RESONATORS, MODES

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Analytic method for the construction of the fundamental mode of a resonator in the form of a Gaussian beam with complex astigmatism

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***Abstract*. Explicit formulas are obtained for a resonator with the fundamental mode in the form of a Gaussian beam with complex astigmatism. The formulas describe the parameters of the beam directly in terms of the ray matrix without using the procedure of finding its eigenvectors. An example is considered.**

Keywords: *Gaussian beam, astigmatism, fundamental resonator mode.*

1. The propagation of a light field in resonators, in which a Gaussian beam with complex astigmatism is formed, was considered in many papers (see, for example, [1, 2]). In this case, the function describing the transverse distribution of the fundamental mode field has the form

$$u(r) = c\exp(\mathrm{i}kr^{\mathrm{t}}Hr/2),$$

where

$$r = \begin{pmatrix} x \\ y \end{pmatrix};\ r^{\mathrm{t}} = (x\ \ y);\ H = \begin{pmatrix} 1/q_x & 1/q_{xy} \\ 1/q_{xy} & 1/q_y \end{pmatrix}$$

is the quadratic matrix. The matrix H is symmetric and has the positive definite imaginary part for a beam concentrated in the vicinity of the resonator axis. This matrix satisfies the matrix equation [2]

$$H = (C + DH)(A + BH)^{-1}, \tag{1}$$

where A, B, C, and D are the real 2×2 matrices (for a passive resonator without losses). These matrices form the 4×4 ray matrix of the round trip in the resonator (monodromy matrix [1])

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

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The matrix T is symplectic [1, 2], which is equivalent to the fulfilment of the condition

$$T^{-1} = \begin{pmatrix} D^{\mathrm{t}} & -B^{\mathrm{t}} \\ -C^{\mathrm{t}} & A^{\mathrm{t}} \end{pmatrix}. \tag{2}$$

The resonator is stable if all the eigenvalues of the matrix T are modulo unity and they have no associated vectors [1]. In this case, Eqn (1) has a symmetric solution with the positive imaginary part. Such a solution is usually constructed by using the components of the eigenvectors of the monodromy matrix. In this paper, we propose the alternative method for solving Eqn (1) in which the matrix H is expressed directly in terms of matrices A, B, C, and D. In this case, there is no need to seek the eigenvectors of the matrix T.

Note that this method can be applied, along with ring resonators, to linear two-mirror resonators with elliptic (hyperbolic) mirrors and (for the three-dimensional manifold) to the problem of a Gaussian beam concentrated in the vicinity of the closed geodesic considered in [1].

2. Let the matrix H – the symmetric solution of Eqn (1) with the positive definite imaginary part, be related to a matrix H' by the expression

$$H = (\tilde{C} + \tilde{D}H')(\tilde{A} + \tilde{B}H')^{-1}, \tag{3}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ are the blocks of a real symplectic matrix T. Then, H' is also a symmetric matrix with the positive definite imaginary part, and it in turn satisfies the equation

$$H' = (C' + D'H')(A' + B'H')^{-1}, \tag{4}$$

which is similar to (1), where A', B', C', and D' are the blocks of the symplectic matrix T' related to the matrix T by the similarity transformation

$$T' = \tilde{T}^{-1}T\tilde{T}. \tag{5}$$

The further strategy of solving Eqn (1) involves the selection of the sequence of transformations of type (5) reducing the general case to matrices T' of the special form for which Eqn (4) can be solved directly. Figure 1 presents the block diagram of the algorithm for solving the problem. This algorithm can be used both for analytic and numerical solution of the problem.

3. The characteristic polynomial of the symplectic matrix is reciprocal, i.e. if λ is an eigenvalue of T, then λ^{-1} is also an eigenvalue of this matrix. Then,

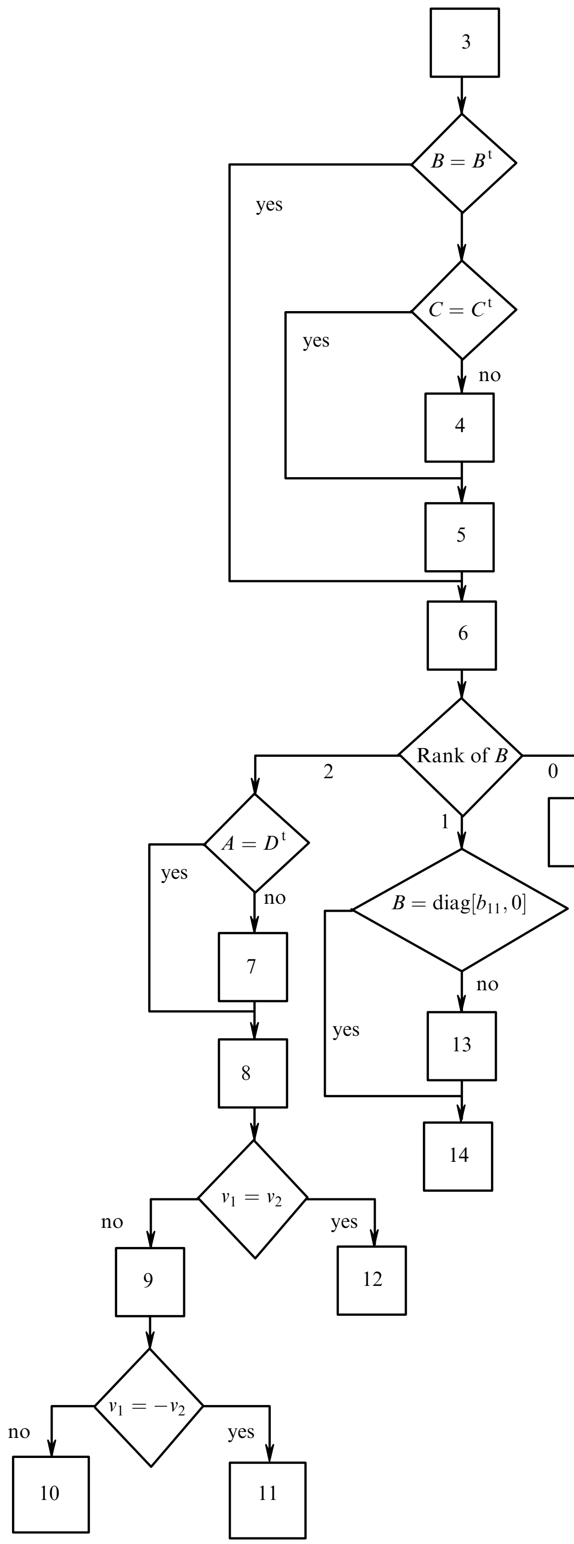

Figure 1. Logical scheme of the algorithm for solving the problem. Numbers in rectangles correspond to sections 3–15 where the corresponding constructions are described.

