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Optical systems for measuring the Wigner function of a laser beam by the method of phase-spatial tomography

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Abstract. **The geometrical and wave approaches to the transformation of a family of the illumination distributions, measured at different sections of a laser beam focused by an optical system, to a family of different-aspect projections of the Wigner function of this beam required for the reconstruction of this function by the method of computer tomography are considered from a unified point of view by using the mathematical apparatus of matrix optics.**

Keywords: *Wigner function, phase-spatial tomography, fractional linear Fourier transform, matrix optics.*

1. Introduction

The Wigner function has been used in quantum mechanics beginning from the 1930s. In 1968 Walter began to use the Wigner function in optics and showed that the two division of optics, which had been considered before independent – the theory of the partial spatial coherence and radiometry of non-Lambert sources, have in fact the general subject for investigations, namely, the radiation energy transfer from sources with the non-Lambert directivity diagram, in particular, from laser sources [1]. The Wigner function is used in modern international standards to describe the energy structure of a laser beam because only this function can provide a unified description of experimentally controllable beam parameters such as its diameter, divergence, M^2 factor, etc. [2 – 4].

The photosensitive area of an array detector (for example, a charge-coupled device (CCD) [5]) is not angle-selective and detects only the radiation intensity distribution in the laser beam cross section where it is placed. Therefore, such a photodetector cannot be used to measure directly the Wigner function of a laser beam specified in the coordinate – spatial frequency phase plane, where the coordinate corresponds to the coordinate of the cross section plane. The illumination distribution measured with a detector can be treated as the orthogonal projection of the Wigner function on the coordinate axis.

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There exists a special class of optical systems in which the Wigner function of a propagated laser beam turns in the phase plane through some angle φ depending on the optical system parameters [6]. Therefore, a photodetector whose photosensitive area is placed in the output plane of such an optical system will measure the projection of the Wigner function specified in the input plane of the optical system with the projection (aspect) angle φ.

In the simplest one-dimensional case of the meridional section of a laser beam, the Wigner function is a three-dimensional object specified in the phase plane at the optical system input, while the intensity distribution detected with a detector at the optical system output is its two-dimensional projection with the aspect angle φ. A three-dimensional object can be reconstructed from a family of its different-aspect two-dimensional projections by the method of computer tomography [7]. A variant of computer tomography in which the required Wigner function of a laser beam in the input plane of the optical system is calculated from a family of illumination distributions measured in the output plane of the optical system by varying its parameters is called the method of phase-spatial tomography [8 – 10]. The specific feature of this method is that the mechanical rotation of an object in the real space is replaced by the optical rotation of the Wigner function in the imaginary phase space.

However, to create an optical system allowing the rotation of the Wigner function in the phase space, it is necessary to have a variable-focus lens [6]. The role of such a lens is fulfilled by a multilens objective or a deformable mirror. The use of such nonstandard optical elements leads to a considerably increase in the cost of the device and, which is most important, to the distortion of the laser beam structure. For this reason, simpler optical systems with usual lenses are employed in practical studies. The illumination distribution in the output plane of such systems can be recalculated to the projection of the required Wigner function of a laser beam [8 – 10]. Calculations of this type can be performed by using two alternative approaches. The first one is based on wave optics and uses the mathematical apparatus of Fourier optics [8, 9], while the second one is based on geometrical optics and employs the mathematical apparatus of matrix optics [10].

The aim of this paper is to consider from a unified point of view the known geometrical and wave approaches to the calculation of optical systems intended for practical realisation of the method of phase-spatial tomography by using the mathematical apparatus of matrix optics.

2. Wigner function of a laser beam

We will restrict our consideration to the scalar quasi-monochromatic approximation, by neglecting polarisation effects and the spectrum of radiation, and will study for simplicity the one-dimensional case corresponding to the meridional cross section of a laser beam in an optical system consisting of cylindrical lenses with parallel generatrices. Consider a scalar quasi-monochromatic one-dimensional light wave propagating along the z axis. The wave field specified in the orthogonal cross section z can be described by the complex amplitude $U(x)$ or complex spectrum $\tilde{U}(u)$, where x and u are the spatial coordinate and frequency, respectively. These descriptions are equivalent:

$$\tilde{U}(u) = \mathrm{F}_{x\to u}\{U(x)\}, \quad U(x) = \mathrm{F}^{-1}_{u\to x}\{\tilde{U}(u)\}, \tag{1}$$

because they are related by the Fourier transform

$$\mathrm{F}_{x\to u}\{\ldots\} \equiv \int_{-\infty}^{\infty}\{\ldots\}\exp(-\mathrm{i}ux)\mathrm{d}x,$$

$$\mathrm{F}^{-1}_{u\to x}\{\ldots\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\ldots\}\exp(\mathrm{i}ux)\mathrm{d}u.$$

