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Eigenfrequencies and *Q* factor in the geometrical theory of whispering-gallery modes

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***Abstract*.** **It is shown that the quasi-geometrical approximation permits a quite accurate calculation of the eigenfrequency spectrum of axially symmetric dielectric resonators with whispering-gallery modes and also gives expressions for the radiative *Q* factor and *Q* factor related to surface losses.**

Keywords: *microresonators, whispering-gallery modes, eikonal, eigenfrequencies, Q factor.*

High-Q fused silica resonators were first demonstrated in paper [1]. In succeeding years, the interest in miniature optical dielectric whispering-gallery-mode (WGM) resonators continuously increased and their manufacturing technology was considerably improved. The characteristic features of WGM resonators such as their small size (0.01 – 10 mm), high Q factor (up to 10^{11}), an arbitrary profile of the generator surface (torus, spheroid [2, 3], special profile [4]), the possibility of using crystal materials, including nonlinear ones, makes these resonators very attractive for wide applications in experimental physics and applied problems of optoelectronics (see reviews [5] and [6] devoted to the theoretical problems of WGMs and also review [7] devoted to experimental applications of WGM resonators), in particular, to stabilise semiconductor lasers [8, 9].

In the case of a sphere or a cylinder, it is possible to find the eigenmodes and corresponding eigenfrequencies exactly, to calculate the field distribution inside and outside the resonator, and estimate energy losses. However, in the general case of a resonator representing an arbitrary body of revolution, this cannot be done because exact analytic solutions do not exist, whereas numerical methods, for example, the method of finite elements loses its efficiency when the resonator size considerably exceeds the wavelength. The geometrical optic approximation (eikonal) is one of the most efficient asymptotic methods for estimating the eigenfrequencies of whispering-gallery modes if exact solutions cannot be found [10]. We obtained in [11] quite accurate approximations for eigenfrequencies in a spheroid and showed that this method can be applied to an arbitrary body of revolution. In this paper, we refine the results and obtain expressions for the Q factors determined by losses depending on the shape of axially symmetric dielectric WGM resonators.

The geometrical optics approximation is described by the eikonal equation

$$(\nabla S)^2 = \epsilon(\boldsymbol{r}), \tag{1}$$

where ϵ is the dielectric constant. For a homogeneous resonator in vacuum, ϵ is independent of coordinates, $\epsilon = n^2$ inside the resonator and is unity outside (n is the refractive index). To reduce the eikonal equation to the eigenvalue problem, we can seek at the solutions for rays and sew them together at the dielectric boundary. But there exists a more descriptive and at the same time more efficient method to find the resonance conditions for WGMs – quasi-classical quantisation of closed optical paths [10 – 15].

Consider a model problem of the eigenmodes of a dielectric spheroid. The spheroidal coordinate system for oblate and prolate spheroids has the form

$$x = \frac{d}{2}[(\xi^2 - s)(1 - \eta^2)]^{1/2}\cos\phi,$$
$$y = \frac{d}{2}[(\xi^2 - s)(1 - \eta^2)]^{1/2}\sin\phi, \tag{2}$$
$$z = \frac{d}{2}\xi\eta,$$

where d is the distance between foci. The parameter $s = 1$ in the system of prolate spheroidal coordinates (ξ, η, ϕ) for $\xi \in [1, \infty)$ corresponds to prolate spheroids and for $\eta \in [-1, 1]$ – to two-sheeted hyperboloids of revolution. Correspondingly, if $s = -1$, then $\xi \in [0, \infty)$ corresponds to oblate spheroids and $\eta \in [-1, 1]$ corresponds to one-sheeted hyperboloids. We are interested in WGMs, when the field is concentrated near the surface and equatorial plane of the resonator. In the spheroidal coordinate system, the eikonal equation is separable if we assume that $S = S_1(\xi) + S_2(\eta) + S_3(\phi)$:

$$S_1(\xi) = \frac{nd}{2}\int\frac{[(\xi^2 - \xi_c^2)(\xi^2 - s\eta_c^2)]^{1/2}}{\xi^2 - s}\mathrm{d}\xi,$$

$$S_2(\eta) = \frac{nd}{2}\int\frac{[(\eta_c^2 - \eta^2)(\xi_c^2 - s\eta^2)]^{1/2}}{1 - \eta^2}\mathrm{d}\eta, \tag{3}$$