$$v = \frac{\lambda + \lambda^{-1}}{2}$$

is an eigenvalue of the matrix

$$\Xi = \frac{1}{2}(T + T^{-1}) = \frac{1}{2}\begin{pmatrix} A + D^t & B - B^t \\ C - C^t & D + A^t \end{pmatrix} \tag{6}$$

and satisfies the equation

$$v^2 + pv + q = 0\,, \tag{7}$$

where

$$p = -\frac{\mathrm{tr}(A + D^t)}{2}\,;$$

$$q = \frac{\det(A + D^t) + (b_{12} - b_{21})(c_{12} - c_{21})}{4}\,.$$

We represent the roots of Eqn (7) in the form $v_{1,2} = \cos\theta_{1,2}$, where $\theta_{1,2}$ are some numbers; then the eigenvalues of the matrix T can be represented in the form $e^{\pm i\theta_j}$, $j = 1, 2$. These numbers are modulo unity if both of the roots of Eqn (7) are real and their moduli do not exceed unity, then the values of $\theta_{1,2}$ are real.

When the values of v_1 and v_2 coincide ($v_1 = v_2 = v$), the eigenvalues prove to be multiple, and to verify the stability, it is necessary to make sure that the matrix T has no associated vectors. For $v = \pm 1$, the multiplicity of the eigenvalue is four. In this case, if the matrix T has no associated vectors, it obviously coincides with $\pm E_4$ (E_4 is the 4×4 unit matrix). In the case $v \neq \pm 1$, the eigenvalues of the matrix T are doubly degenerate, but for the matrix Ξ (6) v is the fourfold degenerate eigenvalue, so that in this case the fulfilment of the relation $\Xi = vE_4$ is necessary (and sufficient, as can be shown) for stability. This means that B and C are symmetric and the matrix

$$G = \frac{A + D^t}{2} \tag{8}$$

coincides accurate to the factor v with the 2×2 unit matrix E: $G = vE$. Another case of the multiple eigenvalues of T, when one or both values of v_j become ± 1 but do not coincide with each other, is more complicated and requires a special consideration.

Note that if the matrix B or C proves to be symmetric, the solution of Eqn (1) is substantially simplified. In particular, the roots of Eqn (7) coincide with the eigenvalues of the matrix G (8). The eigenvectors of G in this case can be obviously chosen real.

4. Consider the general case when $B \neq B^t$, $C \neq C^t$ (otherwise we pass to section 6 or 5, respectively). To simplify the equation, we first symmetrise the block C. We seek the matrix $\tilde{T}$ in the special form

$$\tilde{T} = \begin{pmatrix} E & O \\ z\Phi & E \end{pmatrix},$$

where $\Phi = \Phi^t$ is a symmetric real 2×2 matrix; O is the 2×2 zero matrix; and z is the real factor to be determined. Then,

$$T' = \begin{pmatrix} A + zB\Phi & B \\ C + z(D\Phi - \Phi A) - z^2\Phi B\Phi & D - z\Phi B \end{pmatrix},$$

and

$$H = H' + z\Phi. \tag{9}$$

The condition of the matrix symmetry

$$C' = C + z(D\Phi - \Phi A) - z^2 \Phi B \Phi$$

gives the equation for z

$$(c_{12} - c_{21}) + [-(a_{12} + d_{21})\phi_{11} + (a_{11} - a_{22} + d_{11} - d_{22})\phi_{12}$$

$$+ (a_{21} + d_{12})\phi_{22}]z - (b_{12} - b_{21})(\det \Phi)z^2 = 0;$$

the choice of the matrix Φ should provide its solvability. In particular, when the signs of $\det \Phi$ and $(b_{12} - b_{21})(c_{12} - c_{21})$ coincide, the discriminant of this equation will be certainly positive and the roots – real. Each of these roots allows one, by making substitution (9), to pass to Eqn (4) with the symmetric block C'. For example, in the case $(b_{12} - b_{21}) \times (c_{12} - c_{21}) > 0$, we can set $\Phi = E$, and if $(b_{12} - b_{21}) \times (c_{12} - c_{21}) < 0$, then

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Of interest is also the case when Φ is the degenerate symmetric matrix which has the form accurate to a factor

$$\Phi_\varphi = \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & 1 - \cos(2\varphi) \end{pmatrix}. \quad (10)$$

In this case, the equation for z proves to be linear and has the only solution if the coefficient at z in nonzero. It can be always achieved by a proper choice of φ (in particular, $\varphi = 0$, $\pi/2$ or $\pi/4$) except the case

$$a_{21} + d_{12} = a_{12} + d_{21} = a_{11} - a_{22} + d_{11} - d_{22} = 0,$$

when the antisymmetric part of the matrix $A\Phi - \Phi D$ vanishes for any symmetric matrix Φ, so that the equation for z does not contain a linear term.

The choice of the matrix Φ is in fact the only informal moment in our paper and it determines the level of complexity and awkwardness of further calculations, especially if we seeking the solution analytically rather than numerically. However, this choice will not affect the final result if only the equation for z has real roots.