The spatial coherence of this light wave is characterised by two-point correlation functions – the mutual intensity

$$\Gamma(x_1, x_2) \equiv \langle U(x_1)U^*(x_2)\rangle \tag{2a}$$

or the mutual spectrum

$$\tilde{\Gamma}(u_1, u_2) \equiv \langle \tilde{U}(u_1)\tilde{U}^*(u_2)\rangle. \tag{2b}$$

Here, the angle brackets denote averaging over an ensemble and the asterisk denotes complex conjugation.

By introducing mean variables $x = (x_1 + x_2)/2$, $u = (u_1 + u_2)/2$ and difference variables $\zeta = x_2 - x_1$, $\xi = u_2 - u_1$ and using the Fourier transform from difference variables, it is easy to obtain from correlation functions (2a) and (2b) the mathematically equivalent to them but more convenient characteristic of the spatial coherence of radiation – the Wigner distribution function [1, 11, 12]:

$$W(x, u) \equiv \mathrm{F}_{\zeta\to u}\left\{\Gamma\left(x + \frac{\zeta}{2}, x - \frac{\zeta}{2}\right)\right\}$$

$$= \mathrm{F}^{-1}_{\xi\to x}\left\{\tilde{\Gamma}\left(u + \frac{\xi}{2}, u - \frac{\xi}{2}\right)\right\}. \tag{3}$$

In particular, by combining expressions (1)–(3), we obtain the Wigner function in the explicit form in the coordinate representation:

$$W(x, u) \equiv \int_{-\infty}^{\infty} U\left(x + \frac{\zeta}{2}\right)U^*\left(x - \frac{\zeta}{2}\right)\Big\rangle \exp(-\mathrm{i}u\zeta)\mathrm{d}\zeta. \tag{4}$$

The Wigner function has many useful properties. By integrating this function with respect to the coordinate x and spatial frequency u, we obtain the radiation flux Φ

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x, u)\mathrm{d}x\mathrm{d}u = \Phi,$$

by integrating it with respect to the coordinate x, we obtain the radiation intensity distribution $I(u)$ over spatial frequencies:

$$\int_{-\infty}^{\infty} W(x, u)\mathrm{d}x = \left\langle \left|\tilde{U}(u)\right|^2 \right\rangle = I(u), \tag{5a}$$

and by integrating this function with respect to the spatial frequency u, we obtain the illumination distribution $E(x)$:

$$\int_{-\infty}^{\infty} W(x, u)\mathrm{d}u = \langle |U(x)|^2 \rangle = E(x). \tag{5b}$$

It is known that Wigner function (4) can take negative values, but illumination distribution (5b) is always a nonnegative function.

3. Projection of the Wigner function with the aspect angle φ and the fractional Fourier transform

By choosing the system of units in which the wavelength λ of monochromatic radiation is fixed and equal to 2π, for example, we pass to the dimensionless coordinates x and u. Consider the Cartesian coordinate system $x_\varphi u_\varphi$ in the dimensionless phase space, which is turned through an arbitrary angle φ in the counterclockwise direction with respect to the initial coordinate system xu. These coordinate systems are related by the transformation

$$x_\varphi = x\cos\varphi + u\sin\varphi, \quad u_\varphi = -x\sin\varphi + u\cos\varphi, \tag{6a}$$

or by the transformation

$$\begin{pmatrix} x_\varphi \\ u_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} \tag{6b}$$

in the matrix form.

