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Statistics of light deflection in a random two-phase medium

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***Abstract*.** **The statistics of the angles of light deflection during its propagation in a random two-phase medium with randomly oriented phase interfaces is considered within the framework of geometrical optics. The probabilities of finding a randomly walking photon in different phases of the inhomogeneous medium are calculated. Analytic expressions are obtained for the scattering phase function and the scattering phase matrix which relates the Stokes vector of the incident light beam with the Stokes vectors of deflected beams.**

Keywords: *light scattering, phase function, phase matrix, inhomogeneous medium.*

1. Introduction

The interaction of electromagnetic waves with inhomogeneous media has been extensively studied for the last century. Interest in these studies is considerably related to practical needs of radio communication, radiolocation, space reconnaissance, astrophysics, etc. The main results of these studies are reported in many papers and monographs [1 – 4]. The most universal approach to the problem is based on the solution of the Maxwell equations with the corresponding boundary and initial conditions [5]. In the case of the inhomogeneous medium, these conditions become statistic, which substantially complicates the problem. Therefore, different approximations are used, which yield analytic solutions or considerably simplify computer calculations [6]. Thus, if the characteristic size of the medium inhomogeneities is rather large compared to the wavelength, the approximation of geometrical optics is quite suitable [7]. To estimate the angular deflections of light beams in a random medium, eikonal equations together with the correlation function of the refractive index [8] or Einstein – Fokker diffuse equations [9 – 11] were mainly used. The development of computers made it possible to perform complex calculations of statistics of the beam propagation and to simulate different random media. In biomedical optics, Monte-Carlo methods or diffuse models are often used, which are in fact based on the geometrical concept of light propagation. These methods assume the knowledge of the main optical properties of random media such as absorption, scattering and anisotropy coefficients. The anisotropy coefficient is defined as a mean cosine of the angle of beam deflection in a single scattering event. It depends on the scattering phase function $p(\boldsymbol{s}, \boldsymbol{s}')$, which is the probability density of a change in the direction vector $\boldsymbol{s}$ of a beam by the vector $\boldsymbol{s}'$ upon scattering [3]. Therefore, the radiation propagation in an inhomogeneous medium is determined to a great extent by the type of the scattering phase function.

Unfortunately, it is difficult or impossible to measure the scattering phase function in many real cases [12 – 16]. Therefore, the phenomenological Henyey – Greenstein phase function has become widely used in biomedical optics [17]. In fact, this function is the sum of the first two expansion coefficients of the generalised phase function in series in Legendre polynomials with $\cos\gamma$ as an argument, where γ is the angle between vectors $\boldsymbol{s}$ and $\boldsymbol{s}'$ [18]. The Henyey – Greenstein phase function usually lays the basis of the medium model upon measuring the optical properties. Because in the following applications it is used similarly, despite the absence of relations with the physical nature of the scattering medium, the final result is often quite reasonable. Other types of phase functions are encountered rather seldom [19 – 25]. Some of these functions were critically considered in [20] from the point of view of their possible use in optics of biological tissues. All the known phase functions have substantial drawbacks because they are not related to the physical nature of biological tissues. In addition, the majority of these functions are not sensitive to the light wave polarisation, and that is why they are often used together with the Mie theory, which allows one to calculate the scattering of polarised light by spherical particles. It is evident that simulation of the biological tissue by spherical particles is far from reality.

In other papers, the light scattering by randomly deformed particles was simulated by using ray tracing by the Markov method [6]. In this case, reflection and refraction on the surface of particles were taken into account in accordance with the statistical weights equal to the Fresnel coefficients. The shape of the particles was generated with the help of two parameters: the standard deflection of radius and the correlation length of angular variations. The spatial orientation of particles was chosen randomly. The similar approach was used to calculate the scattering phase function in hexagonal ice crystals in atmosphere [26 – 28]. For particles with fixed statistical

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parameters, this approach in principle allows the calculation of all elements of the phase matrix relating the angular distributions of the Stokes parameters of a scattered wave to the Stokes parameters of an incident plane wave [29]. The relation with real media and the possibility to verify experiments with polarised light make this method very attractive. However, it requires complex calculations, especially if the size and shape of the particles vary considerably [30].

The light is usually scattered due to the interaction with particles, fluctuations of the medium density or due to reflection (refraction) on a rough surface. The optically inhomogeneous medium can be often represented by several phases separated by randomly deformed surfaces. Thus, for example, many biological tissues consist mainly of water and organic matrix. When the light propagates in this medium, many events of the radiation interaction with surfaces of the phase interface occur due to which the beam direction randomly changes. This process is characterised by the probability density of the beam deflection by a specified angle, this density being equivalent to the phase function of light scattering by particles. A part of the flux is reflected and the other is refracted in each such event. According to the Fresnel formulas, the ratio of energies in these fluxes, depends on the angle of incidence, the relative refractive index, and the direction of the polarisation vector. The fact that the light flux is multiply split considerably complicates the consideration. However, if we consider the spatial orientation of the phase interface as a statistical process, the light refraction and reflection can be treated as independent events with statistical weights equal to Fresnel coefficients. One should only take into account that the light deflection in the given direction is possible for two different orientations of the surface, in the first case refraction occurring, while in the second case – reflection.