5. The next step is the transformation of the matrix with the symmetric block C to the matrix with the symmetric block B. By selecting

$$\tilde{T} = \begin{pmatrix} O & E \\ -E & O \end{pmatrix},$$

we obtain

$$T' = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix},$$

and

$$H = -(H')^{-1}.$$

Note that such a transition for a Gaussian beam corresponds to the Fourier transform over transverse coordinates.

6. We reduced the problem to a particular case of the matrix with the symmetric block B (recall that the matrix B relates the transverse projection of the unit vector directed along the axial beam emerging from the origin of coordinates and the radius vector of the point through which this beam will pass after the round trip in the resonator). The condition of the matrix symmetry means that the matrix B has two real mutually orthogonal eigenvectors. Note that some problems can have this property initially, in particular, the problem of a linear resonator with elliptic mirrors if the monodromy matrix is considered for the cross section adjacent to one of the mirrors; in this case, the directions of eigenvectors B coincide with the principal directions of the curvature of the opposite mirror S_2 (Fig. 2). Indeed, if the transverse projection of the wave vector is directed along one of these eigenvectors, the initial and singly reflected beams will remain in the plane formed by this vector and the optical axis, so that the radius vector of the point of intersection of the reflected beam with the initial plane will be collinear to it. The properties of another mirror S_1 do not affect the matrix B: reflection from it will determine the propagation direction of the beam but will not affect the position of the point considered.

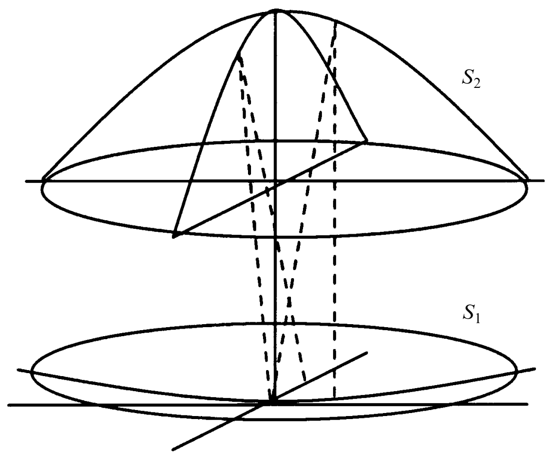

Figure 2. Two-mirror resonator with elliptic mirrors. The dashed straight lines show the beams located in the mutually orthogonal planes. Each of these planes contains the optical axis and one of the eigenvectors of the symmetric matrix B.

Our further operations depend on the rank of the matrix B. Consider first the main case when the matrix $B = B^{\mathrm{t}}$ is nondegenerate (the cases $\det B = 0$, $B \neq O$ and $B = O$ are considered in sections 13 and 15, respectively). The solution for this case was obtained in paper [3]. Let us formulate the results of this paper by changing somewhat the line of the solution in accordance with the method used here.

7. The next step is the passage to the matrix in which the block D is obtained from A by transposition (the case $A = D^{\mathrm{t}}$ is considered in section 8). Consider the symplectic matrix

$$\tilde{T} = \begin{pmatrix} E & O \\ (DB^{-1} - B^{-1}A)/2 & E \end{pmatrix}$$

and perform the transformation of type (5). The resulting matrix will have the form

$$T' = \begin{pmatrix} G & B \\ G^{t}B^{-1}G - B^{-1} & G^{t} \end{pmatrix},$$

the matrix G will be determined by relation (8), while the expression for C' follows from condition (2) for the symplectic matrix (note here that the block C' proves to be symmetric). Matrices H and H', being the solutions of the initial and transformed problems, are related by transformation (3), which takes the form

$$H = \frac{DB^{-1} - B^{-1}A}{2} + H'$$

in this case. By running ahead, note that except for one special case, the term $\tilde{C}$ separated by us coincides with the real part ReH of the matrix H determining the wavefront shape. Correspondingly, the matrix H' proves to be purely imaginary and is responsible for the decay of the field with distance from the resonator axis.

8. Consider now the solution of the problem in the case when $B = B^{t}$ is the nondegenerate symmetric matrix, $A = D^{t} = G$, and $|v_{1,2}| \leqslant 1$ (we will show below that in this case the strict inequality should be fulfilled). Then,

$$T = \begin{pmatrix} G & B \\ G^{t}B^{-1}G - B^{-1} & G^{t} \end{pmatrix},$$

and Eqn (1) takes the form

$$HBH + HG - G^{t}H - G^{t}B^{-1}G + B^{-1} = O. \tag{11}$$

Equation (11) gives two equations

$$HBH - G^{t}B^{-1}G + B^{-1} = O, \quad HG - G^{t}H = O$$

for symmetric and antisymmetric parts, which after multiplying from the left by B, taking condition (2) into account, takes the form

$$(BH)^2 - G^2 + E = O, \tag{12}$$

$$BHG - GBH = O. \tag{13}$$

It follows from Eqn (12) that, when one or both eigenvalues of the matrix G become ± 1, the matrix $(BH)^2$ and, therefore, H proves to be degenerate. Therefore, in this case, to provide the stability of the resonator, both moduli of $v_{1,2}$ should be strictly smaller than unity (unlike the case of the degenerate matrix B, which will be considered below).

9. First we solve the system of equations (12), (13) in the case of $G \neq vE$, $v_1 \neq v_2$ (the opposite case is considered in section 12). It follows from (13) that the matrices BH and G commute and, therefore, BH has the same eigenvectors as G and can be represented as a linear combination of G and the unit matrix, while the matrix H – as a linear combination of $B^{-1}G$ and B^{-1}. Therefore, our task is to find the coefficients of this linear combination.

