

PACS numbers: 42.65.Ky; 42.65.Yj; 42.65.Tg; 05.45.–a
DOI: 10.1070/QE2007v037n03ABEH013467

Nonlinear Schrödinger equation and multicomponent cnoidal waves in parametric frequency conversion

V.M. Petnikova, V.V. Shuvalov

***Abstract*. It is shown that the exact analytic solution of the problem of stationary parametric frequency conversion, including second harmonic generation and parametric amplification in a medium with quadratic nonlinearity, in the approximation of three interacting modes is reduced to the solution of three independent systems of nonlinear equations. Each of these systems consisting of two nonlinear Schrödinger equations is related to other systems only by the boundary conditions and describes a multicomponent cnoidal wave containing two noninterfering components. The problem can be represented in this way because the competition of two processes (the merging and decomposition of quanta) proceeding simultaneously on the second-order nonlinearity can be described through the effective cascade cubic nonlinearity.**

Keywords: *frequency conversion, quadratic nonlinearity, cascade cubic nonlinearity, nonlinear Schrödinger equation, multicomponent cnoidal wave.*

1. Introduction

Despite numerous papers devoted to the analysis of multicomponent self-consistent periodic solutions of the nonlinear Schrödinger equation (NSE), Korteweg–de Vries, sin-Gordon, and other equations [1–7], such solutions in laser physics have been considered so far as somewhat exotic. The matter is that, although these equations in optics have the universal nature because they take into account the lowest (cubic) terms in the expansion of nonlinear polarisation in the wave equation, it is generally accepted that solutions of this type – multicomponent cnoidal waves (MCWs) are important for a limited scope of problems. These are one-dimensional (1D) problems of the soliton-like propagation of pulse trains in optical fibres [3–6, 8] and of parametric generation of pulse trains upon synchronous pumping [9] taking dispersion into account, as well as two (2D)- and three-dimensional (3D) problems of the nondiffractive propagation of beams with a special periodic transverse structure through photorefractive crystals [7, 10] and crystals with quadratic nonlinearity [11]. At the same time, MCWs have become quite popular in other fields of physics. The concept of MCWs is widely used in nonlinear hydrodynamics [1, 12], plasma physics [2, 13], in the description of coupled wave packets – quasiparticles (excitons, biexcitons, superconducting pairs, etc.) formed by electronic wave functions, in the physics of 1D chains (conjugated polymers) [14] and 2D planes (ferromagnetics and high-temperature superconductors) [15].

We will show below that NSE solutions in the form of MCWs play a key role in a classical problem of nonlinear optics – the description of parametric up- and down frequency conversion, including second harmonic generation (SHG) and parametric amplification in nonlinear crystals, i.e. media with the quadratic nonlinearity [16].

2. Parametric conversion and nonlinear Schrödinger equations

Consider a simple case of the collinear interaction of three plane monochromatic waves. Two of them have the fundamental frequency $\omega_{1,2} = \omega$, the amplitudes $A_{1,2}$ and the wave vectors $\boldsymbol{k}_{1,2}$, while the third wave has the second harmonic frequency $\omega_3 = 2\omega$, the amplitude A_3, and the wave vector $\boldsymbol{k}_3$. The waves propagate from the plane $z = 0$ along the z axis in a medium with quadratic nonlinearity (a nonlinear crystal). By neglecting anisotropy and absorption, we assume that the nonlinear crystal occupies the half-space $z \geqslant 0$ and the so-called type II parametric process (the oee-interaction) is realised in it. This process is described by the well-known system of equations for the amplitudes of three coupled waves (modes) [16]

$$\frac{\partial A_1}{\partial z} = -\mathrm{i}\beta A_2^* A_3 \exp(-\mathrm{i}\Delta z), \tag{1a}$$

$$\frac{\partial A_2}{\partial z} = -\mathrm{i}\beta A_1^* A_3 \exp(-\mathrm{i}\Delta z), \tag{1b}$$

$$\frac{\partial A_3}{\partial z} = -\mathrm{i}2\beta A_1 A_2 \exp(\mathrm{i}\Delta z). \tag{1c}$$

Here, β is the nonlinear coupling constant and $\Delta = k_1 + k_2 - k_3$ is the wave mismatch. System of equations (1) has two integrals of motion

$$I_1(z) + I_2(z) + I_3(z) = I_{10} + I_{20} + I_{30}, \tag{2a}$$

V.M. Petnikova, V.V. Shuvalov International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119992 Moscow, Russia; e-mail: vsh@vsh.phys.msu.su

Received 30 October 2006
Kvantovaya Elektronika **37** (3) 266–272 (2007)
Translated by M.N. Sapozhnikov

$$I_1(z) - I_2(z) = I_{10} - I_{20}, \tag{2b}$$

where $I_i(z) = A_i(z)A_i^*(z)$ is the variable proportional to the energy flux density of the ith wave ($i = 1 - 3$), which we will simply call the intensity; $I_{i0} = I_i(z = 0)$. The first integral describes the law of conservation of the energy density flux, while the second one represents the so-called Manley–Rowe relations [16].