This randomly oriented surface is in fact an ideal physical object with a statistical generator of spatial orientations of the phase interface. It is characterised by two parameters: the relative refractive index and the probability density of realising the given angle between the normal to the phase interface and the direction vector of the incident beam. If we assume that different spatial orientations of the interface are equally probable, the only parameter characterising it is the relative refractive index. In this case, one can determine the phase function of light scattering for a particular medium by using only the first principles. Because the proposed model uses Fresnel formulas, it is also applicable for polarised light.

Earlier, many authors described the light scattering by rough surfaces by using the facet models [31 – 34], which ideologically are close to the proposed two-phase model of a medium with randomly oriented phase interfaces. However, they are mainly limited to the case when the light is incident on a rough surface from one of its sides. The specific feature of a multiphase inhomogeneous medium is that photons in it fall on the phase interface at random angles from both sides. In this case, a part of the flux is reflected remaining in the same phase, while a part being refracted, penetrates into the other phase. Therefore, it is necessary to determine in which of the medium phases is the photon when it reaches the surface of the phase interface, performing the sampling according to the Fresnel formulas. Another approach to the problem consists in determining the probability of finding the stochastic photons in each phase of the medium and in using them as the statistical weight of the corresponding terms of the scattering phase function. It is the solution of this problem that forms the novelty of the present paper.

Because the model is based on the ray representation of light, it is limited by the geometrical optics, i.e. the characteristic size of optical inhomogeneities of the medium should be great enough compared to the wavelength. For many biological tissues the scattering coefficient in the visible and near-IR regions lies between 100 and 1000 cm^{-1}. Optical inhomogeneities of size 100 – 10 μm correspond to this range, which substantially exceeds the wavelength of ~ 1 μm. This implies that the geometrical approximation can be used for simulating the light propagation in biological tissues. Unlike the wave model, the geometrical approximation does not take into account light diffraction and interference. However, we can assume that upon multiple scattering of stochastic photons in an inhomogeneous medium these effects are averaged, so that the results obtained in the geometrical approximation satisfactorily describe the situation.

2. Scattering phase function

Consider the light propagation in an optically inhomogeneous medium consisting of two phases with the refractive indices n_1 and n_2. Let us assume that a thin light beam, which is in phase with the refractive index n_1, intersects the phase interface. We will locate the centre of the three-dimensional Cartesian coordinate system in the assumed intersection point of the beam with the phase interface of the medium and direct the z axis towards the incident beam. We will also introduce the equivalent spherical coordinate

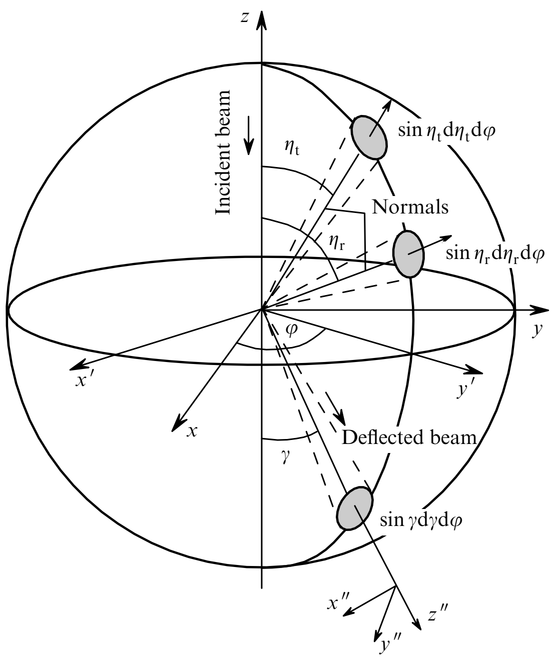

Figure 1. Diagram of the light beam deflection by a randomly oriented phase interface. The interface element is located at the origin of the coordinates and the normal to it can be directed to any solid angle of the upper hemisphere, while the deflected beam can have any direction. Two possible positions of the normal determined by the angles η_r and η_t for which the beam is reflected or refracted, correspond to the specified angle γ of the beam deflection. Other peculiarities of the diagram are explained in the text.

system with the polar angle, which is measured with respect to the z axis, and the azimuthal angle, which is measured with respect to the x axis. The similar coordinate system is commonly used when considering the light scattering by particles [29, 35]. Let η, φ and γ, ψ be the angular coordinates of the normal to the phase interface and of the direction vector of the beam deflected in the spherical coordinate system. A hemisphere ($z > 0$), whose centre is in the origin of the coordinates, is the geometrical locus of the ends of the normal, while a complete sphere is the geometrical locus of the ends of the direction vector of the deflected beam (Fig. 1). For the normal, we have $\eta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi]$ while for the direction vector of the deflected beam, $\gamma \in [0, \pi]$ and $\psi \in [0, 2\pi]$. In our case, the normal to the surface, incident, refracted and reflected beams lie in the same plane, hence, $\psi = \varphi$, and, the angles γ and φ are independent. Let $\sigma(\eta, \varphi)$ be the probability density of realising the direction of this normal to the phase interface, and $p(\gamma, \varphi)$ be the required probability density of this direction of the deflected beam. By definition, the probability of finding the deflected beam within the element of the solid angle in the direction specified by angles γ and φ is $p(\gamma, \varphi) \sin\gamma d\gamma d\varphi$, while the probability of finding the normal to the phase interface within the element of the solid angle in the direction specified by angles η and φ is $\sigma(\eta, \varphi) \sin\eta d\eta d\varphi$.