The orthogonal projection $P_\varphi(x_\varphi)$ of the Wigner function $W(x, u)$ with the aspect angle φ is defined by the expression (Fig. 1a)

$$P_\varphi(x_\varphi) = \int_{-\infty}^{\infty} W(x_\varphi\cos\varphi - u_\varphi\sin\varphi, x\sin\varphi + u_\varphi\cos\varphi)\mathrm{d}u_\varphi. \tag{7}$$

Note that the projection (7) of the Wigner function with the aspect angle $\varphi = 0$ corresponds to the illumination distribution and the square of the complex amplitude (5b), while the projection with the aspect angle $\varphi = 90^\circ$ – to the radiation intensity distribution over spatial frequencies and the square of the complex spectrum (5a) (Fig. 1c):

$$P_0(x_0) = E(x) = \langle U(x)U^*(x)\rangle,$$

$$P_{90^\circ}(x_{90^\circ}) = I(u) = \langle \tilde{U}(u)\tilde{U}^*(u)\rangle.$$

The projection of Wigner function (7) with an arbitrary aspect angle φ can be also written as the square of the corresponding amplitude, i.e.,

$$P_\varphi(x_\varphi) = \langle |U_\varphi(x_\varphi)|^2 \rangle = \langle |U_\varphi(x_\varphi)U^*_\varphi(x_\varphi)|\rangle, \tag{8}$$

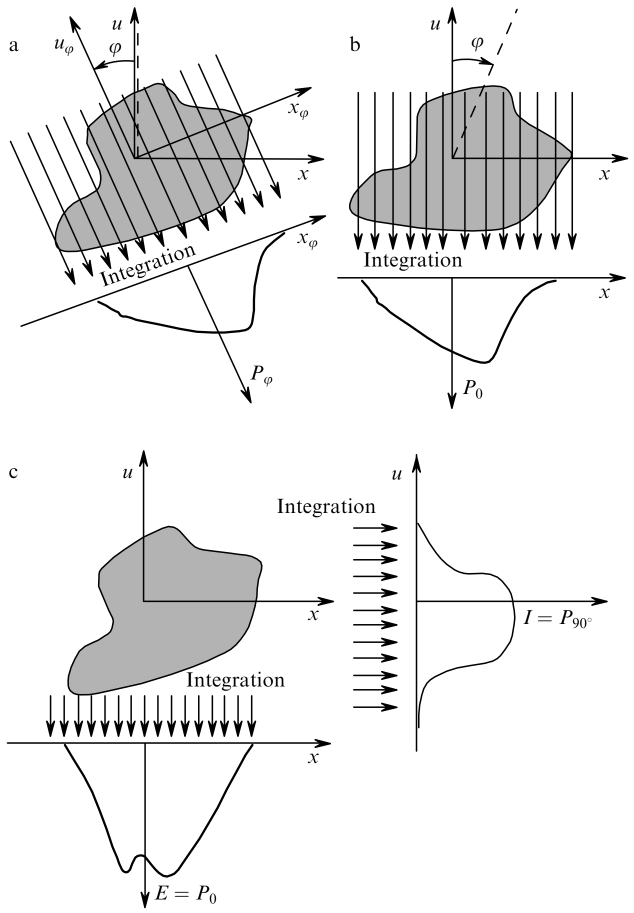

Figure 1. Projection of the Wigner function with the aspect angle φ on an arbitrarily oriented axis x_φ (a); projection of the Wigner function rotated through the angle $-\varphi$ on the coordinate axis x (b); and projection of the Wigner function on the coordinate axes x and u (c). The projection of the Wigner function with the aspect angle φ on an arbitrarily oriented axis x_φ (a) is equivalent to the projection of the Wigner function rotated through the angle $-\varphi$ on the coordinate axis x (b).

where $U_\varphi(x_\varphi)$ is an intermediate amplitude between the initial amplitude $U(x)$ and its Fourier transform $\tilde{U}(u)$, which is called the fractional Fourier transform of degree $m = \varphi/90\,^\circ$[6]. The initial signal $U(x)$ has the degree $m = 0$, the 'inverted' signal $U(-x)$ has the degree $m = 2$, while traditional Fourier transform (1) of this signal has the degree $m = 1$.

Let us find the explicit form of the fractional Fourier transform of degree m. For this purpose, we express the spatial frequencies u and u_φ in (6a) in terms of coordinates x and x_φ:

$$u = \frac{x_\varphi}{\sin\varphi} - \frac{x}{\tan\varphi},$$

$$u_\varphi = \left(\frac{x_\varphi}{\sin\varphi} - \frac{x}{\tan\varphi}\right)\cos\varphi - x\sin\varphi = \frac{x_\varphi}{\tan\varphi} - \frac{x}{\sin\varphi}.$$