By using (2), we can reduce system (1) to three closed nonlinear equations describing self-consistent periodic solutions for the complex amplitudes $A_i(z)$ of the waves interacting in the nonlinear crystal. For this purpose, by making the change of variables

$$A_j(z) = \tilde{A}_j(z)\exp(-\mathrm{i}\alpha_j z) \tag{3}$$

and selecting constant α_{1-3} so that

$$\alpha_1 + \alpha_2 - \alpha_3 - \Delta = 0, \tag{4}$$

we rewrite system (1) in the form

$$\frac{\partial \tilde{A}_1}{\partial z} - \mathrm{i}\alpha_1\tilde{A}_1 = -\mathrm{i}\beta\tilde{A}_2^*\tilde{A}_3, \tag{5a}$$

$$\frac{\partial \tilde{A}_2}{\partial z} - \mathrm{i}\alpha_2\tilde{A}_2 = -\mathrm{i}\beta\tilde{A}_1^*\tilde{A}_3, \tag{5b}$$

$$\frac{\partial \tilde{A}_3}{\partial z} - \mathrm{i}\alpha_3\tilde{A}_3 = -\mathrm{i}2\beta\tilde{A}_1\tilde{A}_2. \tag{5c}$$

Then, after some simple transformations taking relation (2a) into account, we obtain for the amplitude $\tilde{A}_1\tilde{A}_2$ of the nonlinear polarisation wave at the frequency ω_3 the equation

$$\frac{\partial(\tilde{A}_1\tilde{A}_2)}{\partial z} = \mathrm{i}(\alpha_1 + \alpha_2)\tilde{A}_1\tilde{A}_2 - \mathrm{i}\beta(I_{10} + I_{20} + I_{30} - \tilde{A}_3\tilde{A}_3^*)\tilde{A}_3. \tag{6}$$

By differentiating (5c) and substituting (6) into the result obtained, we find the expression

$$\frac{\partial^2 \tilde{A}_3}{\partial z^2} - \mathrm{i}(\alpha_1 + \alpha_2 + \alpha_3)\frac{\partial \tilde{A}_3}{\partial z} + 2\beta^2 \times[I_{10} + I_{20} + I_{30} - (\alpha_1 + \alpha_2)\alpha_3 - \tilde{A}_3\tilde{A}_3^*]\tilde{A}_3 = 0. \tag{7}$$

Note that the second term in (7) can be easily eliminated. For this purpose, taking into account that a choice of the specific values of α_{1-3} is not yet unique [see expression (4)], we should set

$$\alpha_1 + \alpha_2 = \Delta/2, \tag{8a}$$

$$\alpha_3 = -\Delta/2. \tag{8b}$$

Then, we obtain finally

$$\frac{\partial^2 \tilde{A}_3}{\partial z^2} + 2\beta^2\left(I_{10} + I_{20} + I_{30} + \frac{\Delta^2}{8\beta^2} - \tilde{A}_3\tilde{A}_3^*\right)\tilde{A}_3 = 0, \tag{9}$$

i.e. the closed equation for the complex amplitude $\tilde{A}_3$ in the NSE form. Note that because (9) is the second-order equation, we are interested only in the solutions satisfying the boundary condition

$$\left.\frac{\partial \tilde{A}_3}{\partial z}\right|_{z=0} = -\mathrm{i}\frac{\Delta}{2}\tilde{A}_{30} - \mathrm{i}2\beta\tilde{A}_{10}\tilde{A}_{20}, \tag{10}$$

which follows from Eqn (5c). Here, $\tilde{A}_{i0} = \tilde{A}_i(z = 0)$.

By repeating the procedure of successive transformations described above, we obtain the equation

$$\frac{\partial(\tilde{A}_1^*\tilde{A}_3)}{\partial z} = -\mathrm{i}(\alpha_1 - \alpha_3)\tilde{A}_1^*\tilde{A}_3 + \mathrm{i}\beta(-2I_{10} + 4I_{20} + I_{30} - 4A_2A_2^*)\tilde{A}_2 \tag{11}$$

for the amplitude $\tilde{A}_1^*\tilde{A}_3$ of the nonlinear polarisation wave at the frequency ω_2. Then, by differentiating (5b) taking (11) into account and selecting α_{1-3} so that

$$\alpha_1 - \alpha_3 = \Delta/2, \tag{12a}$$

$$\alpha_2 = \Delta/2, \tag{12b}$$

we find the similar closed NSE for the amplitude $\tilde{A}_2$

$$\frac{\partial^2 \tilde{A}_2}{\partial z^2} - \beta^2\left(-2I_{10} + 4I_{20} + I_{30} - \frac{\Delta^2}{4\beta^2} - 4\tilde{A}_2\tilde{A}_2^*\right)\tilde{A}_2 = 0. \tag{13}$$

As in the previous case, we are interested only in the solutions of (13) satisfying the boundary condition

$$\left.\frac{\partial \tilde{A}_2}{\partial z}\right|_{z=0} = \mathrm{i}\frac{\Delta}{2}\tilde{A}_{20} - \mathrm{i}\beta\tilde{A}_{10}^*\tilde{A}_{30}, \tag{14}$$

which follows from Eqn (5b).