3. Probability density of the beam deflection angle in the given direction

The incident beam can be deflected in the direction specified by the angles γ, φ, if the normal to the surface has only definite directions specified by the angles η_r, φ and η_t, φ (Fig. 1). The subscripts r and t correspond to reflection and refraction, respectively, η_r and η_t are the angles between the normals to the phase interfaces and the incident beam, and the beam deflection angle is γ. We assume that the probability of finding a photon in the solid angle $\sin\gamma d\gamma d\varphi$ is equal to the sum of probabilities of finding the normal to the phase interface in the elementary solid angles $\sin\eta_r d\eta_r d\varphi$ and $\sin\eta_t d\eta_t d\varphi$ taking into account the statistic weights of the reflection and refraction events. We also assume that the statistical weights of these events are equal to the Fresnel coefficients for the intensities of the reflected $[R(\eta_t, \varphi)]$ and transmitted $[T(\eta_t, \varphi)]$ light. These assumptions lead to the equation:

$$p(\gamma, \varphi) \sin\gamma d\gamma = R(\eta_r, \varphi)\sigma(\eta_r, \varphi) \sin\eta_r d\eta_r + $$
$$+ T(\eta_t, \varphi)\sigma(\eta_t, \varphi) \sin\eta_t d\eta_t. \quad (1)$$

To perform transformations of random variables, it is necessary to determine the relation between angles η_r, η_t and γ. They follow from the laws of reflection and refraction:

$$\eta_r(\gamma) = (\pi - \gamma)/2, \quad (2)$$

$$\eta_t(\gamma) = \arctan\left(\frac{n \sin\gamma}{1 - n\cos\gamma}\right), \quad (3)$$

where $n = n_2/n_1$ is the relative refractive index. Figure 2 shows these dependences. Here, the region of admissible deflection angles is expanded from $-\pi$ to π, which, in principle, corresponds to the two-dimensional medium. It is assumed also that the clockwise deflection with respect to the initial direction gives negative values of the angle γ. In the case of refraction, the region of admissible values of the modulus of the deflection angle $|\gamma|$ is restricted by the quantities $\pi/2 - \arcsin(1/n)$ for $n \geqslant 1$ and $\pi/2 - \arcsin n$ for $n < 1$. Correspondingly, in the case of $n < 1$ the region of angles of incidence upon refraction is limited by the critical angle $-\arcsin n$. By substituting the angles $\eta_r(\gamma)$ and $\eta_t(\gamma)$ and the absolute values of derivatives $d\eta_r/d\gamma$ and $d\eta_t/d\gamma$ into (1), we obtain the expression for the probability density of the beam deflection in the given direction:

$$p(\gamma, n) = \frac{R(\eta_r(\gamma)) \sin(\eta_r(\gamma))}{2\sin\gamma}\sigma((\eta_r(\gamma))$$
$$+ \frac{T(\eta_t(\gamma)) \sin(\eta_t(\gamma))}{\sin\gamma}\left|\frac{(n^2 - \sin^2\gamma)^{1/2}}{\cos\gamma - (n^2 - \sin^2\gamma)^{1/2}}\right|\sigma((\eta_t(\gamma)). \quad (4)$$

Hereafter, the angle φ is omitted assuming the azimuthal symmetry. The Fresnel coefficients in (4) depend on the relative refractive index of the medium phases and the direction of the polarisation vector of the incident light with respect to the plane of incidence.

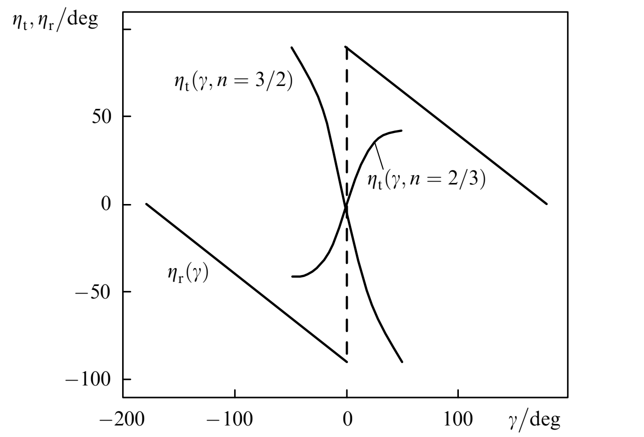

Figure 2. Dependences of the angles η_r and η_t of incidence of light beams on the deflection angle γ upon reflection and refraction on the phase interface.