10. Let us assume that tr$G \neq 0$, $v_1 \neq -v_2$ (otherwise see section 11) and $G^2 \neq v^2 E$. Then, it follows from (21) that tr$(BH) \neq 0$. In addition, it also follows from this equation that $-(BH)^2$ is the matrix with positive eigenvalues equal to $1 - v_{1,2}^2$. Taking into account that the eigenvectors of this matrix are real, BH is a purely imaginary matrix and therefore,

$$H = \mathrm{i}|H|,$$

where $|H|$ is the positive definite real matrix. Equation (12) takes the form

$$(B|H|)^2 = E - G^2. \tag{14}$$

This means, by the way, that if Eqn (1) has form (11) initially but is not reduced to this form by similarity transformations, then for $v_1 \neq \pm v_2$ it characterises the transverse distribution of the filed in the cross section where a Gaussian beam has the waist and the wave front is plane.

Taking into account that an arbitrary 2×2 matrix M satisfies its characteristic equation

$$M^2 - \mathrm{tr}M \times M + \det M \times E = O, \tag{15}$$

and applying (15) to the matrices $B|H|$ and G in Eqn (14), we obtain

$$\mathrm{tr}(B|H|) \times B|H| = (\det(B|H|) + \det G + 1)E - \mathrm{tr}G \times G, \tag{16}$$

and because tr$(BH) \neq 0$ in the case under study, to obtain the required expression for $|H|$, we should find only $\det(B|H|)$ and tr$(B|H|)$.

Let us find the determinants of matrices in the left- and right-hand sides of Eqn (14):

$$\det{}^2(B|H|) = \det(E - G^2) = (1 + \det G)^2 - \mathrm{tr}^2 G;$$

then

$$\det(B|H|) = [\det(E - G^2)]^{1/2} \mathrm{sign} \det B.$$

Let us now calculate trace (16):

$$\mathrm{tr}^2(B|H|) = 2(\det(B|H|) + \det G + 1) - \mathrm{tr}^2 G,$$

which gives

$$|\mathrm{tr}(B|H|)| = \{2(\det(B|H|) + \det G + 1) - \mathrm{tr}^2 G\}^{1/2}.$$

Then, the final expression for the matrix H has the form

$$H = \frac{\mathrm{i}}{|\mathrm{tr}(B|H|)|} \hat{H} \,\mathrm{sign}\, \mathrm{tr}\hat{H},$$

where

$$\hat{H} = [\det(B|H|) + \det G + 1] \times B^{-1} - \mathrm{tr}G \times B^{-1}G.$$

11. Consider now a particular case when G is a traceless matrix: $v_1 = -v_2 = v = \sqrt{-\det G}$, tr$G = 0$. Then,

$$G^2 = v^2 E = -\det G \times E$$

and Eqn (12) is transformed to

$$-(BH)^2 = (1 - v^2)E. \tag{17}$$

After the substitution

$$H = \mathrm{i}h\sqrt{1-v^2}, \tag{18}$$

where h is the symmetric matrix with the positive definite real part, Eqn (12) is transformed to

$$(Bh)^2 = E. \tag{19}$$

It follows from Eqn (19) that the eigenvalues of the matrix Bh are equal to ± 1, and if the eigenvalues coincide, the associated vectors should be absent. This means that either Bh coincides with $\pm E$ or it is a traceless matrix with the determinant equal to -1. Which of these two possibilities is realised depends on the sign of $\det B$. Consider two cases.

(i) Let the matrix B be of fixed sign, $\det B > 0$. The matrix $\mathrm{Re}h$ is positive definite ($\det \mathrm{Re}\, h > 0$) and, therefore, $\det \mathrm{Re}(Bh) > 0$. Then, the signs of the eigenvalues of Re(Bh) coincide and $\mathrm{tr}\mathrm{Re}(Bh) \neq 0$, i.e. Bh is not a traceless matrix. In this case,

$$Bh = \pm E,$$

and (20)

$$h = \pm B^{-1}$$

is a purely real matrix whose sign is determined from the condition of its positive definition and coincides with the sign of $\mathrm{tr}B$. For the matrix H, we have finally

$$H = \mathrm{i}\sqrt{1-v^2}B^{-1}\mathrm{sign}\,\mathrm{tr}B. \tag{21}$$

(ii) Let now the matrix B not be of fixed sign, $\det B < 0$. Then, (20) is no longer the required solution because it is also not a matrix of fixed sign. Therefore, in this case, Bh is the matrix with the eigenvalues $+1$ and -1, the zero trace, and the determinant equal to -1. Because, according to (13), this matrix commutes with G, it should coincide with G accurate to a factor. The final expression for the matrix H has the form

$$H = \mathrm{i}\frac{\sqrt{1-v^2}}{v}B^{-1}G\,\mathrm{sign}\,\mathrm{tr}(B^{-1}G).$$

In both cases, the matrix H again proves to be purely imaginary.

12. Consider now the case when $v = v_1 = v_2$. In this case, as pointed out above, the condition of the resonator stability is the equality $G = vE$. Then, Eqn (13) is fulfilled automatically and (12) again takes form (17). By following section 11, we again obtain expression (21) for $\det B > 0$, while the case $\det B < 0$, Bh again proves to be a matrix with the zero trace and the determinant equal to -1. However, now we have no additional information on the form of this matrix, and the solution of this problem is certainly ambiguous.

To describe a family of the corresponding matrices, we represent B as a linear combination of the unit matrix E and the traceless matrix σ with the determinant equal to -1:

$$B = b_0E + b\sigma, \quad b_0 = \frac{\mathrm{tr}B}{2}, \quad b = (b_0^2 + d)^{1/2},$$

$$d = |\det B| = -\det B, \quad \sigma = \begin{pmatrix} c & s \\ s & -c \end{pmatrix},$$

where c and s are some numbers and $c^2 + s^2 = 1$. Consider also the matrix

$$\sigma' = \begin{pmatrix} s & -c \\ -c & -s \end{pmatrix}.$$

Then, the required family of solutions of Eqn (19) has the form

$$h = \frac{(1+\zeta^2)(bE - b_0\sigma) + 2\zeta\sqrt{d}\sigma'}{(1-\zeta^2)d}, \tag{22}$$

where ζ is a complex parameter. Matrix (22) has the positive definite real part for $|\zeta| < 1$; correspondingly, matrix H (18) has the positive definite imaginary part for the same ζ.