Taking the symmetry of the problem into account, the closed NSE for the amplitude $\tilde{A}_1$ can be now obtained by a simple interchange of subscripts $1 \leftrightarrow 2$. Therefore,

$$\frac{\partial^2 \tilde{A}_1}{\partial z^2} - \beta^2\left(4I_{10} - 2I_{20} + I_{30} - \frac{\Delta^2}{4\beta^2} - 4\tilde{A}_1\tilde{A}_1^*\right)\tilde{A}_1 = 0 \tag{15}$$

for

$$\alpha_2 - \alpha_3 = \Delta/2, \tag{16a}$$

$$\alpha_1 = \Delta/2 \tag{16b}$$

and the boundary condition

$$\left.\frac{\partial \tilde{A}_1}{\partial z}\right|_{z=0} = \mathrm{i}\frac{\Delta}{2}\tilde{A}_{10} - \mathrm{i}\beta\tilde{A}_{20}^*\tilde{A}_{30}. \tag{17}$$

The possibility of reducing system (1) to the three closed NSEs for the complex amplitudes $\tilde{A}_{1-3}$ is amazing at first glance. Indeed, it is well known that periodic and aperiodic NSE solutions, the so-called cnoidal waves and solitons expressed in terms of the Jacobi elliptic functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ [17] and hyperbolic functions $\cosh\xi$ and $\tanh\xi$ (where ξ is a variable proportional to z), describe the self-consistent solutions of numerous problems in a variety of fields in physics [3–6]. The possibility of writing equations in the NSE form is usually attributed to the presence of cubic nonlinearity in the medium [3–6]. However, there is no paradox in this case because, by having passed to closed equations (9), (13), and (15), we in fact simply began to describe the result of competition of two processes simultaneously proceeding on the second-order nonlinearity [merging ($\omega_1+\omega_2\to\omega_3$) and decomposition ($\omega_3\to\omega_1+\omega_2$) of quanta] in terms of an efficient cascade cubic nonlinearity [18].

Note also that NSEs (9), (13) and (15) obtained above are, however, related to each other by boundary conditions (10), (14) and (17), and, which is more important, the field amplitudes $\tilde{A}_i$ in these equations are complex in the general case. Therefore, unlike many other nonlinear problems described by NSEs, the dependences of the moduli and phases of the field amplitudes $\tilde{A}_i$ on the coordinate z in our case can be very complicated. Because of this, the known analytic solutions of NSEs [16, 19], which are proportional to the Jacobi functions sn ξ, cn ξ and dn ξ, do not exhaust all the possible solutions of the initial problem (1) but determine only those branches of these solutions for which the phase of the amplitude $\tilde{A}_i$ is fixed at least for one of the interacting waves, while the phase of the amplitude A_i changes linearly during the wave propagation according to (8b), (12b) or (16b).

To avoid this problem, we can separate the real and imaginary parts of the amplitudes $\tilde{A}_i(z)$ of all the waves interacting in a nonlinear crystal by introducing the three pairs of real functions $\tilde{A}'_i(z)$ and $\tilde{A}''_i(z)$, so that

$$\tilde{A}_i(z)=\tilde{A}'_i(z)+\mathrm{i}\tilde{A}''_i(z). \tag{18}$$

By substituting now (18) into (9), (13) and (15) and into boundary conditions (10), (14) and (17), we obtain the three systems of equations for the real functions $\tilde{A}'_i(z)$ and $\tilde{A}''_i(z)$

$$\frac{\partial^2\tilde{A}'_3}{\partial z^2}+2\beta^2\left\{I_{10}+I_{20}+I_{30}+\frac{\Delta^2}{8\beta^2}-[(\tilde{A}'_3)^2+(\tilde{A}''_3)^2]\right\}\tilde{A}'_3=0, \tag{19a}$$

$$\frac{\partial^2\tilde{A}''_3}{\partial z^2}+2\beta^2\left\{I_{10}+I_{20}+I_{30}+\frac{\Delta^2}{8\beta^2}-[(\tilde{A}'_3)^2+(\tilde{A}''_3)^2]\right\}\tilde{A}''_3=0, \tag{19b}$$

$$\frac{\partial^2\tilde{A}'_2}{\partial z^2}-\beta^2\left\{-2I_{10}+4I_{20}+I_{30}-\frac{\Delta^2}{4\beta^2}-4[(\tilde{A}'_2)^2+(\tilde{A}''_2)^2]\right\}\tilde{A}'_2=0, \tag{20a}$$

$$\frac{\partial^2\tilde{A}''_2}{\partial z^2}-\beta^2\left\{-2I_{10}+4I_{20}+I_{30}-\frac{\Delta^2}{4\beta^2}-4[(\tilde{A}'_2)^2+(\tilde{A}''_2)^2]\right\}\tilde{A}''_2=0, \tag{20b}$$

$$\frac{\partial^2\tilde{A}'_1}{\partial z^2}-\beta^2\left\{4I_{10}-2I_{20}+I_{30}-\frac{\Delta^2}{4\beta^2}-4[(\tilde{A}'_1)^2+(\tilde{A}''_1)^2]\right\}\tilde{A}'_1=0, \tag{21a}$$

$$\frac{\partial^2\tilde{A}''_1}{\partial z^2}-\beta^2\left\{4I_{10}-2I_{20}+I_{30}-\frac{\Delta^2}{4\beta^2}-4[(\tilde{A}_1)^2+(\tilde{A}''_1)^2]\right\}\tilde{A}''_1=0, \tag{21b}$$