Let us assume that all positions of the phase interface with respect to the incident beam are equally probable, i.e. $\sigma(\eta, \varphi) = 1/(2\pi)$. Then, Eqn (4) is simplified:

$$p(\gamma, n) = \frac{R(\eta_r(\gamma), n) \sin(\eta_r(\gamma))}{4\pi \sin\gamma}$$
$$+ \frac{T(\eta_t(\gamma), n) \sin(\eta_t(\gamma))}{2\pi\sin\gamma}\left|\frac{(n^2 - \sin^2\gamma)^{1/2}}{\cos\gamma - (n^2 - \sin^2\gamma)^{1/2}}\right|. \quad (5)$$

The two terms in the right-hand side of Eqn (5) correspond to the reflection and refraction events in which the deflection angle γ is the same. We will denote them for convenience as $p_r(\gamma, n)$ and $p_t(\gamma, n)$, respectively. In fact, these are the probability densities of reflection and refraction events, which occur for different orientations of the normal to the phase interface. Therefore, they are statistically independent. In addition, the reflection and refraction events are independent of any surface position, because their sampling is performed according to the statistical weights, the sum of which is equal to unity. It follows from here that the total numbers of photons transmitted through the randomly oriented surface and reflected from it in all possible directions are also statistically independent.

The Fresnel coefficients in (5) depend on the polarisation direction of the incident light wave with respect to the plane of incidence. The nonpolarised light beam can be decomposed into two orthogonally linearly polarised incoherent beams of equal intensity. Therefore,

$$R(\eta_{\rm r}(\gamma), n) = 0.5[R_{\rm p}(\eta_{\rm r}(\gamma), n) + R_{\rm s}(\eta_{\rm r}(\gamma), n)], \tag{6}$$

$$T(\eta_{\rm t}(\gamma), n) = 0.5[T_{\rm p}(\eta_{\rm t}(\gamma), n) + T_{\rm s}(\eta_{\rm t}(\gamma), n)], \tag{7}$$

where the subscripts p and s correspond to the light polarisation parallel and perpendicular to the plane of incidence, respectively. The dependences of the Fresnel coefficients on the incidence angle are presented in the Appendix. The dependences of the probability density $p(\gamma)$ for two relative refractive indices ($n = 2/3$ and 3/2) are presented in Fig. 3. One can see from the comparison of the curves that in the case of $n = 2/3$, a sharper increase in the curve for $\gamma \to 0$ and simultaneously a higher plateau for large deflection angles are observed. This is caused by the effect of total internal reflection. As γ tends to zero, both curves tend to infinity, while the integrals $\int_0^{\pi} p(\gamma, n){\rm d}\gamma$ converge. By substituting expressions (6) and (7) into (5), it is easy to see that the normalisation condition

$$\int_0^{2\pi}\int_0^{\pi} p(\gamma, n)\sin\gamma{\rm d}\gamma{\rm d}\varphi = 1 \tag{8}$$

is fulfilled for the probability density $p(\gamma, n)$ independently of the value of n. Therefore, the probability density $p(\gamma, n)$ for a two-phase medium plays the same role as the scattering phase function for particles.

In many cases the angle-integrated probability density of the beam deflection by the specified angle γ, i.e. the function $p(\gamma, n)2\pi\sin\gamma$ is of interest. Its behaviour is shown in Fig. 4 for the same relative refractive indices as in Fig. 3 ($n = 2/3$ and 3/2). It demonstrates many specific features of the scattering phase function under study. Now, as the deflection angle γ approaches zero, both curves tend to the finite value of 0.5 and for $\gamma \neq 0$ they have maxima. There also exist two characteristic deflection angles for which the functions $p(\gamma, n)2\pi\sin\gamma$ have breaks. The first break is observed for $\gamma \approx 48°$, which corresponds to the maximum deflection angle $\gamma_{\rm t}^{\rm max}$ upon refraction, while the second break occurs for $\gamma \approx 96°$, which corresponds to the deflection angle $\gamma_{\rm c}$ at the critical angle of incidence. It is interesting that these angles are related by the expression $2\gamma_{\rm t}^{\rm max} = \gamma_{\rm c}$. All the peculiarities mentioned here are caused by the physical nature of the multiphase scattering medium and are absent in other known scattering phase functions.

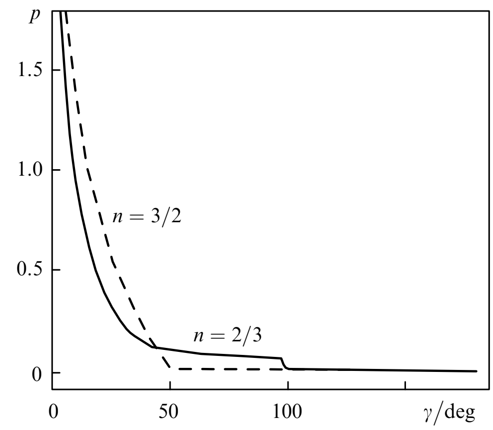

Figure 3. Probability density of the light beam deflection by the angle γ by the randomly oriented phase interface.