By using these relations and making the change of variables in (7), we obtain

$$P_\varphi(x_\varphi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(x + \frac{\zeta}{2}\right)U^*\left(x - \frac{\zeta}{2}\right)\times$$

$$\times\exp\left[-\mathrm{i}\zeta\left(\frac{x_\varphi}{\sin\varphi} - \frac{x}{\tan\varphi}\right)\right]\mathrm{d}\left(-\frac{x}{\sin\varphi}\right)\mathrm{d}\zeta. \tag{9}$$

By using the variables $x = (x_1 + x_2)/2$, $\zeta = x_2 - x_1$, and the relation

$$\mathrm{d}x\mathrm{d}\zeta = \mathrm{d}\left(\frac{x_1 + x_2}{2}\right)\mathrm{d}(x_2 - x_1)$$

$$= \det\begin{vmatrix}\partial x/\partial x_2 & \partial\zeta/\partial x_2\\ \partial x/\partial x_1 & \partial\zeta/\partial x_1\end{vmatrix}\mathrm{d}x_2\mathrm{d}x_1$$

$$= \det\begin{vmatrix}1/2 & 1/2\\ 1 & -1\end{vmatrix}\mathrm{d}x_2\mathrm{d}x_1 = -\mathrm{d}x_2\mathrm{d}x_1,$$

we reduce (9) to the expression with separable variables

$$P_\varphi(x_\varphi) = \frac{1}{\sin\varphi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_1)U^*(x_2)$$

$$\times\exp\left[-\mathrm{i}\left(\frac{x_2x_\varphi}{\sin\varphi} - \frac{x_1x_\varphi}{\sin\varphi} - \frac{x_2^2 - x_1^2}{2\tan\varphi}\right)\right]\mathrm{d}x_2\mathrm{d}x_1,$$

which allows us to represent the required projection as the product of two integrals

$$P_\varphi(x_\varphi) = \frac{1}{\sin\varphi}\int_{-\infty}^{\infty} U(x_2)\exp\left[-\mathrm{i}\left(\frac{x_2x_\varphi}{\sin\varphi} - \frac{x_2^2}{2\tan\varphi}\right)\right]\mathrm{d}x_2$$

$$\times\int_{-\infty}^{\infty} U^*(x_1)\exp\left[\mathrm{i}\left(\frac{x_1x_\varphi}{\sin\varphi} - \frac{x_1^2}{2\tan\varphi}\right)\right]\mathrm{d}x_1.$$

By comparing this relation with (8), we see that the required fractional Fourier transform of degree $m = \varphi/90^\circ$ has the form [6, 8, 9, 13]

$$U_\varphi(x_\varphi) = \frac{1}{(\sin\varphi)^{1/2}}\int_{-\infty}^{\infty} U(x)$$

$$\times\int_{-\infty}^{\infty} U^*(x_1)\exp\left[-\mathrm{i}\left(\frac{xx_\varphi}{\sin\varphi} - \frac{x^2}{2\tan\varphi}\right)\right]\mathrm{d}x. \tag{10}$$

4. Phase brightness and the optical φ system

If a plane monochromatic wave with the wavelength λ in vacuum is normally incident on a sinusoidal diffraction grating with period d, the diffracted wave is deflected by the angle θ (Fig. 2). In this case, the projection $k_\perp = kn\sin\theta$ of the wave vector $\boldsymbol{k}$ of the diffracted wave on the diffraction grating plane in an optically homogeneous medium with the refractive index n is equal to the spatial frequency of this grating $u = 2\pi/d$ [14]:

$$u = k_\perp = kp, \tag{11}$$

where $p = n\sin\theta$ is the 'momentum' (optical unit vector) and $k \equiv |\boldsymbol{k}| = 2\pi/\lambda$ is the wave number in vacuum. When the condition $k > u$ is satisfied, the 'momentum' p of the diffracted wave is proportional to the spatial frequency u of the diffraction gating. This proportionality, characterising the 'double meaning of spatial frequencies' [14], allows one to pass easily from the concepts of wave optics to those of Hamiltonian optics and radiometry.

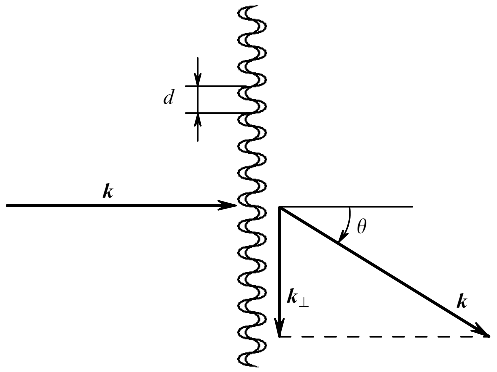

Figure 2. Scheme illustrating the double meaning of spatial frequencies.