and the boundary conditions corresponding to these tree systems

$$\tilde{A}'_{30}=\mathrm{Re}\tilde{A}_{30}, \tag{22a}$$

$$\tilde{A}''_{30}=\mathrm{Im}\tilde{A}_{30}, \tag{22b}$$

$$\left.\frac{\partial\tilde{A}'_3}{\partial z}\right|_{z=0}=\frac{\Delta}{2}\tilde{A}''_{30}+2\beta(\tilde{A}'_{10}\tilde{A}''_{20}+\tilde{A}''_{10}\tilde{A}'_{20}), \tag{22c}$$

$$\left.\frac{\partial\tilde{A}''_3}{\partial z}\right|_{z=0}=-\frac{\Delta}{2}\tilde{A}'_{30}-2\beta(\tilde{A}'_{10}\tilde{A}'_{20}-\tilde{A}''_{10}\tilde{A}''_{20}), \tag{22d}$$

$$\tilde{A}'_{20}=\mathrm{Re}\tilde{A}_{20}, \tag{23a}$$

$$\tilde{A}''_{20}=\mathrm{Im}\tilde{A}_{20}, \tag{23b}$$

$$\left.\frac{\partial\tilde{A}'_2}{\partial z}\right|_{z=0}=-\frac{\Delta}{2}\tilde{A}''_{20}+\beta(\tilde{A}'_{10}\tilde{A}''_{30}-\tilde{A}''_{10}\tilde{A}'_{30}), \tag{23c}$$

$$\left.\frac{\partial\tilde{A}''_2}{\partial z}\right|_{z=0}=\frac{\Delta}{2}\tilde{A}'_{20}-\beta(\tilde{A}'_{10}\tilde{A}'_{30}+\tilde{A}''_{10}\tilde{A}''_{30}), \tag{23d}$$

$$\tilde{A}'_{10}=\mathrm{Re}\tilde{A}_{10}, \tag{24a}$$

$$\tilde{A}''_{10}=\mathrm{Im}\tilde{A}_{10}, \tag{24b}$$

$$\left.\frac{\partial\tilde{A}'_1}{\partial z}\right|_{z=0}=-\frac{\Delta}{2}\tilde{A}''_{10}+\beta(\tilde{A}'_{20}\tilde{A}''_{30}-\tilde{A}''_{20}\tilde{A}'_{30}), \tag{24c}$$

$$\left.\frac{\partial\tilde{A}''_1}{\partial z}\right|_{z=0}=\frac{\Delta}{2}\tilde{A}'_{10}-\beta(\tilde{A}'_{20}\tilde{A}'_{30}+\tilde{A}''_{20}\tilde{A}''_{30}). \tag{24d}$$

It is easy to verify that each of systems (19), (20) and (21) is still closed (with an accuracy to its boundary conditions) and is formed by a pair of coupled NSEs describing a two-component cnoidal wave with the noninterfering components $\tilde{A}'_i(z)$ and $\tilde{A}''_i(z)$. Because now both these components are real, there is no need to present here the procedure for constructing the self-consistent periodic and aperiodic analytic solutions for them because this procedure is similar to that described in detail in our paper [10].

3. Peculiarities of analytic MCW solutions

Note that in situations when we are not interested in variations in phases of all the interacting waves during their propagation, i.e. in all the particular dependences $\tilde{A}'_i(z)$ and $\tilde{A}''_i(z)$, it is sufficient to obtain the analytic solution only for one of the systems of equations (19), (20) or (21). The dependences of the intensities of the two remaining waves on z can be found from integrals (2). For example, by solving the system of equations (19) taking (2) into account, we can immediately obtain

$$I_1(z) = \frac{1}{2}\left\{2I_{10} + I_{30} - [\tilde{A}_3'(z)]^2 - [\tilde{A}_3''(z)]^2\right\}, \tag{25a}$$

$$I_2(z) = \frac{1}{2}\left\{2I_{20} + I_{30} - [\tilde{A}_3'(z)]^2 - [\tilde{A}_3''(z)]^2\right\}, \tag{25b}$$

and by solving systems (20) and (21), we obtain

$$I_1(z) = I_{10} - I_{20} + [\tilde{A}_2'(z)]^2 + [\tilde{A}_2''(z)]^2, \tag{26a}$$

$$I_3(z) = 2I_{20} + I_{30} - 2\{[\tilde{A}_2'(z)]^2 + [\tilde{A}_2''(z)]^2\}, \tag{26b}$$

and

$$I_2(z) = I_{20} - I_{10} + [\tilde{A}_1'(z)]^2 + [\tilde{A}_1''(z)]^2, \tag{27a}$$

$$I_3(z) = 2I_{10} + I_{30} - 2\{[\tilde{A}_1'(z)]^2 + [\tilde{A}_1''(z)]^2\}, \tag{27b}$$

respectively.