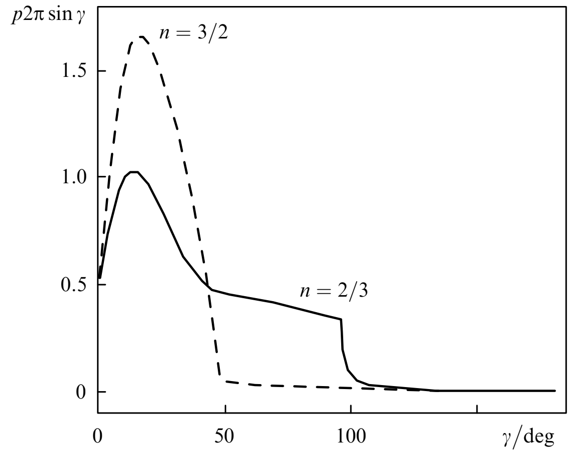

Figure 4. Probability density of the light beam deflection by the angle γ by the randomly oriented phase interface integrated over the azimuthal angle.

4. Light propagation

4.1 Distribution of light between phases

When the light propagates in a two-phase medium, the probabilities of finding a photon in different phases are not equal due to the so-called waveguide effect, when a part of photons, which are in the medium with a higher refractive index, will experience total internal reflection during the interaction with the phase interface and will return back. At the same time, this effect is absent for the photons in the medium with a smaller refractive index. The probability that the photon from the medium with the refractive index n_1 (phase 1) will pass into the medium with the refractive index n_2 (phase 2) is

$$T_{12} = \int_0^{\pi}\int_0^{2\pi} p_{\rm t}(\gamma, n_2/n_1)\sin\gamma{\rm d}\gamma{\rm d}\varphi,$$

where the function $p_{\rm t}(\gamma, n)$ is the second term in the right-hand side of Eqn (5), which is responsible for refraction. Similarly,

$$T_{21} = \int_0^{\pi}\int_0^{2\pi} p_{\rm t}(\gamma, n_1/n_2)\sin\gamma{\rm d}\gamma{\rm d}\varphi.$$

The probability of finding a stochastic photon in the phase with the refractive index n_1 is

$$\chi\left(\frac{n_2}{n_1}\right) = \frac{T_{21}}{T_{12} + T_{21}}. \tag{9}$$

The dependences $\chi(n)$ and $\chi(n^{-1})$ on the relative refractive index are presented in Fig. 5. They quantitatively predict a higher probability of finding a photon in the phase with a higher refractive index. Note that even small differences in the refractive indices can lead to large differences in the probabilities of finding a photon in different phases of the medium. The ratio of the residence times of photons in phases 1 and 2 can be estimated as $\chi(n)V_1^{1/3}n_1/[\chi(n^{-1})V_2^{1/3}n_2]$, where V_1 and V_2 are the specific volumes of phases 1 and 2, respectively. Here, the difference in the speed of light in media is taken into account. The density of photons in the phase with the refractive index n_1 is proportional to the probability $\chi(n_2/n_1)$ and inversely proportional to the specific volume of this phase. Consideration of the probabilities of the photon residence in individual phases of the random medium can be very important in many applications, for example, when solving the radiation transfer problems or in fluorescence measurements.

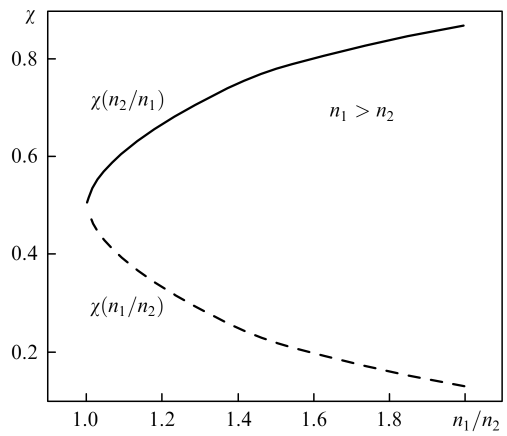

Figure 5. Probabilities of finding photons in different phases of the inhomogeneous medium during their interaction with the phase interface as a function of the relative refractive index.

Taking into account the probabilities of finding a photon in different phases of the medium, we can generalise the expression for the scattering phase function in the form:

$$p'(\gamma, n) = \chi(n)p(\gamma, n) + \chi(n^{-1})p(\gamma, n^{-1}). \quad (10)$$

This expression already includes the statistical sampling of the phase and, hence, there is no need to find out in which specific phase of the medium is the photon when determining the deflection angles of the light beam.