The simplest example of a resonator in which such an ambiguous solution is realised is the three-mirror resonator [4] with two plane and one elliptic mirror with radii of curvature selected so that to provide the coincidence of the eigenvalues of the monodromy matrix.

The consideration of cases for which B is a nondegenerate symmetric matrix is now completed.

13. Let now $B = B^{\mathrm{t}}$ be a degenerate symmetric nonzero matrix. In this case, it can be represented in the form

$$B = \mathrm{tr}B \times \Phi_\varphi,$$

where Φ_φ is the matrix of type (10), $\mathrm{tr}B \neq 0$, and $\varphi = \arctan(b_{12}/b_{11})$ (or $\varphi = \pi/2$ for $b_{12} = b_{11} = 0$).

The degeneracy of the matrix B means physically that geometric optical rays propagate along one of the directions in the focusing cross section under study. In particular, the matrix B is degenerate in the above-mentioned problem of a two-mirror resonator if one of the radii of curvature of the opposite mirror is equal to the distance between the mirrors. In this case, the beams for which the transverse projection of the ray vector is parallel to the corresponding principal direction of the curvature will return after reflection to the origin of coordinates.

Let us diagonalise the matrix B by rotating the coordinate axes through angle φ (we assume that $\varphi \neq 0$; otherwise see section 14). For this purpose, we consider the symplectic matrix

$$\tilde{T} = \begin{pmatrix} U_\varphi & O \\ O & U_\varphi \end{pmatrix},$$

where

$$U_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{23}$$

is the matrix of rotation through angle φ, and perform the transformation of type (5). The resulting matrix will have the form

$$T' = \begin{pmatrix} U_{-\varphi}AU_\varphi & U_{-\varphi}BU_\varphi \\ U_{-\varphi}CU_\varphi & U_{-\varphi}DU_\varphi \end{pmatrix}.$$

It is easy to verify that

$$B' = U_{-\varphi}BU_\varphi = \begin{pmatrix} \mathrm{tr}B & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrices H and H', being the solutions of the initial and transformed problems, are related by transformation (3), which takes in this case the form

$$H = U_{\varphi} H' U_{-\varphi}.$$

14. The problem with the degenerate nonzero matrix $B = B^{\mathrm{t}}$ is reduced to the case when the only matrix element b_{11} is nonzero. In this case, it follows from the condition that the matrix T is symplectic that, in particular,

$$a_{21} = d_{12} = 0, \quad a_{11}d_{11} - b_{11}c_{11} = a_{22}d_{22} = 1. \tag{24}$$

The matrix G (8) proves to be upper triangular, so that the roots of Eqn (7) coincide with its diagonal elements, i.e., $v_i = (a_{ii} + d_{ii})/2$, $i = 1, 2$. In addition, it follows from the structure (24) of the matrix T that the matrix elements a_{22} and d_{22} are its eigenvalues, while the other two eigenvalues of T are the eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix}.$$

Let us formulate the conditions providing the resonator stability. First of all the values of a_{22} and d_{22} should be equal to ± 1 and coincide: $a_{22} = d_{22} = v_2 = \pm 1$ (unlike the case of the nondegenerate matrix $B = B^{\mathrm{t}}$). Thus, one of the eigenvalues of T is doubly degenerate. The condition of the absence of associated vectors has the form

$$2(v_1 - v_2)c_{22} = a_{12}c_{21} + d_{21}c_{12}.$$

At the same time, the modulus of $v_1 = \cos\theta$ should be strictly smaller than unity because when the eigenvalues become ± 1, the nonzero value of b_{11} results in the appearance of associated vectors. (Note that for a two-mirror resonator it follows from the stability conditions, in particular, that one of the principal radii of curvature for the second mirror should be also equal to the distance between the mirrors).

Let the stability conditions be fulfilled. By rewriting matrix equation (1) in the form of the system of algebraic equations for elements of the matrix H, we can find that the two of them are quadratic equations (one for h_{11} and another for $h_{12} = h_{21}$), while the other two contain both these elements and allow one to express linearly one of the elements in terms of another. The system of equations is compatible when the symplectic and stability conditions are fulfilled and it has two solutions differing in the signs of imaginary parts. Let us write one of the solutions:

$$h_{11} = \frac{1}{b_{11}}\left(\frac{d_{11} - a_{11}}{2} + \mathrm{i}\sin\theta\right),$$
$$h_{12} = \frac{1}{b_{11}}\left[\frac{d_{21} - a_{12}}{2} + \mathrm{i}\frac{a_{12} + d_{21}}{2(v_1 - v_2)}\sin\theta\right]. \tag{25}$$

By selecting the sign of θ in expressions (25) coinciding with that of b_{11}, we obtain $\operatorname{Im} h_{11} > 0$, which is necessary for $\operatorname{Im} H$ to be positive definite.

Equation (1) in the case under study imposes no restrictions on h_{22}, so that expressions (25) for all possible values of this element determine one of the two families of solutions of this equation. In this case, $\operatorname{Im} H$ is positive definite if

$$\operatorname{Im} h_{22} > \frac{(a_{12} + d_{21})^2}{4(v_1 - v_2)^2}\sin\theta.$$

15. Consider now the case $B = B^{\mathrm{t}} = O$. In this case, all the beams emerging from the origin of coordinates are focused to one point after the round trip in the resonator. In the problem of a two-mirror resonator, this corresponds to the situation when the opposite mirror is spherical and its radius is equal to the distance between mirrors. Because the matrix T is symplectic, we have $AD^{\mathrm{t}} = E$, $A^{\mathrm{t}}C = C^{\mathrm{t}}A$. Equation (1) takes the form

$$HA - DH = C. \tag{26}$$

The eigenvalues of T in this case are the eigenvalues of matrices A and D and are equal to unity for the stable matrix. It follows from this, in particular, that the determinants of these matrices are equal to $+1$ or -1. In the first case, the eigenvalues are $\mathrm{e}^{\pm\mathrm{i}\theta}$, and A and D are unimodular matrices with the trace $2\cos\theta$. In the second case, their eigenvalues are $+1$ and -1, while matrices A and D are traceless. Consider these situations separately.