Note also that, as shown in [10], the functional type of any of the solutions of systems (19), (20) and (21) is limited by the fundamental solutions of the first- and second-order Lame equations [20] , i.e. $\tilde{A}_i'(z)$ and $\tilde{A}_i''(z)$ should be proportional to one of the elliptic functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, $\mathrm{dn}\,\xi$, $\mathrm{dn}^2\xi + \gamma_{1,5}^{(2)}$, $\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi$, $\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi$ and $\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi$ (here, $\gamma_{1,5}^{(2)}$ are constants, see [10]). In the cases when the two MCW components $\tilde{A}_i'(z)$ and $\tilde{A}_i''(z)$, obtained by solving some of the systems (19), (20) or (21), are proportional to the same elliptic function, the solution is degenerate and is reduced to the MCW of the so-called Manakov type [10]. In this case, the phase of the amplitude $\tilde{A}_i$ for one of the MCW components remains constant, while the phase of A_i changes linearly during the wave propagation according to (8b), (12b) or (16b). Note that due to the orthogonality of the fundamental solutions of the Lame equation, the MCWs describing the solution of the problems in which the intensity of any of the interacting fields in a nonlinear crystal vanishes at some points on the z axis should have namely this type.

In fact, the solutions of this type are usually presented in all papers devoted to the search for exact analytic solutions of the problem under study [16, 19]. If this is not the case, the type of MCWs due to boundary conditions (22), (23) and (24) begins to change with changing the relation between the initial (at the nonlinear crystal input) phases $\Delta\varphi_0 = \varphi_{10} + \varphi_{20} - \varphi_{30}$ of the complex amplitudes A_{i0} of all the three interacting waves. This is not surprising because this process is coherent and therefore it is extremely sensitive to the phases of the initial fields A_{i0} in the cases when they are present at the nonlinear crystal input (in the plane $z = 0$). To solve the initial problem in this case by traditional methods is apparently difficult [16, 19] because it is difficult to imagine the form of a closed equation describing the variation in the phase, which is determined by the arctangent of the ratio of two different elliptic functions.

Note also that the types of solutions of system (19) on the one hand and systems (20) and (21) on the other are fundamentally different, which determines the admissible character of MCWs of the Manakov type for waves A_3 and $A_{1,2}$, respectively. This can be explained with the help of a simple analogy with the problem of self-action on the so-called Kerr nonlinearity, in which system (19) corresponds to the propagation of radiation through a medium with a defocusing nonlinearity, whereas systems (20) and (21) correspond to the propagation of radiation in a medium with a focusing nonlinearity (see, for example, [10]).

4. Examples of analytic solutions

As an example of the realisation of the approach described above, we will analyse one of the known simplest solutions of problem (1) corresponding to the Manakov MCW and describing SHG in the absence of a wave with the amoplitude A_3 at the frequency $\omega_3 = 2\omega$ at the nonlinear crystal input (i.e. for $I_{30} = 0$). Taking this into account, we will seek the solution of NSE (9) in the form

$$\tilde{A}_3 = B_3 \mathrm{sn}\,(\gamma z). \tag{28}$$

By substituting (28) into (9), we obtain at once two necessary conditions

$$\gamma^2(1 + k^2) = 2\beta^2\left(I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2}\right), \tag{29a}$$

$$\gamma^2 k^2 = \beta^2 B_3 B_3^*. \tag{29b}$$

Here, k is the modulus of elliptic functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$, whose value should lie in the interval $1 \geqslant k \geqslant 0$.

The situations when $k \to 0$ and $k \to 1$ correspond to the limiting cases of harmonic [$\mathrm{sn}\,(\gamma z) \to \sin(\gamma z)$, $\mathrm{cn}\,(\gamma z) \to \cos(\gamma z)$, $\mathrm{dn}\,(\gamma z) \to 1$] and aperiodic [$\mathrm{sn}\,(\gamma z) \to \tanh(\gamma z)$, $\mathrm{cn}\,(\gamma z) \to 1/\cosh(\gamma z)$, $\mathrm{dn}\,(\gamma z) \to 1/\cosh(\gamma z)$] solutions. This gives

$$\gamma^2 = \beta^2\left[2(I_{10} + I_{20}) + \frac{\Delta^2}{4\beta^2} - B_3 B_3^*\right], \tag{30a}$$

$$k^2 = \frac{B_3 B_3^*}{2(I_{10} + I_{20}) + \Delta^2/(4\beta^2) - B_3 B_3^*}. \tag{30b}$$

Note that in the case under study, we always have $1 \geqslant k^2 \geqslant 0$ because, taking (2a) into account, the obvious relation

$$I_{10} + I_{20} \geqslant B_3 B_3^* \tag{31}$$

is always fulfilled. In this case, the oscillations of $\tilde{A}_3(z)$ cannot be harmonic ($k = 0$) because for $B_3 B_3^* = 0$, we obtain, taking (25) into account, the trivial solution

$$I_1(z) = I_{10}, \tag{32a}$$

$$I_2(z) = I_{20}, \tag{32b}$$

$$A_3(z) = A_{30} = 0. \tag{32c}$$

It follows from (10) that $A_{10} = 0$ or $A_{20} = 0$, i.e. at least one of the fields at the frequency ω in the plane $z = 0$ is also absent. At the same time, the solution in form (28) can be aperiodic ($k = 1$), but only when energy transfer to the mode at frequency ω_3 is complete ($I_{10} = I_{20}$, $B_3 B_3^* = 2I_{10}$) and the SHG process is synchronous ($\Delta = 0$).