4.2 Anisotropy coefficient

The knowledge of the scattering phase function allows one to estimate the anisotropy coefficient, or in other words, the mean cosine of the deflection angle of photons during their propagation in an inhomogeneous medium:

$$g(n) \equiv \langle \cos\gamma \rangle = \int_0^{\pi} \int_0^{2\pi} p'(\gamma, n) \sin\gamma \cos\gamma \mathrm{d}\gamma \mathrm{d}\varphi. \quad (11)$$

This coefficient is usually employed as one of the main parameters characterising the scattering properties of the random medium. It is commonly calculated from the experimental data by simulating the light propagation in a medium with the help of the phenomenological Henyey–Greenstein phase function. Figure 6 shows the dependence of the anisotropy coefficient on the relative refractive index. Note that for $n = 1.5/1.33$, the anisotropy coefficient g is 0.933 and for $n = 1.5$ it is 0.721. These values of n are chosen not accidentally, they are often used by different authors to simulate the light propagation in scattering media with the help of the Mie theory. The value $n = 1.5/1.33$ approximately corresponds to the ratio of refractive indices of organic components of biological tissues and water in the optical spectrum. It is interesting to compare the behaviour of the function $p'(\gamma, n)$ and the Henyey–Greenstein phase function for equal anisotropy coefficients. Figure 7 shows the Henyey–Greenstein function calculated for $g = 0.721$ and the function $p'(\gamma, n)$ calculated for $n = 1.5$. Both curves are close to each other, which, in principle is not surprising but can serve as an additional argument in favour of the proposed alternative approach. First of all, its advantages are the physical basis and the possibility to take into account the light polarisation, which will be described below.

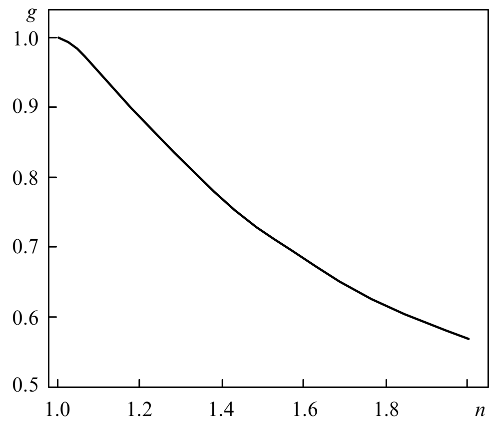

Figure 6. Anisotropy coefficient $g(n)$ calculated by (11) for different relative refractive indices n.

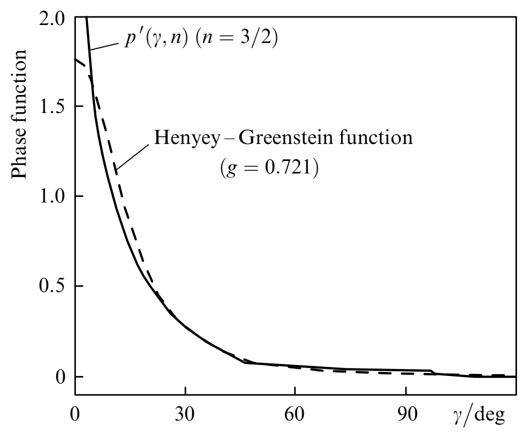

Figure 7. Comparison of the function $p'(\gamma, n)$ with the scattering Henyey–Greenstein phase function for the same value of the g factor ($n = 3/2$, $g = 0.721$).

4.3 Sampling

If the probability density $p'(\gamma, n)$ is known, the sampling of the deflection angle is not difficult by using modern computers. To do this, it is sufficient to solve the equation for the unknown parameter x:

$$2\pi \int_0^x p'(\gamma, n) \sin\gamma \mathrm{d}\gamma = \mathrm{RND}, \tag{12}$$

where RND is a random number between 0 and 1 with the homogeneous distribution function. This sampling can be required to simulate the light propagation in a medium by the Monte-Carlo method by using the proposed phase function. The azimuthal angle is generated simply as $\varphi = 2\pi\mathrm{RND}$. The sampling of the photon free path L between the scattering events is performed according to the usual scheme: $L = -\ln(\mathrm{RND})/\mu_s$, where μ_s is the scattering coefficient.

5. Scattering matrix

Let us assume that a light beam incident on a randomly oriented phase interface is partially polarised (Fig. 1) and has the Stokes vector $\boldsymbol{I}_i$ in the laboratory Cartesian coordinate system xyz. We will define the Stokes vector $\boldsymbol{I}_s(\gamma, \varphi)$ for the beam deflected in the direction specified by the angles γ, φ, in the virtual coordinate system $x''y''z''$, in which the z'' axis is directed along the deflected beam, the y'' axis lies in the plane of incidence and the x'' axis is perpendicular to the plane of incidence. Let us introduce additional virtual coordinate axes x' and y' lying in the plane xy and directed perpendicular and parallel to the plane of the light incidence on the phase interface, respectively (see Fig. 1). The counter-clockwise rotation of the coordinate system by the angle $\pi/2 - \varphi$ yields the first transformation of the Stokes vector:

$$\boldsymbol{I}'_i = K(\varphi)\boldsymbol{I}_i, \tag{13}$$

where

$$K(\varphi) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{14}$$

is the transformation matrix during the rotation of the coordinate axes.