In the one-dimensional Hamiltonian radiometry, the radiation flux (power) Φ from a source is distributed over the coordinate x and 'momentum' $p \in [-n, n]$. The distribution of the radiation flux Φ in the phase plane xp in the cross section z is described by the phase brightness [15]

$$d^2\Phi = \mathscr{B}(x,p)\mathrm{d}x\mathrm{d}p. \tag{12}$$

In the general case, taking into account relation (11), the phase brightness $\mathscr{B}(x,p)$ in (12) is related to the Wigner function $W(x,u)$ (3) by the similarity transformation

$$\mathscr{B}(x,p) = kW(x,kp)\chi_n(p),$$

where

$$\chi_n(p) \equiv \begin{cases} 1 & \text{for } |p| \leqslant n, \\ 0 & \text{for } |p| > n. \end{cases}$$

It is convenient to normalise the wavelength λ so that $k = 1$; then, the phase brightness $\mathscr{B}(x,p)$ is proportional to the Wigner function $W(x,p)$:

$$\mathscr{B}(x,p) = W(x,p)\chi_n(p), \tag{13}$$

which allows the use of the phase brightness in experimental studies instead of the Wigner function.

By describing the transformation of a pencil of light rays in an optical system by the methods of matrix optics, it is convenient to represent its phase brightness (12) as a function of the column matrix

$$\mathscr{B}\left\{\begin{pmatrix} x \\ p \end{pmatrix}\right\} \equiv \mathscr{B}(x,p). \tag{14}$$

The basic mathematical model of a pencil of rays in geometrical optics is a homocentric or parallel pencil of light rays, which represent lines in the phase plane. Because such lines have no area in the phase space, the homocentric or parallel pencil of light rays cannot transfer energy from the radiometric point of view. In radiometry, the radiation energy transfer can be calculated by using a convenient mathematical model of the simplest uniformly diverging (transform-limited) pencil of light rays described by phase brightness (14) in the form [13, 16]

$$\mathscr{B}\left\{\begin{pmatrix} x - x_0 \\ p - p_0 \end{pmatrix}\right\} = \mathrm{rect}\left(\frac{x - x_0}{\sigma_x}\right)\mathrm{rect}\left(\frac{p - p_0}{\sigma_p}\right), \tag{15}$$

where

$$\mathrm{rect}x \equiv \begin{cases} 1 & \text{for } |x| < 1/2, \\ 1/2 & \text{for } |x| = 1/2, \\ 0 & \text{for } |x| > 1/2 \end{cases}$$

is the rectangular function. The parameters σ_x and σ_p (root-mean-square deviations) characterise the width of the pencil, respectively. Such a uniformly diverging pencil of light rays is formed, for example, in the symmetry plane of the waist of a focused laser beam. The beam converges in front of this plane and diverges behind it.

Note that these pencils of light rays for $\sigma_x \to 0$ are continuously transformed to homocentric pencils, while a uniformly diverging (converging) pencil is transformed for $\sigma_p \to 0$ to a parallel pencil.

It is known that the position and orientation of a light ray in the meridional plane of an optical system can be characterised by a point in the phase plane xp. It is convenient to describe the transformation of the light ray in the optical system by the matrix method. A light ray is described in matrix optics [16, 17] by the column matrix, and its transformation in the optical system is described by the $ABCD$ matrix

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}. \tag{16}$$

where $AD - BC = 1$. Hereafter, the primed variables correspond to the output reference plane of the optical system (image plane), while variables without primes – to the input reference plane (object plane).

Any optical $ABCD$ system can be represented as a cascade of lenses (modulators – linear systems invariant in the frequency domain) and air gaps (filters – linear systems invariant in the frequency domain) [16].