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$$S_3(\phi) = \mu \int \mathrm{d}\phi,$$

where

$$\eta_c^2 = \frac{(1+s\gamma) - [(1+s\gamma)^2 - 4s\gamma\eta_0^2]^{1/2}}{2s\gamma};$$
$$\xi_c^2 = \frac{(1+s\gamma) + [(1+s\gamma)^2 - 4s\gamma\eta_0^2]^{1/2}}{2\gamma} = \frac{1+s\gamma}{\gamma} - s\eta_c^2; \quad (4)$$

$\gamma = n^2d^2/(4v^2)$; $\eta_0^2 = 1 - \mu^2/v^2$; μ and v are separation constants.

The eikonal equation describes the rays propagating inside the resonator, reflecting from its surface. The rays touch the internal caustic, a spheroid ξ_c, and propagate along the geodesic lines on it. In an ideal sphere, all the rays of the same family lie in the same plane, but even a small eccentricity removes this degeneracy and causes the precession of the ray path around the z axis [16]. The geodesic lines acquire the form of unclosed spirals winding on a caustic spheroid, the upper and lower points of these trajectories specifying another caustic corresponding to η_c, which is a two-sheet hyperboloid if the resonator is a prolate spheroid or a hyperboloid of one sheet if the spheroid is oblate.

Within the framework of a classical ray interpretation of the eikonal method, it is necessary to use the phase-matching conditions during the cyclic change of each of the coordinate functions S_j, which gives the equations

$$2kS_1|_{\xi_c}^{\xi_s} = 2\pi(q - 1/4),$$
$$2kS_2|_{-\eta_c}^{\eta_c} = 2\pi(p + 1/2), \quad (5)$$
$$kS_3|_0^{2\pi} = 2\pi|m|$$

for the eigenvalues of the problem, where k is the wave number; ξ_s is a spheroid corresponding to the resonator surface; and q, p, and m are integers. The equations are written taking into account the changes in the phase of a ray after touching caustics and reflection from surfaces. Each contact of the ray with the caustic changes its phase by $\pi/2$, while reflection changes the ray phase by π [17].

The same equations can be obtained by the method proposed in [14]; in this case, the eikonal equations obtained formally can be clearly interpreted. The integral S_1 corresponds to the difference of two geodesic paths between points $P_1(\xi_c, -\eta_c, \phi_1)$ and $P_2(\xi_c, -\eta_c, \phi_2)$ on the surface $-\eta_c$ (Fig. 1). The first path begins from a circle at which caustics ξ_c and $-\eta_c$, intersect, passes along $-\eta_c$ to the resonator boundary ξ_s, is reflected from it and returns back to the same circle, while the second path passes along the circle arc between points P_1 and P_2. The integral S_2 corresponds to the difference of two paths, of which the first one passes from the point P'_1 over the surface ξ_c, descends on the surface $-\eta_c$ and returns to the point P'_2 on the surface η_c, while the second one passes along the circle arc between points P'_1 and P'_2. The third integral S_3 simply corresponds to the length of the intersection circle of caustics ξ_c and η_c. As a result, we have for S_1 one caustic phase shift by $\pi/2$ on the surface ξ_c and one reflection from the spheroid surface, for S_2 – one caustic phase shift by $2(\pi/2)$ on the surfaces η_c and $-\eta_c$, whereas S_3 has no additional phase shifts. Such an interpretation is a more general and is valid also in case when the eikonal solution cannot be written in the explicit form.

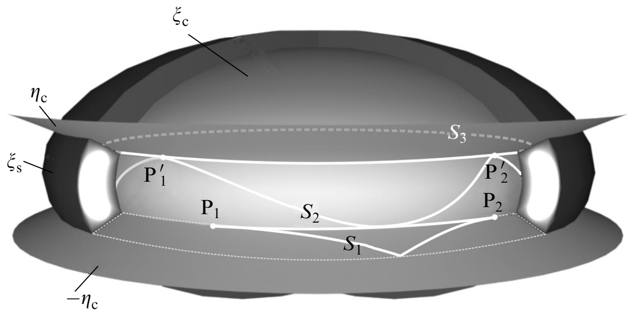

Figure 1. View of caustics and geodesic curves of a spheroidal resonator. The field of a whispering-gallery mode is concentrated in the equatorial region near the resonator surface and fills the space restricted by this surface ξ_s and caustics ξ_c and $\pm\eta_c$. The field can be represented as a set of geometrical rays reflected from the surface, tangential to caustics, and adjacent to geodesic lines on these caustics.