If the incident beam is deflected by the angle γ, the Stokes vector changes in accordance with the corresponding Müller matrices for reflection $R(\eta_r(\gamma))$ and refraction $T(\eta_t(\gamma))$ [36–38]:

$$R(\eta_r(\gamma)) = \frac{1}{2} \begin{vmatrix} r_p r_p^* + r_s r_s^* & r_p r_p^* - r_s r_s^* & 0 & 0 \\ r_p r_p^* - r_s r_s^* & r_p r_p^* + r_s r_s^* & 0 & 0 \\ 0 & 0 & 2\mathrm{Re}(r_p^* r_s) & 0 \\ 0 & 0 & 0 & 2\mathrm{Re}(r_p^* r_s) \end{vmatrix}, \tag{15}$$

$$T(\eta_t(\gamma)) = \frac{\mathrm{Re}[n\cos(\eta_t(\gamma))]}{2\cos\gamma} \times \begin{vmatrix} t_p t_p^* + t_s t_s^* & t_p t_p^* - t_s t_s^* & 0 & 0 \\ t_p t_p^* - t_s t_s^* & t_p t_p^* + t_s t_s^* & 0 & 0 \\ 0 & 0 & 2\mathrm{Re}(t_p^* t_s) & 0 \\ 0 & 0 & 0 & 2\mathrm{Re}(t_p^* t_s) \end{vmatrix}, \tag{16}$$

where r_s, r_p, t_s and t_p are the corresponding amplitude Fresnel coefficients depending on the deflection angle γ via the functions $\eta_r(\gamma)$ and $\eta_t(\gamma)$ (see Appendix); the asterisk means complex conjugation.

In our case, the light scattered in the given direction consists of two statistically independent beams, which are produced during reflection and refraction of the incident light beam on the randomly oriented phase interface. Because the components of the Stokes vector obey the superposition principle [36], the required Stokes vector $\boldsymbol{I}_s(\gamma, \varphi)$ is equal to the sum of the Stokes vectors of reflected and refracted beams. We will take into account that the angular distribution of the scattered light intensity and the probability density of the deflection angles are equal within a factor. Therefore, the equation for the Stokes vector $\boldsymbol{I}_s(\gamma, \varphi)$ in the matrix form should have the same form as Eqn (5) for the probability density of the deflection angle, but instead of the Fresnel coefficients we should substitute the corresponding Müller matrices $R(\eta_r(\gamma))$ and $T(\eta_t(\gamma))$:

$$\boldsymbol{I}_s(\gamma, \varphi) = [Q_r(\gamma)R(\eta_r(\gamma)) + Q_t(\gamma)T(\eta_t(\gamma))]K(\varphi)\boldsymbol{I}_i. \tag{17}$$

For convenience we introduce here functions $Q_r(\gamma)$ and $Q_t(\gamma)$, which are cofactors of the Fresnel coefficients in (5). Eqn (17) also includes the phase matrix of a randomly oriented phase interface

$$Z(\gamma, \varphi) = [Q_r(\gamma)R(\eta_r(\gamma)) + Q_t(\gamma)T(\eta_t(\gamma))]K(\varphi). \tag{18}$$

The first element (Z_{11}) of this matrix is a scattering phase function [29], and one can easily see that it coincides with the expression for $p(\gamma, n)$ [see Eqn (5)]. The phase matrix for non-spherical particles is usually expressed by the amplitude matrix, which relates the amplitude of electric components of the incident plane wave with the amplitudes of the electric components of the scattered light [30]. For a randomly oriented phase interface, this procedure probably cannot be fulfilled, because the deflected light beam is a superposition of two incoherent polarised beams formed at different angular positions of the interface. In this connection, the transformation of the Stokes vector for a partially polarised light by the proposed method seems very convenient.

6. Conclusions

The formation of an optically inhomogeneous medium has been described in this paper as a statistical process of orientation of the normal to the phase interface with respect to the light beam direction. An analytic expression has been obtained for the probability density of the light beam deflection in any solid angle, which is a physical analogue of the phase function of light scattering by particles. It consists of two terms describing the reflection and refraction of the light flux in the given direction by the randomly oriented surface. Due to the use of Fresnel reflection and transmission coefficients as statistical weights of the random process, different photon fluxes interacting with the phase interfaces are independent. This allows the additive approach to be used to study the interaction of stochastic photons with phase interfaces.

We have calculated the probabilities of finding the randomly walking photon in different phases of the medium, which allowed us to include in the statistical process of the slope of randomly oriented surface the choice of the phases of the medium from which the light falls on this surface and to generalise the expression for the probability density of the beam deflection in the given direction. The comparison of the phase function obtained in this paper with the Henyey–Greenstein function has shown that for the same anisotropy coefficients their behaviour is similar except the region of small deflection angles, where the proposed model predicts a sharp peak.

Apart from the physical foundation, the undoubted advantage of this model is the possibility to calculate the propagation of polarised light in an inhomogeneous medium. The expression for the scattering matrix has been obtained, which relates the parameters of the Stokes vectors of deflected beams and the incident beam. This allows one to abandon the Mie theory in simulating the propagation of polarised light in such media as biological tissues.