The transformation of the phase brightness after passage through a modulator (lens with the focal distance f) is described in the quadratic approximation by the expression

$$\mathscr{B}_M\left\{\begin{pmatrix} x' \\ p' \end{pmatrix}\right\} = \mathscr{B}\left\{\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\right\} = \mathscr{B}\left\{M\begin{pmatrix} x' \\ p' \end{pmatrix}\right\},$$

i.e the matrix

$$M \equiv \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

produces a linear shift in the spatial frequency (Fig. 3c). The transformation of the phase brightness after passage through a filter (layer of an optically homogeneous medium with the refractive index $n = 1$ and thickness l) is described in the quadratic approximation by the expression

$$\mathscr{B}_F\left\{\begin{pmatrix} x' \\ p' \end{pmatrix}\right\} = \mathscr{B}\left\{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ p' \end{pmatrix}\right\} = \mathscr{B}\left\{F\begin{pmatrix} x' \\ p' \end{pmatrix}\right\},$$

i.e. the matrix

$$F \equiv \begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}$$

causes a linear displacement over the coordinate (Fig. 3b).

The author of [6] proposed to create a new class of optical systems rotating the phase brightness distribution of

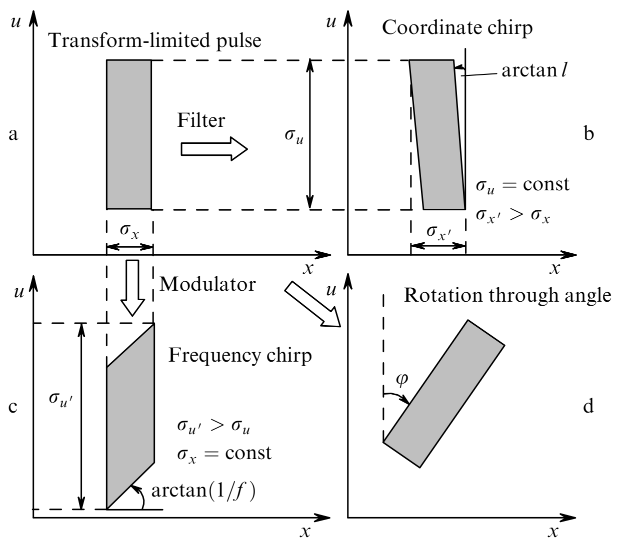

Figure 3. Rectangular signal (a), its transformation by a quadratic filter (b) and a quadratic modulator (c), and the rotation of the signal through the angle φ in the phase plane (d).

a laser beam through the specified angle φ [see (6b)] in the phase plane (Fig. 3d) upon its transfer from the input reference plane to the output reference plane, which is described by the expression

$$\mathscr{B}_{\varphi}'\left\{\begin{pmatrix} x' \\ p' \end{pmatrix}\right\} = \mathscr{B}\left\{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x_{\varphi} \\ p_{\varphi} \end{pmatrix}\right\}. \quad (17)$$

For example [6, 13], such a system can be created by using a cascade containing a free-space layer (filter) of thickness l, a lens (modulator) with the focal distance f, and another free-space layer (filter) of the same thickness (Fig. 4a). This system can be described by the expression

$$FMF = \begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - l/f & -l(2 - l/f) \\ 1/f & 1 - l/f \end{pmatrix},$$

or, in a more symmetric form, by the expression

$$FMF = \begin{pmatrix} 1 - l/f & -f[1 - (1 - l/f)][1 + (1 - l/f)] \\ 1/f & 1 - l/f \end{pmatrix}. \quad (18)$$

Indeed, by assuming that $1 - l/f = \cos\varphi$ and $1/f = \sin\varphi$, expression (18) is simplified:

$$FMF = \begin{pmatrix} \cos\varphi & -f(1 - \cos^2\varphi) \\ 1/f & \cos\varphi \end{pmatrix},$$

and is transformed to the required rotation matrix [see expression (6b)]:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

If $l = 2f$, then the $ABCD$ matrix of optical φ system (18) is transformed to the matrix of an ideal optical system with the magnification -1:

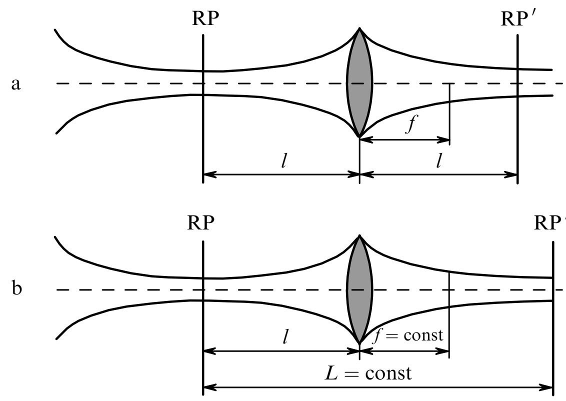

Figure 4. Optical systems for rotating the Wigner function in the phase plane (a) and performing the fractional Fourier transform (b); RP and RP′ are the input and output reference planes, respectively.