The amplitude B_3 can be now found from condition (10), which, taking (30a) into account, immediately gives

$$(B_3B_3^*)^2 - 2\left(I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2}\right)B_3B_3^* + 4I_{10}I_{20} = 0. \quad (33)$$

This leads to the final result

$$I_1(z) = I_{10} - \frac{1}{2}\left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} - \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}\mathrm{sn}^2(\gamma z), \quad (34a)$$

$$I_2(z) = I_{20} - \frac{1}{2}\left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} - \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}\mathrm{sn}^2(\gamma z), \quad (34b)$$

$$A_3(z) = \left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} - \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20}) \times\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}^{1/2}\mathrm{sn}(\gamma z)\exp\left(\mathrm{i}\frac{\Delta}{2}z\right) \quad (34c)$$

for

$$\gamma^2 = \beta^2\left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} + \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}, \quad (35)$$

$$k^2 = \left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} - \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20}) \times\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}\Big/\left\{I_{10} + I_{20} + \frac{\Delta^2}{8\beta^2} + \left[(I_{10} - I_{20})^2 + (I_{10} + I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2\right]^{1/2}\right\}. \quad (36)$$

It is quite natural that solution (34) coincides with the known analytic solution for the simplest SHG case considered above [16, 19]. However, we emphasise once more that our method allows us to obtain the analytic solution of the initial problem (1) for any boundary conditions. Let us illustrate this by the second example describing another solution of problem (1), which also corresponds to the Manakov MCW in the case of SHG with a complete depletion of the field with the amplitude A_2 due to transfer of its energy to the field with the amplitude A_3. Taking this into account, we will seek the solution of Eqn (13) in the form

$$\tilde{A}_2 = B_2\mathrm{cn}(\gamma z). \quad (37)$$

By substituting (31) into (13), we also obtain the two requirements

$$\gamma^2 = \beta^2\left(2I_{10} - 4I_{20} - I_{30} + \frac{\Delta^2}{4\beta^2} + 4B_2B_2^*\right), \quad (38a)$$

$$\gamma^2k^2 = 2\beta^2B_2B_2^*, \quad (38b)$$

which, taking the boundary conditions into account, give immediately the required solution

$$I_1(z) = I_{10} - I_{20}\mathrm{sn}^2(\gamma z), \quad (39a)$$

$$A_2(z) = A_{20}\mathrm{cn}(\gamma z)\exp\left(-\mathrm{i}\frac{\Delta}{2}z\right), \quad (39b)$$

$$I_3(z) = 2I_{20}\left[\frac{\Delta^2}{8\beta^2I_{10}} + \mathrm{sn}^2(\gamma z)\right] \quad (39c)$$

for

$$\gamma^2 = 2\beta^2I_{10}\left[1 + \frac{\Delta^2}{8\beta^2I_{10}}\left(1 - \frac{I_{20}}{I_{10}}\right)\right], \quad (40a)$$

$$k^2 = I_{20}\Big/\left[I_{10} + \frac{\Delta^2}{8\beta^2}\left(1 - \frac{I_{20}}{I_{10}}\right)\right]. \quad (40b)$$

It follows from (39) that the solution of this type exists only for $I_{10} \geqslant I_{20}$ and the amplitude A_{30} of the wave at frequency ω_3 at the nonlinear crystal input can be arbitrary small only upon synchronous interaction ($\Delta \to 0$). In this case, when the intensities of the waves with amplitudes A_1 and A_2 are equal at the nonlinear crystal input ($I_{10} = I_{20}$), solution (39) becomes aperiodic ($k = 1$) and

$$I_1(z) = I_2(z) = \frac{I_{10}}{\cosh^2(\gamma z)}, \quad (41a)$$

$$I_3(z) = \frac{\Delta^2}{4\beta^2} + 2I_{10}\tanh^2(\gamma z) \quad (41b)$$

for

$$\gamma^2 = 2\beta^2I_{10}. \quad (42)$$

As far as we know, analytic solution (39) of initial problem (1) has not been reported in the literature so far.

In conclusion, we consider another quite interesting exact analytic solution of problem (1), which we have not found in the literature, and which also corresponds to MCWs of the Manakov type in the case of SHG with an incomplete depletion of the field with the amplitude A_2 due to transfer of its energy to the field with the amplitude A_3. Taking this into account, we will seek the solution of Eqn (13) in the form

$$\tilde{A}_2 = B_2\mathrm{dn}(\gamma z). \quad (43)$$

By substituting (43) into (13), we obtain the two requirements

$$\gamma^2k^2 = \beta^2\left(2I_{10} - 4I_{20} - I_{30} + \frac{\Delta^2}{4\beta^2} + 4B_2B_2^*\right), \quad (44a)$$

$$k^2(\gamma^2 - 2\beta^2B_2B_2^*) = 0. \quad (44b)$$

Equality (44b) can be fulfilled either in the trivial case

$$k^2 = 0, \tag{45}$$

or when the condition

$$\gamma^2 = 2\beta^2 B_2 B_2^* \tag{46}$$

is fulfilled. In the first case, taking (44a) and condition (14) into account, we obtain