In the geometrical optics approximation, WGMs represent a set of rays reflecting from the internal surface of a dielectric resonator at an angle greater than the total internal reflection angle. To pass from a spheroid with simple zero boundary conditions described by the obtained equations to a real dielectric spheroid within the framework of the geometric optics approximation, we should take into account the additional phase shift upon total internal reflection. The amplitude reflection (r_F) and transmission (t_F) coefficients for a plane wave incident on a plane interface of two media are obtained from the known Fresnel formulas (see section 1.5 in [17]):

$$E_r = r_F E_i = \frac{\chi n_i \cos\theta_i - n_t \cos\theta_t}{\chi n_i \cos\theta_i + n_t \cos\theta_t} E_i,$$
$$E_t = t_F E_i \frac{2\sqrt{\chi} n_i \cos\theta_i}{\chi n_i \cos\theta_i + n_t \cos\theta_t} E_i. \quad (6)$$

Here, E_i, E_t, and E_r are the electric field amplitudes for the incident, transmitted, and reflected waves, respectively; $\theta_i = \theta_r$, θ_t are the angles of incidence, transmission, and reflection waves, respectively (Fig. 2a); n_i and n_t are the refractive indices in two media; and χ is the coefficient depending on the wave polarisation. For the wave with the vector $\boldsymbol{E}$ perpendicular to the plane of incidence (transverse electric TE wave) and parallel to the interface, the coefficient $\chi = 1$. For the TM wave with the vector $\boldsymbol{E}$ lying in the plane of incidence and directed in the first medium at an angle θ_i to the interface, $\chi = (n_t/n_i)^2$. The angle of refraction θ_t is related to the angle of incidence by the Snell law

$$\frac{n_i \sin\theta_i}{n_t \sin\theta_t} = 1. \quad (7)$$

Fresnel formulas can be easily obtained from the continuity condition at the interface of the longitudinal components of the vector $\boldsymbol{E}$ and transverse components of the vector $\boldsymbol{D} = n^2\boldsymbol{E}$. The Snell and reflection laws follow from the

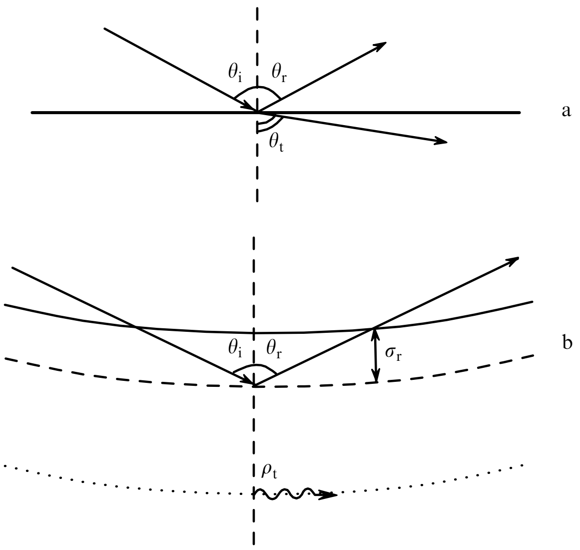

Figure 2. Incidence of a ray on a plane (a) and curved (b) surfaces.

continuity of the longitudinal component of the wave vector at the interface: $k_\tau = n_i k \sin\theta_i = n_i k \sin\theta_r = n_t k \sin\theta_t$.