In the first case, the eigenvalues are fourfold ($\theta = 0, \pi$) or doubly degenerate. In the case of the fourfold degeneracy, the conditions $A = D = \pm E$, $C = O$ should be fulfilled to provide the matrix stability, and then an arbitrary symmetric matrix H with the positive definite imaginary part will satisfy Eqn (26). It is this situation that is realised in a two-mirror resonator in which the second mirror also should be spherical with the same radius of curvature (confocal resonator). Such a resonator was considered in [5].

Consider now the case $\theta \neq 0, \pi$. In this case, it follows from the condition of the absence of associated vectors, which simultaneously provides the solvability of Eqn (26) that the matrix C should be symmetric; taking the symmetry of $A^{\mathrm{t}}C$ into account, this means that either $C = 0$ or $\det C < 0$. Taking into account that the matrix is symplectic, the solution of the system of linear equations following from (26) can be written in the form

$$H = \frac{AC - CD}{(a_{11} - a_{22})^2 + 2(a_{12}^2 + a_{21}^2)} + \zeta\begin{pmatrix} -2a_{21} & a_{11} - a_{22} \\ a_{11} - a_{22} & 2a_{12} \end{pmatrix}. \tag{27}$$

Expression (27) at different values of the complex parameter ζ determines a family of the solutions of Eqn (26). The imaginary part of H is positive definite when the sign of $\operatorname{Im}\zeta$ coincides with that of $a_{12} - a_{21}$ (here, this value does not vanish).

Note that because the matrix C is symmetric in this case, the transformation considered in section 5 allows us to reduce the problem to the problem considered earlier (if $C \neq O$).

Consider now the second case when the determinants of matrices A and D are equal to -1. The eigenvalues of these matrices are equal to ± 1, the eigenvalues of the matrix T are also equal to ± 1 and doubly degenerate. In this case, $D = A^{\mathrm{t}}$, and it follows from the absence of associated vectors, which provides the resonator stability and solvability of Eqn (26), that the matrix C should be already antisymmetric. The solution of Eqn (26) has the form

$$H = \frac{CA}{2} + \zeta_1(E + DA) + \zeta_2(A + D). \quad (28)$$

Expression (28) for different values of complex parameters ζ_1 and ζ_2 determines the two-parametric family of solutions of Eqn (26). The imaginary part of H is positive definite when $\mathrm{Im}\,\zeta_1 > |\mathrm{Im}\,\zeta_2| \geqslant 0$.

16. It follows from the above analysis that for stable matrices T with the symmetric block B there exists the relation between the rank of the block and the eigenvalues of T: for the nondegenerate matrix B, both values $|v_{1,2}|$ are smaller than unity (and $\theta \neq 0, \pi$); for the degenerate nonzero matrix, one of the values of $|v|$ is equal to unity, while another is smaller than unity; finally, for $B = O$, either $v_1 = v_2 = v \neq \pm 1$, or $|v_{1,2}| = 1$. It also follows from this observation that, if the matrix T with the antisymmetric block is initially unstable, we can predict from the values of $v_{1,2}$ the rank of the symmetric matrix B' after transformation (5) despite the fact that this transformation is certainly not the only one. The uncertainty remains only in the case $v_1 = v_2 \neq \pm 1$, when the resulting matrix B' proves to be either nondegenerate or zero. Of course, the same can be said about the rank of the block C upon its symmetrisation.

Note also that in all the cases transforming to the problem with the degenerate symmetric block B, in particular, each time when at least one of the values of v is modulo unity, the solution of the problem proves to be not the only one (when the stability conditions are fulfilled). The family of solutions can also appear in the case of the nondegenerate block $B = B^t$ when the values of $v_{1,2}$ coincide for $\det B < 0$.

17. Let us illustrate the above discussion by a model example. Consider a multi-mirror ring resonator with the nonplanar axial contour providing the spatial rotation of an image through the angle φ (see, for example, [2, 6]). We assume that the number of mirrors is even and one of them is spherical (or elliptic), the one of the principal direction of the curvature lying in the plane of incidence of the beam; the rest of the mirrors are plane. Let the length of the axial contour be L. For the contour passing along the edges of a tetrahedron in Fig. 3, the angle φ is equal to the sum of dihedral angles between the faces intersecting along these edges. The propagation along the contour is described by the matrix

$$T_L = \begin{pmatrix} E & LE \\ O & E \end{pmatrix},$$

the rotation through the angle φ – by the matrix

$$T_\varphi = \begin{pmatrix} U_\varphi & O \\ O & U_\varphi \end{pmatrix},$$

where U_φ is matrix (23), and reflection from the elliptic mirror is described by the matrix

$$T_{2\Psi} = \begin{pmatrix} E & O \\ -2\Psi & E \end{pmatrix},$$

where

$$\Psi = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}; \quad \psi_1 = \frac{1}{R_1 \cos\alpha}; \quad \psi_2 = \frac{\cos\alpha}{R_2};$$

α is the angle of incidence and $R_{1,2}$ are the radii of curvature. The monodromy matrix for the cross section located at a distance of $L/2$ from the mirror is calculated by the expression

$$T = T_{\varphi/2} T_{L/2} T_{2\Psi} T_{L/2} T_{\varphi/2}$$
$$= \begin{pmatrix} \gamma U_\varphi + \delta I & \frac{L}{2}[(\gamma+1)U_\varphi + \delta I] \\ \frac{2}{L}[(\gamma-1)U_\varphi + \delta I] & \gamma U_\varphi + \delta I \end{pmatrix}, \quad (29)$$

where

$$\gamma = 1 - \frac{(\psi_1 + \psi_2)L}{2}; \quad \delta = \frac{(\psi_2 - \psi_1)L}{2}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