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Appendix. Fresnel formulas as functions of the deflection angle

Recall the expressions for the amplitude Fresnel coefficients for the reflected and transmitted electromagnetic waves:

$$r_{\rm p}(\theta_{\rm i}) = \frac{\cos\theta_{\rm t} - n\cos\theta_{\rm i}}{\cos\theta_{\rm t} + n\cos\theta_{\rm i}}, \tag{A1}$$

$$t_{\rm p}(\theta_{\rm i}) = \frac{2\cos\theta_{\rm i}}{\cos\theta_{\rm t} + n\cos\theta_{\rm i}}, \tag{A2}$$

$$r_{\rm s}(\theta_{\rm i}) = \frac{\cos\theta_{\rm i} - n\cos\theta_{\rm t}}{\cos\theta_{\rm i} + n\cos\theta_{\rm t}}, \tag{A3}$$

$$t_{\rm s}(\theta_{\rm i}) = \frac{2\cos\theta_{\rm i}}{\cos\theta_{\rm i} + n\cos\theta_{\rm t}}, \tag{A4}$$

where $\theta_{\rm i}$ is the angle of incidence on the interface of two media; and $\theta_{\rm t}$ is the angle of refraction. Strictly speaking, the value of $\cos\theta_{\rm t}$ is complex and can be calculated by using the Snell law [38]:

$$\cos\theta_{\rm t} = \begin{cases} (1 - \sin^2\theta_{\rm i}/n^2)^{1/2} & \text{for } \sin\theta_{\rm i} \leqslant n, \\ -{\rm i}(\sin^2\theta_{\rm i}/n^2 - 1)^{1/2} & \text{for } \sin\theta_{\rm i} > n \end{cases} \tag{A5}$$

Expression (A5) for $\cos(\theta_{\rm t})$ allows one to calculate the reflection and transmission coefficients as:

$$R_{\rm p,s}(\theta_{\rm i}) = r_{\rm p,s}(\theta_{\rm i})r^*_{\rm p,s}(\theta_{\rm i}), \tag{A6}$$

$$T_{\rm p,s}(\theta_{\rm i}) = {\rm Re}\left[t_{\rm p,s}(\theta_{\rm i})t^*_{\rm p,s}(\theta_{\rm i})\frac{n\cos\theta_{\rm t}}{\cos\theta_{\rm i}}\right]. \tag{A7}$$

By replacing the angle of incidence $\theta_{\rm i}$ by functions $\eta_{\rm r}(\gamma)$ and $\eta_{\rm t}(\gamma)$ in the right-hand sides of Eqns (A6) and (A7), respectively, we obtain the reflection and transmission coefficients as functions of the deflection angle. Figures 1A and 2A show their behaviour for $n = 3/2$ and $2/3$. Note that reflection and refraction events here occur at different deflection angles.

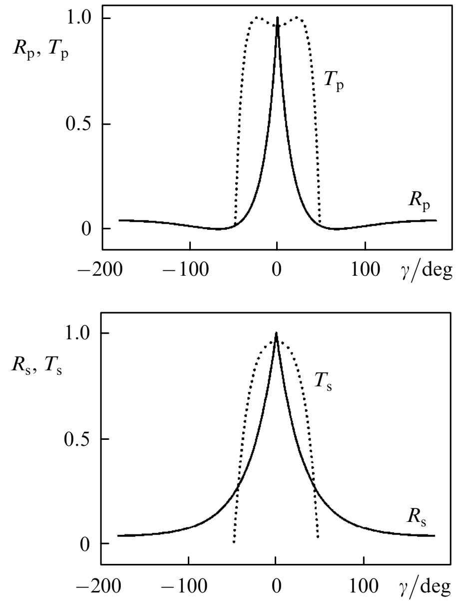

Figure 1A. Fresnel reflection ($R_{\rm p,s}$) and transmission ($T_{\rm p,s}$) coefficients for the phase interface with the relative refractive index $n = 3/2$ as a function of the deflection angle for parallel ($R_{\rm p}$, $T_{\rm p}$) and perpendicular ($R_{\rm s}$, $T_{\rm s}$) polarisations with respect to the plane of incidence of light.

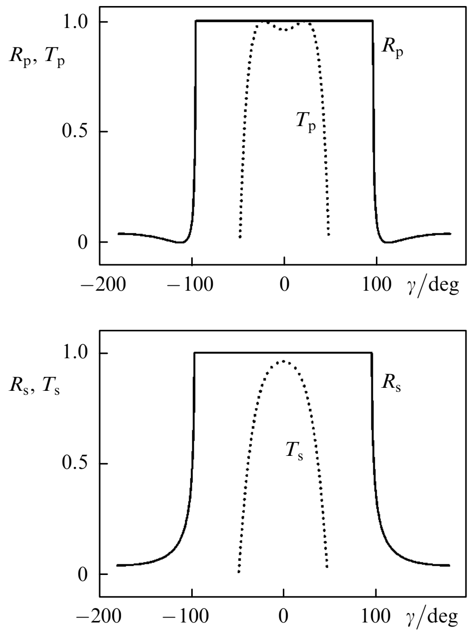

Figure 2A. Same as in Fig. 1A but for $n = 2/3$.

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