$$FMF = \begin{pmatrix} -1 & 0 \\ 1/f & -1 \end{pmatrix},$$

i.e. the image of the phase brightness turns through the angle $\varphi = 180°$ in the phase plane. If the input plane of the optical φ system is located in the object focal plane of the lens ($l = f$), then the $ABCD$ matrix of optical φ system (18) is transformed to the Fourier-converter matrix

$$FMF = \begin{pmatrix} 0 & -f \\ 1/f & 0 \end{pmatrix},$$

which rotates the phase brightness through the angle $\varphi = 90°$.

Because the photosensitive area of an array photodetector is not angular-selective and integrates radiation incident on it at different angles θ, such a photodetector with the photosensitive area located in the output reference plane of the optical φ system measures directly the illumination distribution $E_{\varphi}'(x')$ in this section related to the phase brightness distribution rotated clockwise through the angle φ by a simple expression

$$E_{\varphi}'(x') = \int_{-n}^{n} \mathscr{B}_{\varphi}'\left\{\begin{pmatrix} x' \\ p' \end{pmatrix}\right\} dp'. \quad (19)$$

Because rotation is a relative motion, the projection of initial phase brightness (19) with the aspect angle φ read off counterclockwise is equal to the illumination distribution measured with an array photodetector in the output reference plane of the optical φ system, but the angle φ should be read off clockwise in this case [13] (Fig. 1b):

$$E_{\varphi}'(x') = P_{\varphi}(x_{\varphi}).$$

The practical realisation of this method for measuring the projection of the phase brightness with the aspect angle φ is prevented by the necessity to change the focal distance of a lens used in the system. This requires the use of a multilens objective with a variable focal distance instead of a simple lens, which complicates the system and increases its cost.

Note that a single projection cannot provide the unambiguous reconstruction of an object; for example, contours of the axial projections of a cylinder and sphere of the same radius are indiscernible. This ambiguity allows one to obtain the illumination distribution in the plane of the photosensitive area of a photodetector, which is proportional to the required projection $P_\varphi(x_\varphi)$, with the help of a usual lens with a constant focal distance [8–10]. Taking into account that the obtained solution is constructive, we consider two its proofs in the contexts of geometrical and wave optics.

4.1 Proof in the context of geometrical optics

Let us construct an ideal optical system that produces the illumination distribution in the output reference plane, which is proportional to the illumination distribution in the input reference plane, but with the coordinate system rotated through the angle φ. The matrix of such an optical system can be obtained by combining the $ABCD$ matrix from (16) and the rotation matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

of the coordinate system:

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_\varphi \\ p_\varphi \end{pmatrix}$$

$$= \begin{pmatrix} A\cos\varphi + b\sin\varphi & -A\sin\varphi + B\cos\varphi \\ C\cos\varphi + D\sin\varphi & -C\sin\varphi + D\cos\varphi \end{pmatrix} \begin{pmatrix} x_\varphi \\ p_\varphi \end{pmatrix}. \quad (20)$$

For the illumination distributions in the input $(x_\varphi p_\varphi)$ and output $(x'p')$ reference planes of this optical system to be proportional, the right upper element of the matrix should be equal to zero [15, 16]:

$$-A\sin\varphi + B\cos\varphi = 0.$$

This means that the rotation angle φ is related to the elements of the $ABCD$ matrix of the optical system focusing the laser beam by the expression $\tan\varphi = B/A$. In this case, matrix (18) takes the form of the matrix of an ideal optical system with the magnification $(A^2+B^2)^{1/2}$:

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} (A^2+B^2)^{1/2} & 0 \\ \dfrac{AC+BD}{(A^2+B^2)^{1/2}} & \dfrac{1}{(A^2+B^2)^{1/2}} \end{pmatrix} \begin{pmatrix} x_\varphi \\ p_\varphi \end{pmatrix}. \quad (21)$$

It is important that the two-dimensional projections required for tomography can be measured by using the optical scheme (Fig. 4b) with a usual lens, while the scaling factor $(A^2+B^2)^{1/2}$ can be easily taken into account in the preliminary mathematical data processing because the illumination distribution $E'_\varphi(x')$ experimentally measured in the output plane of the optical system is related to the required projection $P_\varphi(x_\varphi)$ of the Wigner function by the similarity relation

$$P_\varphi(x_\varphi) = (A^2+B^2)^{1/2} E'_\varphi\big(x_\varphi(A^2+B^2)^{1/2}\big). \quad (22)$$