$$I_1(z) = I_{10}, \tag{47a}$$

$$I_2(z) = I_{10}\left(1 + \frac{8\beta^2 I_{10}}{\Delta^2}\right), \tag{47b}$$

$$I_3(z) = 2I_{10}\left(1 + \frac{\Delta^2}{8\beta^2 I_{10}}\right), \tag{47c}$$

which corresponds to the regime of the so-called parametric bleaching, i.e. to the situation in which the rates of processes $\omega_1 + \omega_2 \to \omega_3$ and $\omega_3 \to \omega_1 + \omega_2$ are the same. In the second case, taking boundary conditions (14) into account, we have

$$I_1(z) = I_{10}\mathrm{cn}^2(\gamma z) + \frac{\Delta^2}{8\beta^2 I_{10}}(I_{20} - I_{10})\mathrm{sn}^2(\gamma z), \tag{48a}$$

$$A_2(z) = A_{20}\mathrm{dn}(\gamma z)\exp\left(-\mathrm{i}\frac{\Delta}{2}z\right), \tag{48b}$$

$$I_3(z) = \frac{\Delta^2}{4\beta^2 I_{10}} I_{20}\mathrm{cn}^2(\gamma z) + 2I_{10}\left(1 + \frac{\Delta^2}{8\beta^2 I_{10}}\right)\mathrm{sn}^2(\gamma z) \tag{48c}$$

for

$$\gamma^2 = 2\beta^2 I_{20}, \tag{49a}$$

$$k^2 = \frac{I_{10}}{I_{20}} - \frac{\Delta^2}{8\beta^2 I_{10}}\frac{I_{20} - I_{10}}{I_{20}}. \tag{49b}$$

It follows from the requirement $0 \leqslant k \leqslant 1$ and boundary condition (14) that this new solution of the initial problem (1) exists only under the condition

$$I_{10} \leqslant I_{20} \leqslant I_{10}\left(1 + \frac{8\beta^2 I_{10}}{\Delta^2}\right). \tag{50}$$

5. Conclusions

We have shown that the exact analytic solution of the problem of stationary parametric frequency conversion, including SHG and parametric amplification in a medium with quadratic nonlinearity, in the approximation of three interacting modes is reduced to the solution of three independent systems of nonlinear equations. Each of these systems consisting of two NSEs is related to other systems only by its boundary conditions and describes a MCW containing two noninterfering components. This very efficient approach is based on the description of the competition between processes of merging ($\omega_1 + \omega_2 \to \omega_3$) and decomposition ($\omega_3 \to \omega_1 + \omega_2$) of quanta, which proceed simultaneously on the second-order nonlinearity in terms of the efficient cascade cubic nonlinearity.

In this case, due to well-developed methods for solving such systems of equations, the analytic solutions of the initial problem in the MCW form for any boundary conditions are constructed by using standard algorithms based on a finite set of elliptic functions – the fundamental solutions of the first- and second-order Lame equations. Moreover, the approach described in the paper allows one not only to classify such solutions, by simplifying considerably *a priori* choice of elliptic functions that are the most convenient for each particular situation, but also to displace them along the z axis, thereby changing the type of boundary conditions. The latter circumstance allows the use of the same solution form both for SHG and parametric amplification. For example, the shift of the argument $\xi \to \xi + K$ of elliptic functions by a quarter of a period (where K is the complete elliptic integral of the first kind) is described by the transformation [17]

$$\mathrm{sn}\,\xi \to \frac{\mathrm{cn}\,\xi}{\mathrm{dn}\,\xi}, \quad \mathrm{cn}\,\xi \to -(1-k^2)^{1/2}\frac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi},$$

$$\mathrm{dn}\,\xi \to (1-k^2)^{1/2}\frac{1}{\mathrm{dn}\,\xi}.$$

Note also that solutions in the MCW form can be extrapolated to the half-space $z < 0$ (by filling in fact this half-space with the same nonlinear medium) and then set into motion at a constant velocity v along the z axis (by performing the transformation $z \to \eta$, where $\eta = z - vt$ is the running coordinate, and t is the time). This follows (at least without the consideration of the problem of correctness of using multipole expansions at relativistic velocities [21]) directly from the invariance of initial equations (1) (in fact, the wave equation) with respect to the Lorentz transformations. Therefore, the question arises of whether these solutions are 'true' soliton-like solutions or we are dealing here with a mathematical analogy. It seems that the answer to this question is not simple because the presence of a defocusing nonlinearity in one of the pairs of coupled NSEs (19) prevents the 'collision' of two aperiodic solutions moving at different velocities $v_{1,2}$, which could separated by an infinite distance after the collision.