The coefficient r_F relating E_r and E_i in the first Fresnel formula in (6) is simply equal to the amplitude reflection coefficient R because it relates directly the amplitudes of incident and reflected waves. The relation between the amplitude transmission coefficient T of the wave with the second Fresnel formula and t_F can be found from the expressions for the optical wave power

$$R = \frac{\chi n_i \cos\theta_i - n_t \cos\theta_t}{\chi n_i \cos\theta_i + n_t \cos\theta_t},$$
$$T = t_F \left[\frac{n_t \mathrm{Re}(\cos\theta_t)}{n_i \cos\theta_i}\right]^{1/2} = \frac{2[\chi n_i n_t \cos\theta_i \mathrm{Re}(\cos\theta_t)]^{1/2}}{\chi n_i \cos\theta_i + n_t \cos\theta_t}. \tag{8}$$

When light is incident from an optically denser medium to a less optically dense medium ($n_i > n_t$), the angle of refraction remains real only for angles of incidence satisfying the condition $\sin\theta_i < n_i/n_t$. For larger angles of incidence, the angle of refraction becomes complex (the real part is $\pi/2$) and its cosine $[\cos\theta_i = (1 - \sin\theta_i n_i/n_t)^{1/2}]$ and, hence, the transverse component of the wave vector $k_{\tau z} = n_t k \cos\theta_t$ become purely imaginary. This means that the wave in the second medium with the amplitude proportional to $\exp(-\mathrm{i}kn_2\cos\theta_2)$ transforms to a decaying wave with the amplitude proportional to $\exp[-k(n_i^2 \sin^2\theta_i - n_t^2)^{1/2}]$ and not propagating at infinity. In this case, the modulus of the reflection coefficient is $|R| = 1$, and $T = 0$. The field in the second medium, determined by the second Fresnel formula in (6), does not vanish at the boundary due to the presence of such a decaying wave. This effect of the evanescent field upon total internal reflection plays a very important role in the properties of WGMs because it provides coupling with these modes and is responsible for the interaction of modes with the surrounding medium.

The penetration of the field to the second medium can be described in the ray approximation by introducing the imaginary mirror boundary separated by a distance σ_r from the real boundary (Fig. 2). This distance can be obtained from expression (8) for the reflection coefficient by assuming that the additional phase shift in the expansion coefficient is caused by the propagation over this additional path. For almost grazing angles of incidence in WGMs, when the value of $\cos\theta_i$ is small, we obtain from Fresnel formula (6)

$$R = -\mathrm{e}^{\mathrm{i}\phi_r} = \frac{\chi n_i \cos\theta_i - \mathrm{i}(n_i^2 \sin^2\theta_i - n_t^2)^{1/2}}{\chi n_i \cos\theta_i + \mathrm{i}(n_i^2 \sin^2\theta_i - n_t^2)^{1/2}},$$
$$\phi_r = 2\arctan\frac{\chi n \cos\theta_i}{(n^2 \sin^2\theta_i - 1)^{1/2}}$$
$$\simeq \frac{2\chi n}{(n^2-1)^{1/2}}\cos\theta_i + \frac{\chi n^3(3-2\chi^2)}{3(n^2-1)^{3/2}}\cos^3\theta_i$$
$$+ \frac{\chi n^5(15 - 20\chi^2 + 8\chi^4)}{20(n^2-1)^{5/2}}\cos^5\theta_i + O(\cos^7\theta_i), \tag{9}$$

where $n = n_i/n_t$. Rays behave as if they were reflected without displacement for a surface separated from the real surface by the distance

$$\sigma_r = \frac{\phi_r}{2kn\cos\theta} = k^{-1}\left[\frac{\chi}{(n^2-1)^{1/2}} + \frac{\chi n^2(3-2\chi^2)}{6(n^2-1)^{3/2}}\cos^2\theta_i \right.$$
$$\left. + \frac{\chi n^4(15 - 20\chi^2 + 8\chi^4)}{40(n^2-1)^{5/2}}\cos^4\theta_i + O(\cos^6\theta_i)\right]. \tag{10}$$

For the same reason, the longitudinal displacement $\sigma_\tau = \sigma_r \tan\theta_i$ of the reflected beam occurs. This effect is known as the Goos–Hänchen effect [18] and it can be quite large for grazing angles.

