\Delta k} \approx \frac{\alpha}{4}\left(\frac{\omega}{2M} - \frac{2M}{\omega}\right)^2$$

can be derived for the rate of the frequency variation as functions of I and κ, respectively. The numerical estimate gives $I \approx -0.3\Delta\kappa$, whereas $\Delta\omega/\omega \approx 0.03\Delta\kappa$. Thus, the relative change in the frequency is indeed much smaller than the change in the modulation amplitude.

Equations (10) and (11), determining the position of the bifurcation point and the oscillation frequency, can be used to obtain explicitly the parameters of the laser providing the maximum oscillation frequency

$$\omega_i = \frac{\pi}{\tau}\left(2i+1\right) = \left(\omega_{\rm r}^2 + 4M^2\right)^{1/2},$$

where i is an integer. By specifying ω_i and M, we can determine the relaxation frequency. The phase entering the equation for the stationary frequency Ω can be found from the expression

$$\tan(\kappa - \Omega\tau) = \frac{\alpha\omega_{\rm r}^2}{2\omega_i^2 - \omega_{\rm r}^2} \equiv \chi.$$

By using the latter expression and (2), we can find the explicit expression for Ω:

$$\frac{\Omega\tau}{s} = \left(\chi - \frac{1}{\alpha}\right)\left\{(1+\chi^2)\left[1 + \left(\frac{1}{\alpha}\right)^2\right]\right\}^{-1/2}.$$

The pump intensity corresponding to the chosen values of ω_i and M can be found from expression (4) for $\omega_{\rm r}$. For example, for $i = 1$, $M = 0.035$, $\tau = 80$ ($s = 8.85$), and the photon lifetime 1 ps, we have $f_1 = \omega_1/2\pi\tau_{\rm ph} = 18.75$ GHz. This oscillation frequency is achieved for the pump intensity $P \approx 4.49$, which corresponds to the fourfold excess of the pump current over the threshold. The frequency of the stationary solution at the bifurcation point is $\Omega/2\pi\tau_{\rm ph} = 10.5$ GHz and $\kappa \approx 180°$. The numerical calculations confirmed the presence of the bifurcation point and solution oscillating at the frequency 19 GHz. The above-described method for finding bifurcation points with the high oscillation frequency of the obtained solutions should be accompanied by a direct numerical calculation because we are interested only in bifurcations from a stable stationary solution. The expressions derived above also describe the cases when bifurcation occurs from the unstable (i.e. non-realised) stationary solution.

4. Conclusions

By analysing the operation of an external-feedback semiconductor laser in the approximation of Lang – Kobayashi equations, we have described the oscillatory solution appearing at the Hopf bifurcation point when stationary solutions become unstable. The delay differential equation describing the harmonic modulation of the laser radiation power has been derived and the method for its solution has been proposed. A relatively simple transcendental equation has been obtained from this equation, which gives the upper and lower boundaries of the oscillation frequency for the specified parameters of the laser. The procedure has been proposed for determining the parameters of the laser providing the presence of the bifurcation passing to oscillations at the specified frequency. In particular, for the photon lifetime in the diode resonator $\tau_{ph} = 1$ ps, the parameters of the laser are found and its operation regime is indicated in which oscillations occur at a frequency of 19 GHz, which can be of interest for WDM optical communication systems.

References

1. Verduyn Lunel S.M., Krauskopf B., in *'Fundamental Issues of Nonlinear Laser Dynamics'*. Ed. by B. Krauskopf, D. Lenstra (Melville, NY: AIP Conf. Proc., 2000) Vol. 548, p. 66.
2. Ubaidullaev R.R. *Volokonno-opticheskie seti* (Fibreoptic Networks) (Moscow: Eko-Trends, 2001).
3. Lang R., Kobayashi K. *IEEE J. Quantum Electron.*, **16**, 347 (1980).
4. Napartovich A.P., Sukharev A.G. *Kvantovaya Elektron.*, **34**, 630 (2004) [*Quantum Electron.*, **34**, 630 (2004)].
5. Winful H.G., Rahman L. *Phys. Rev. Lett.*, **65** (13), 1575 (1990).
6. Winful H.G., Wang S.S. *Appl. Phys. Lett.*, **53** (20), 1894 (1988).
7. Bespalov V.I., Talanov V.I. *Pis'ma Zh. Eksp. Teor. Fiz.*, **3**, 471 (1966).
8. Olver P.J. *Applications of Lie Groups to Differential Equations* (New York: Springer Verlag, 1993; Moscow: Mir, 1989).