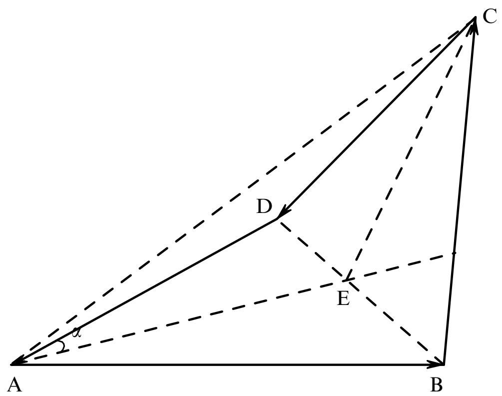

Figure 3. Scheme of the ray contour of the resonator. The arrows show the direction of the round trip. Spherical (elliptic) mirror is located at the point A.

The sufficient conditions for the stability of the matrix T have the form

$$|\delta| < ||\gamma| - |\cos\varphi||, \quad \delta^2 > (\gamma^2 - 1)\sin^2\varphi, \quad |\gamma| < \frac{1}{|\cos\varphi|}. \quad (30)$$

For $\varphi = \pi/3$, set (30) is shown in Fig. 4. The cases when inequalities in (30) are transformed to equalities correspond to the appearance of multiple eigenvalues and require a separate study. We will not do it in this paper and intend to investigate this problem in detail is a separate paper. Here, we restrict ourselves to the simplest case when $(b_{12} - b_{21}) \times (c_{12} - c_{21}) > 0$, i.e. $|\gamma| > 1$. For definiteness, we also assume that the values of γ, δ, $\cos\varphi$ and $\sin\varphi$ are positive.

We start the transformation of matrix (29) from the unnecessary but useful scaling procedure resulting in the simplification of the matrix. Let

$$\tilde{T}_0 = \begin{pmatrix} \sqrt{\frac{L}{2}}E & O \\ O & \sqrt{\frac{2}{L}}E \end{pmatrix}.$$

Then, $T_0 = \tilde{T}_0^{-1} T \tilde{T}_0$ is the matrix with blocks

$$A_0 = D_0 = \gamma U_\varphi + \delta I, \quad B_0 = A_0 + U_\varphi, \quad C_0 = A_0 - U_\varphi,$$

and

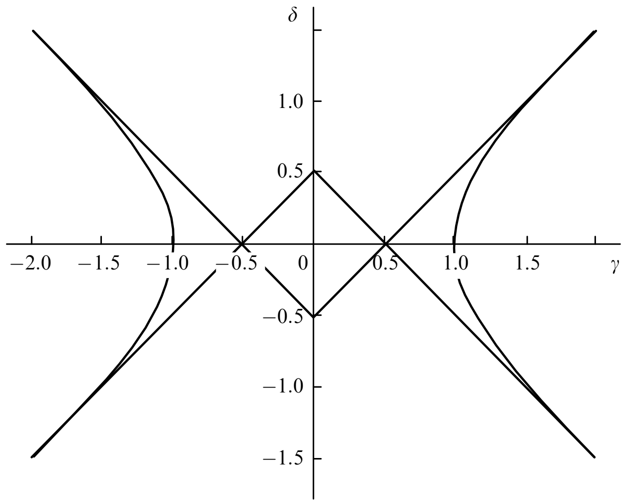

Figure 4. Stability region of the resonator in the plane of parameters δ and γ for $\varphi = \pi/3$ (square and two curvilinear triangles).

$$H = \frac{2}{L} H_0.$$

The next stage is the symmetrisation of the block C (see section 4). We select

$$\tilde{T}_1 = \begin{pmatrix} E & O \\ zE & E \end{pmatrix};$$

the value of z is determined from the condition of the symmetry of the block C of the matrix $T_1 = \tilde{T}_1^{-1} T_0 \tilde{T}_1$. This condition leads to the relations

$$z = \pm\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}, \quad \gamma = \frac{1 + z^2}{1 - z^2};$$

in this case, $H_0 = zE + H_1$. Let us select for definiteness the positive direction of z and write the blocks of the matrix T_1:

$$A_1 = (1 + z)\left(\frac{U_\varphi}{1 - z} + \delta I\right), \quad B_1 = \frac{2U_\varphi}{1 - z^2} + \delta I,$$

$$C_1 = (1 - z^2)\delta I, \quad D_1 = (1 - z)\left(\frac{U_\varphi}{1 + z} + \delta I\right).$$

Then, we pass to the matrix T_2, in which already the block B is symmetric: $A_2 = D_1$, $B_2 = -C_1$, $C_2 = -B_1$, $D_2 = A_1$; and $H_1 = -H_2^{-1}$ (see section 5).

Finally, we perform the last transformation in this series by passing to the matrix T_3 with blocks $A_3 = G$, $D_3 = G^{\mathrm{t}}$, $B_3 = B_2$, $C_3 = G^{\mathrm{t}} B_2^{-1} G - B_2^{-1}$ (see section 7), where

$$G = \frac{A_2 + D_2^{\mathrm{t}}}{2}$$

$$= \begin{pmatrix} \gamma\cos\varphi & (\gamma^2 - 1)^{1/2}\sin\varphi \\ -(\gamma^2 - 1)^{1/2}\sin\varphi & -\gamma\cos\varphi \end{pmatrix} + \delta I;$$

$$B_2^{-1} = -\frac{\gamma + 1}{2\delta} I$$

(here, it is convenient to pass again from z to γ). In this case, the real part is separated in the matrix H_2: $H_2 = \operatorname{Re} H_2 + H_3$, where

$$\operatorname{Re} H_2 = \frac{D_2 B_2^{-1} - B_2^{-1} A_2}{2} = -\frac{\gamma + 1}{2\delta}$$

$$\times\begin{pmatrix} (\gamma^2 - 1)^{1/2}\cos\varphi & \gamma\sin\varphi \\ \gamma\sin\varphi & -(\gamma^2 - 1)^{1/2}\cos\varphi \end{pmatrix} - \frac{(\gamma^2 - 1)^{1/2}}{2} E,$$

and H_3 is the purely imaginary matrix with the positive imaginary part (see section 10) satisfying Eqn (14).