4.2 Proof in the context of wave optics

It was shown in [17, 18] that the $ABCD$ matrix of an optical system unambiguously determines its point eikonal – the path length of a light beam coupling the point x in the input reference plane of this system with the point x' in the output reference plane:

$$S(x,x') = S_0 - \frac{Ax^2 + Dx'^2 - 2xx'}{2B}, \quad (23)$$

where S_0 is the optical path along the optical axis of the system between the input and output reference planes. It is known [19] that the point eikonal determines the form of the kernel of the integral transformation relating the distributions of the wave-field amplitude in the input $[U(x)]$ and output $[U'(x')]$ reference planes of the optical system:

$$U'(x') = \left(\frac{\partial^2 S}{\partial x \partial x'}\right)^{1/2} \int_{-\infty}^{\infty} U(x)\exp[ikS(x,x')]\mathrm{d}x. \quad (24)$$

By combining expressions (23) and (24), we obtain

$$U'(x') = \text{const}\int_{-\infty}^{\infty} U(x)\exp\left[-ik\left(\frac{xx'}{B} - \frac{A}{2B}x^2\right)\right]\mathrm{d}x. \quad (25)$$

Note that expressions (10) and (25) are similar. They coincide if [8, 9]

$$k=1,\ \ \tan\varphi = \frac{B}{A},\ \ x' = \frac{B}{\sin\varphi}x_\varphi = \left(A^2+B^2\right)^{1/2}x_\varphi. \quad (26)$$

5. Example of an optical system

The simplest optical system for measuring the illumination distribution proportional to the required projection of the Wigner function of a laser beam is described in [8, 9]. The optical system consists of a thin lens with the focal distance f surrounded by layers of an optically homogeneous medium of thickness l and $L-l$ (Fig. 4b). By knowing the matrix description of the elements of the optical system, we can easily obtain its integral description in the form of the $ABCD$ matrix:

$$\begin{pmatrix} 1 & l-L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & l-L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ -1/f & 1-l/f \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \dfrac{L-l}{f} & l-(L-l)+\dfrac{l(L-l)}{f} \\ -1/f & 1-l/f \end{pmatrix}.$$

In this case, expression (25) is simplified and can be transformed to the expression

$$U'(x') = \text{const}\int_{-\infty}^{\infty} U(x)\exp\left[-i\left(\frac{xx'}{B} - \frac{x^2}{2R}\right)\right]\mathrm{d}x$$

obtained earlier by the method of Fourier optics [8, 9], where

$$R \equiv \frac{B}{A} = \frac{l-(L-l)+l(L-l)/f}{1+(L-l)/f} = \frac{2lf - Lf + l(L-l)}{f+L-l} =$$

$$= \frac{lf - Lf + l(L - l + f)}{f + L - l} = l - \frac{f(L - l)}{f + L - l} = l - R_0$$

is the radius of curvature of the wave front and

$$R_0 \equiv \frac{f(L - l)}{f + L - l}, \quad \text{i.e.} \quad \frac{1}{R_0} \equiv \frac{1}{L - l} + \frac{1}{f}.$$

In new notations, we have

$$B \equiv (l - R_0)A = (l - R_0)\left(\frac{F + L - l}{f}\right) = \left(\frac{l}{R_0} - 1\right)$$

$$\times \left[\frac{f(L - l)}{f + L - l}\right]\left(\frac{f + L - l}{f}\right) = \left(\frac{l}{R_0} - 1\right)(L - l).$$

6. Conclusions

The spatially coherent and energy properties of a laser beam in a specified section are completely characterised by the Wigner function. This function describes the radiation power distribution in the phase plane, and therefore cannot be directly measured but can be calculated by computer processing a family of illumination distributions measured in different sections of the laser beam focused by the optical system. For this purpose, the method of phase-spatial tomography is used, which has several alternative variants of optical realisation, but all of them can be considered in the context of matrix optics. The mathematical foundations of the method of phase-spatial tomography of a laser beam considered consistently in the paper include the fractional linear Fourier transform, the properties of the Wigner function and the method of its rotation in the phase plane by using a combination of simplest linear operators, to which lenses and gaps of an optically homogeneous medium correspond, and also the calculation of projections of the Wigner function over the illumination distribution.

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