When a ray is incident on a plane interface under the condition $\sin\theta_i > n_i/n_t$, its reflection will be total. However, if the surface is convex outside, this will not be the case (Fig. 2b). The reason can be easily explained, and the effect can be estimated from simple physical considerations. The evanescent field of the wave moves along the curved surface with the radius of curvature ρ_{cv} with the tangential velocity $v_\tau = \omega/k_\tau = c/(n_i \sin\theta_i)$ (ω is the cyclic frequency, c is the speed of light in the surrounding medium), and the phase fronts move away from the surface at a constant angular velocity. However, at a distance of $\rho_t = \rho_{cv} c/v_\tau = \rho_{cv} n \sin\theta_i$ from the centre of curvature, this velocity becomes equal to the speed of light, and the 'tail' of the evanescent field reaching this boundary is emitted tangentially and therefore cannot return back to the first medium [19]. Unlike reflection from a plane, the decay of the evanescent field with ρ occurs not exponentially, but can be easily found. It is described by the law $E = E_i t_F \exp[\mathrm{i}\int k_\rho(\rho)\mathrm{d}\rho]$, where

$$k_\rho(\rho) = \left[k^2 - \left(\frac{k_\tau \rho_{cv}}{\rho}\right)^2\right]^{1/2}$$
$$= \mathrm{i}k\left[\left(\frac{\rho_{cv} n \sin\theta_i}{\rho}\right)^2 - 1\right]^{1/2}. \tag{11}$$

One can see from this expression that the decay ceases at the distance ρ_t, and the imaginary k_ρ becomes real. Thus, by taking the integral, we obtain the final expression for power losses after reflection from the curved surface:

$$|T|^2 = |t_{\mathrm{F}}|^2 \frac{(n^2 \sin^2 \theta_{\mathrm{i}} - 1)^{1/2}}{n \cos \theta_{\mathrm{i}}}$$

$$\times \exp\left\{ -2k \int_{\rho_{\mathrm{cv}}}^{\rho_{\mathrm{t}}} \left[\left(\frac{\rho_{\mathrm{cv}} n \sin \theta_{\mathrm{i}}}{\rho} \right)^2 - 1 \right]^{1/2} \mathrm{d}\rho \right\}$$

$$= \frac{4n\chi \cos \theta_{\mathrm{i}} (n^2 \sin^2 \theta_{\mathrm{i}} - 1)^{1/2}}{n^2 - 1 - n^2(1 - \chi^2) \cos^2 \theta_{\mathrm{i}}} \mathrm{e}^{-2\Psi(\theta_{\mathrm{i}})}, \quad (12)$$

$$\Psi(\theta_{\mathrm{i}}) = k\rho_{\mathrm{cv}}[n \sin \theta_{\mathrm{i}} \,\mathrm{arcosh}\,(n \sin \theta_{\mathrm{i}})$$

$$-(n^2 \sin^2 \theta_{\mathrm{i}} - 1)^{1/2}].$$

By using the expression found for the displacement σ_{r} and calculating asymptotically integrals in the approximation of smallness of η_{c} and $(\xi_{\mathrm{s}} - \xi_{\mathrm{c}})/\xi_{\mathrm{s}}$, we obtain the following approximation for the eigenfrequencies of a spheroidal resonator [11]

$$nka = l + \alpha_q \left(\frac{l}{2} \right)^{1/3} + \frac{2p(a-b) + a}{2b}$$

$$- \frac{\chi n}{(n^2 - 1)^{1/2}} + \frac{3\alpha_q^2}{20} \left(\frac{l}{2} \right)^{-1/3} + \frac{\alpha_q}{12} \left[\frac{2p(a^3 - b^3) + a^3}{b^3} \right.$$

$$\left. + \frac{2n^3 \chi (2\chi^2 - 3)}{(n^2 - 1)^{3/2}} \right] \left(\frac{l}{2} \right)^{-2/3} + O(l^{-1}), \quad (13)$$

where a and b are the semiaxes of the spheroid and $l = |m| + p$ is the mode order. To obtain a better accuracy, the formal solutions $\alpha_q = [\frac{3}{2}\pi(q - \frac{1}{4})]^{2/3}$ of (13) should be replaced by the roots of the equation $\mathrm{Ai}(-\alpha_q) = 0$, ($\alpha_q \simeq 2.3381, 4.0879, 5.5206, \ldots$), where $\mathrm{Ai}(z)$ is the Airy function. The validity of such a substitution follows from the properties of the approximation of the field near the caustic by the Airy functions [12]. The obtained expression with the error $\sim O(l^{-1})$ is considerably more accurate than approximations found earlier [2, 14, 15], whose error is $O(l^{-1/3})$. The accuracy of the approximation obtained here is confirmed by numerical simulation [11].