Let us find H_3 by using expressions from section 10. For this purpose, we write several auxiliary quantities and matrices:

$$B_3^{-1} G = -\frac{\gamma + 1}{2\delta}\begin{pmatrix} \gamma\cos\varphi & (\gamma^2 - 1)^{1/2}\sin\varphi \\ (\gamma^2 - 1)^{1/2}\sin\varphi & -\gamma\cos\varphi \end{pmatrix}$$

$$-\frac{\gamma + 1}{2} E,$$

$$\operatorname{tr} G = 2\gamma\cos\varphi,$$

$$\det G = \gamma^2 - \delta^2 - \sin^2\varphi,$$

$$\det(B_3 | H_3|) = -|\det(B_3 | H_3|)|$$

$$= -\{[(\gamma + \cos\varphi)^2 - \delta^2][(\gamma - \cos\varphi)^2 - \delta^2]\}^{1/2}$$

(the sign in front of the root coincides with that of $\det B_3$),

$$|\operatorname{tr}(B_3 | H_3|)| = \{2[2\gamma\cos\varphi(1 - \gamma\cos\varphi) + [(\gamma - \cos\varphi)^2 - \delta^2]$$

$$- \{[(\gamma + \cos\varphi)^2 - \delta^2][(\gamma - \cos\varphi)^2 - \delta^2]\}^{1/2}]\}^{1/2}$$

$$= \{2[\{\det^2(B_3 | H_3|) + 4\gamma^2\cos^2\varphi[\delta^2 - (\gamma^2 - 1)\sin^2\varphi]\}^{1/2}$$

$$- |\det(B_3 | H_3|)|]\}^{1/2};$$

radicands are positive in the region of parameters under study. The matrix $\hat{H}$ has the form

$$\hat{H} = (\gamma + 1)\left\{\frac{|\det(B_3 | H_3|)| + \delta^2 - \gamma^2 - \cos^2\varphi}{2\delta} I + \frac{\gamma\cos\varphi}{\delta}\right.$$

$$\left.\times\begin{pmatrix} \gamma\cos\varphi & (\gamma^2 - 1)^{1/2}\sin\varphi \\ (\gamma^2 - 1)^{1/2}\sin\varphi & -\gamma\cos\varphi \end{pmatrix} + \gamma\cos\varphi E\right\},$$

and finally,

$$H_3 = \frac{\mathrm{i}}{|\operatorname{tr}(B_3 | H_3|)|}\hat{H};$$

the sign in the last expression is positive because $\operatorname{tr}\hat{H} = 2(\gamma + 1)\gamma\cos\varphi > 0$.

The expression for the matrix H, being the solution of the initial problem, has the form

$$H = \frac{2}{L}\left[\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2} E - (\operatorname{Re} H_2 + H_3)^{-1}\right]$$

in the region of values of parameters under study.

By substituting matrices $\operatorname{Re} H_2$ and H_3 into this expression, we obtain, after quite cumbersome computer-aided transformations, the expression

$$H = \frac{1}{2L[(\gamma+1)(\gamma+\cos^2\varphi)-\delta^2]} \times \left[\frac{\mathrm{i}|\mathrm{tr}(B_3|H_3|)|}{\gamma}(wE+vI)+2\sin\varphi v\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right],$$

where

$$w = \frac{\gamma^2+(1+2\gamma)\cos^2\varphi-\delta^2-\det(B_3|H_3|)}{\cos\varphi};$$

$$v = \frac{(\gamma+1)[\gamma^2-\cos^2\varphi+\det(B_3|H_3|)]-(\gamma-1)\delta^2}{\delta}.$$

This matrix is one of the four symmetric solutions of Eqn (1). Obviously, the second solution is obtained from it by complex conjugation. The rest of the two solutions can be obtained for the opposite sign of $\det(B_3|H_3|)$ (including $|\mathrm{tr}(B_3|H_3|)|$); the fulfilment of (1) is verified directly. Thus, the solution obtained for one of the subregions of set (30) allowed us to find all the solutions of (1), and the choice of the requires solution for other subregions of (30) was determined from the conditions that the diagonal matrix $\mathrm{Im}H$ is positive definite, i.e. its elements are positive.

Note also without the proof that for the points $\delta = 0$ and $|\gamma| = |\cos\varphi| \neq 0$ dividing the subregions, the problem is reduced to the case $\det B = 0$, $B = B^{\mathrm{t}} \neq O$ considered in sections 13 and 14, and for the point $\delta = \gamma = \cos\varphi = 0$ – to the case $B = O$, $\det A = \det D = -1$ considered in section 15. The expressions for solutions in these cases contain one or two complex parameters, respectively.

18. Let us summarise the results of the paper. We have proposed a new method for constructing the fundamental mode of a resonator in the form of a Gaussian beam with complex astigmatism. Unlike the traditional method, in which the beam parameters are expressed in terms of the eigenvectors of the 4×4 monodromy matrix, our procedure does not require the determination of these vectors. Another advantage of the method is, in our opinion, that it allows one to obtain explicit analytic expressions for the beam in terms of the elements of the initial matrix, thereby revealing the dependence of its properties on the parameters of the problem. At the same time, the algorithm proposed in the paper can be also used in numerical calculations.

The procedure proposed in our paper involves the successive simplification of the problem with the help of similarity transformations and its final reduction to one of the basic variants admitting the explicit solution, in the form of a family of functions in a number of cases. The stability condition has been found for each of these variants. The correspondence between these variants and spectral parameters of the initial monodromy matrix has been established.

The possibilities of the method have been illustrated for a resonator with the nonplanar contour performing the spatial rotation of the image.

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