An arbitrary surface of a body of revolution can be often approximated by an equivalent spheroid, taking into account that the WGM field is concentrated near the equatorial plane close to the resonator surface, and the obtained result can be used directly.

In the case of an arbitrary convex body of revolution, we can also construct a more general theory. First we determine the families of caustics. The first family can be found by using the approximation [12]

$$\sigma_{\mathrm{c}}(P) \simeq -\frac{1}{2}\kappa^2 \rho_{\mathrm{cv}}^{1/3}(P) + O(\kappa^4),$$

$$\cos \theta_{\mathrm{i}}(P) = \kappa \rho_{\mathrm{cv}}^{-1/3}(P), \quad (14)$$

where $\sigma_{\mathrm{c}}(P)$ is the normal distance from the point P on the surface body to the caustic; κ is the family parameter; and $\cos \theta_{\mathrm{i}}$ is the angle of ray incidence at the point P. If a caustic of the first family is found, which is specified parametrically as $\rho = g(z)$, we can also find the second family of caustics specified parametrically as $h(z)$ and orthogonal to any surface of the first family for different κ.

The geodesic curve on the surface is described by the expression

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{\rho_{\mathrm{c}}(1 + g'^2)^{1/2}}{g(z)[g^2(z) - \rho_{\mathrm{c}}^2]^{1/2}}, \quad (15)$$

where $\rho_{\mathrm{c}} = g(z_{\max})$ is the radius of a circle on the caustic located at a maximum distance from the equatorial plane. The length of the geodesic curve of the segment is

$$\mathrm{d}L = \frac{g(1 + g'^2)^{1/2}}{(g^2 - \rho_{\mathrm{c}}^2)^{1/2}} \mathrm{d}z. \quad (16)$$

Then, the length of the geodesic curve connecting points ϕ_1 and ϕ_2 is

$$L_1^g = 2 \int_{-z_{\max}}^{z_{\max}} \frac{g(1 + g'^2)^{1/2}}{(g^2 - \rho_{\mathrm{c}}^2)^{1/2}} \mathrm{d}z, \quad (17)$$

and the length of the arc from $\phi_0 = 0$ to $\phi_{\mathrm{c}} = 2 \int_{-\eta_{\mathrm{c}}}^{\eta_{\mathrm{c}}} \frac{\mathrm{d}\phi}{\mathrm{d}u} \mathrm{d}u$ is

$$L_2^g = \rho_{\mathrm{c}} \phi_{\mathrm{c}} = 2 \int_{-z_{\max}}^{z_{\max}} \frac{\rho_{\mathrm{c}}^2 (1 + g'^2)^{1/2}}{g(g^2 - \rho_{\mathrm{c}}^2)^{1/2}} \mathrm{d}z. \quad (18)$$

As a result, we obtain

$$nk(L_1^g - L_2^g) = 2nk \int_{-z_{\max}}^{z_{\max}} \frac{(1 + g'^2)^{1/2} (g^2 - \rho_{\mathrm{c}}^2)^{1/2}}{g} \mathrm{d}z$$

$$= 2\pi(p + 1/2). \quad (19)$$

Similarly, by taking integral over the resonator surface, we obtain for the geodesic curve on the caustic $\rho = h(z)$ of another family

$$nk(L_1^h - L_2^h) = 2nk \int_{z_{\max}}^{z_{\mathrm{s}}} \frac{(1 + h'^2)^{1/2} (h^2 - \rho_{\mathrm{c}}^2)^{1/2}}{h(z)} \mathrm{d}z$$

$$= 2\pi(q - 1/4). \quad (20)$$

The third condition is

$$2\pi nk\rho_{\mathrm{c}} = 2\pi|m|. \quad (21)$$

For the spheroidal coordinate system, this system is equivalent to (5).

In [20], another interesting method was proposed for calculating the WGM eigenfrequencies. The field is represented in the form of the eigensolution for a dielectric cylinder, which is slowly varying along the z axis. The dependence on z is taken into account by the WKB method. A similar method was proposed earlier in [2]. The use of this method for solving the model eigenfrequency problem for a sphere and a spheroid show that its accuracy is of the order of $O(l^{-2/3})$, i.e. worse than that of our method.

To estimate the intrinsic Q factor of WGMs in the quasi-classical approximation, it is necessary to take into account internal ray losses upon each reflection from the resonator surface. The Q factor is determined by a simple expression [1]

$$Q = \frac{2\pi n}{\alpha \lambda}, \quad (22)$$

where α corresponds to losses per unit path length of the ray; and λ is the wavelength. The path can be represented as many segments of a broken line, each of them of length $L_n = 2\rho_{\mathrm{cv}}\cos\theta$. Let energy losses upon reflection on the given segment of the path be $T(\theta)$ and $\alpha_n = T(\theta, \rho_{\mathrm{cv}})/L_n$. By averaging α_n over one coil of the geodesic curve, we obtain the total losses and Q factor

$$Q = \frac{2\pi n L^g}{\lambda}\left[\oint \frac{T(\theta)}{2\rho_{\mathrm{cv}}(\theta)\cos\theta}\mathrm{d}L\right]^{-1}$$

$$= \frac{2\pi n L^g}{\lambda}\left[\int_{-z_{\max}}^{z_{\max}} \frac{T(\theta)}{\rho_{\mathrm{cv}}(\theta)\cos\theta}\frac{\mathrm{d}L}{\mathrm{d}z}\mathrm{d}z\right]^{-1}. \qquad (23)$$

This expression can be used to estimate the Q factors of arbitrary dielectric WGM resonators. Note that not only the radiative Q factor but also scattering and absorption losses on the surface can be calculated. The radiation losses upon reflection from a curved surface were calculated above. They can be also obtained by solving the model problem in a sphere, when $\rho_{\mathrm{cv}} = a$ and $\cos\theta_0 = [1-(l+1/2)^2/(kna)^2]^{1/2} \simeq (\alpha_q)^{1/2}(l/2)^{-1/3}$, and

$$T_0 = \frac{4\pi n a\cos\theta_0}{\lambda Q_0}, \qquad (24)$$

where ρ_{cv}, $\cos\theta_0$, and T_0 are mode constants. By using the Debye approximation for the Neumann function in the expression for the radiative Q factor found from the exact equation for a sphere, we obtain the same expression (12). The Q factor can be calculated, if $T(\theta)$ is known, by using the expressions

$$\rho_{\mathrm{cv}} = \frac{|\boldsymbol{r}'|^3}{|\boldsymbol{r}'\times\boldsymbol{r}''|} = \frac{\rho^3(1+\rho'^2)^{3/2}}{\rho_{\max}^2(1+\rho'^2) - \rho\rho''(\rho^2-\rho_{\max}^2)},$$

$$\frac{\mathrm{d}l}{\mathrm{d}z} = \frac{\rho(z)[1+\rho'(z)^2]^{1/2}}{[\rho^2(z)-\rho_{\max}^2]^{1/2}},$$

$$L^g = 4\int_0^{z_{\max}} \frac{\rho(z)[1+\rho'(z)^2]^{1/2}}{[\rho^2(z)-\rho_{\max}^2]^{1/2}}\mathrm{d}z, \qquad (25)$$

$$\cos\theta \simeq [2\sigma_{\mathrm{c}}/\rho_{\mathrm{cv}}]^{1/2},$$

where $\rho_{\max} = \rho(z_{\max})$ is the distance from the z axis to the highest point of the geodesic curve; and σ_{c} is the normal distance from the sphere surface to the caustic.

From (25), we find the expressions for the intrinsic radiative Q factor of a spheroid:

$$\rho_{\mathrm{cv}} = a\frac{[1+z^2(a^2-b^2)/b^4]^{3/2}}{1+z_{\max}^2(a^2-b^2)/b^4}$$

$$\simeq a\left(1 - \frac{a^2-b^2}{b^2}\eta_{\mathrm{c}}^2 + \frac{3}{2}\frac{a^2-b^2}{b^4}z^2\right), \qquad (26)$$

$$\cos\theta \simeq (\alpha_q)^{1/2}\left(\frac{l}{2}\right)^{-1/3}\left(1 + \frac{a^2-b^2}{2b^2}\eta_{\mathrm{c}}^2 - \frac{a^2-b^2}{2b^4}z^2\right),$$

$$L^g \simeq 2\pi b + \frac{\pi}{2}\frac{a^2-b^2}{b}\eta_{\mathrm{c}}^2.$$

By setting $z = \eta_{\mathrm{c}}\cos\psi$, we obtain

$$Q \simeq \frac{\pi(n^2-1)^{1/2}l}{4\chi n}\left[\int_0^{\pi/2} \mathrm{e}^{-2\Psi(\psi)}\mathrm{d}\psi\right]^{-1}$$

$$\simeq \frac{(n^2-1)^{1/2}l}{2\chi n}\mathrm{e}^{2\psi_0}\frac{\mathrm{e}^{\psi_1}}{\mathrm{I}_0(\psi_1)},$$

$$\Psi_0 = nka\left[1 - \frac{(2p+1)a(a^2-b^2)}{lb^3}\right] \qquad (27)$$

$$\times\left\{\operatorname{arcosh} n\left[1 - \frac{\alpha_q}{2}\left(\frac{l}{2}\right)^{-2/3}\right] - \left(1 - \frac{1}{n^2}\right)^{1/2}\right\},$$

$$\Psi_1 = \Psi_0\frac{3(2p+1)a(a^2-b^2)}{2b^3 l},$$

where $\mathrm{I}_0(z)$ is the Infeld function. Figure 3 shows the dependences of the radiative Q factor on the flatness parameter of a spheroid for the TE and TM modes for $l = m = 100$ (fundamental mode) and $l = 100$, $m = 98$. The flattening $f = 0$ ($a = b$) corresponds to an ideal sphere.

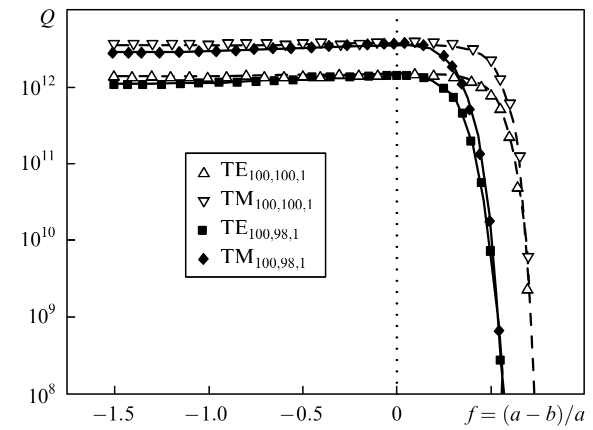

Figure 3. Dependences of the radiative Q factor on the flatness parameter.

We calculate losses caused by scattering from the nonideal surface of the resonator by using the results of calculations obtained for a sphere [21]:

$$T = \frac{32n^2\pi^3B^2\sigma^2}{3\lambda^4}\cos\theta, \qquad (28)$$

where B is the correlation length of inhomogeneities and σ is the inhomogeneity size.

As a result, we obtain the Q factor for surface scattering in the form

$$Q_{ss} = \frac{3\lambda^2 L_g}{16n\pi^2B^2\sigma^2}$$

$$\times\left[\int_{-z_{\max}}^{z_{\max}} \frac{1+z_{\max}^2(a^2-b^2)/b^4}{(z_{\max}^2-z^2)^{1/2}(1+z^2(a^2-b^2)/b^4)}\mathrm{d}z\right]^{-1} =$$

$$= \frac{3\lambda^2 a}{8n\pi^2 B^2 \sigma^2} \frac{1}{[1 + z_{\max}^2 (a^2 - b^2)/b^4]^{1/2}}. \quad (29)$$

Thus, by using the ray approximation, we have obtained accurate approximate expressions for the eigenfrequencies and Q factor of axially symmetric dielectric WGM resonators with an arbitrary profile of the generator surface.

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