References

1. Yuen H.C., Lake B.M. *Adv. Appl. Mech.*, **22**, 67 (1982).
2. Kuznetsov E.A., Rubenchik A.M., Zakharov V.E. *Phys. Rep.*, **142**, 103 (1986).
3. Akhmediev N.N., Ankiewicz A. *Solitons, Nonlinear Pulses and Beams* (London: Chapman and Hall, 1997).
4. Infeld E., Rowlands G. *Nonlinear Waves, Solitons, and Chaos* (Cambridge: Cambridge University Press, 2000).
5. Kamchatnov A.M. *Nonlinear Periodic Waves and Their Modulation* (Singapore: World Scientific, 2000).
6. Kivshar Yu.S., Agrawal G.P. *Optical Solitons: From Fibers to Photonic Crystals* (San Diego: Acad. Press, 2003).
7. Fleischer J.W., Segev M., Efremidis N.K., Christodoulides D.N. *Nature (London)*, **422**, 147 (2003).
8. Mikhailovskii A.B., Kudashev V.R., Lakhin V.P., et al. *Pis'ma Zh. Eksp. Teor. Fiz.*, **40**, 273 (1984); Hioe F.T. *Phys. Rev. Lett.*, **82**, 1152 (1999); Chow K.W., Lai D.W.C. *Phys. Rev. E*, **65**, 026613 (2002); Chow K.W., Nakkeeran K., Malomed B.A. *Opt. Commun.*, **219**, 251 (2003).
9. Ankiewicz A., Maruno K., Akhmediev N. *Phys. Lett. A*, **308**, 397 (2003); Maruno K., Ankiewicz A., Akhmediev N. *Opt. Commun.*, **221**, 199 (2003).
10. Petnikova V.M., Shuvalov V.V., Vysloukh V.A. *Phys. Rev. E*, **60**, 1009 (1999); Vysloukh V.A., Petnikova V.M., Rudenko K.V.,

Shuvalov V.V. *Kvantovaya Elektron.*, **28**, 55 (1999) [*Quantum Electron.*, **29**, 613 (1999)].
11. Karamzin Yu.N., Sukhorukov A.P. *Pis'ma Zh. Eksp. Teor. Fiz.*, **20**, 734 (1974); Bang O., Christiansen P.L., Clausen C.B. *Phys. Rev. E*, **56**, 7257 (1997); Kobyakov A., Darmanyan S., Pertsch T., Lederer E. *J. Opt. Soc. Am. B*, **16**, 1737 (1999); Sukhorukov A.A., Kivshar Yu.S., Bang O., Soukoulis C.M. *Phys. Rev. E*, **63**, 016615 (2001); Malomed B.A., Kevrekidis P.G., Frantzeskakis D.J., et al. *Phys. Rev. E*, **65**, 056606 (2002); Torner L., Barthelemy A. *IEEE J. Quantum Electron.*, **39**, 22 (2003); Kartashov Y.V., Torner L., Vysloukh V.A. *Opt. Lett.*, **29**, 1117 (2004).
12. Zakharov V.E. *Zh. Prikl. Mat. Tekh. Fiz.*, (2), 86 (1968); Martin D.U., Yeun H.C., Saffman P.G. *Wave Motion*, **2**, 215 (1980).
13. Pavlenko V.P., Petviashvili V.I. *Fiz. Plasmy*, **8**, 206 (1982); Fil'chenkov S.E., Freiman G.M., Yunakovskii A.D. *Fiz. Plasmy*, **13**, 961 (1987); Zakharov V.E., Kuznetsov E.A. *Usp. Fiz. Nauk*, **167**, 1137 (1997).
14. Davydov A.S. *Solitons in Molecular Systems* (The Netherlands, Dordrecht: D.Reidel, 1985); Takeuchi S., Yoshizawa M., Matsuda T., et al. *IEEE J. Quantum Electron.*, **28**, 2508 (1992); Takahashi A., Mukamel S. *J. Chem. Phys.*, **100**, 2366 (1994); Hartmann M., Takahashi A., Mukamel S., Schreiber M. *Mol. Cryst. Liq. Cryst. Sci. Techn.*, **256**, 531 (1995); Xu Ji-Zhong, Huang Jing-Ning. *Phys. Lett. A*, **197**, 127 (1995); Voronov A.V., Petnikova V.M., Shuvalov V.V. *Kvantovaya Elektron.*, **33**, 219 (2003) [*Quantum Electron.*, **33**, 219 (2003)].
15. Davydov A.S. *Phys. Stat. Sol. B*, **146**, 619 (1988); Goff J.P., Tennant D.A. *Phys. Rev. B*, **52**, 15992 (1995); Haviland D.B., Delsing P. *Phys. Rev. B*, **54**, R6857 (1996); Voronov A.V., Petnikova V.M., Shuvalov V.V. *Zh. Eksp. Teor. Fiz.*, **120**, 1256 (2001).
16. Armstrong J.A., Bloembergen N., Ducuing J., Pershan P.S. *Phys. Rev.*, **127**, 1918 (1962); Akhmanov S.A., Khokhlov R.V. *Problemy nelineinoi optiki* (Problems of Nonlinear Optics) (Moscow: VINITI, 1964); Blombergen N. *Nonlinear Optics* (New York: Benjamin Press, 1965; Moscow: Mir, 1966); Zernicke F., Midwinter J.E. *Applied Nonlinear Optics* (New York: Wiley, 1973; Moscow: Mir, 1976); Shen Y.R. *The Principles of Nonlinear Optics* (New York: Wiley, 1984; Moscow: Nauka, 1989).
17. Gradshtein I.S., Ryzhik I.M. *Tablitsy integralov, summ, ryadov i proizvedenii* (Tables of Integrals, Sums, Series and Products) (Moscow: Nauka, 1971).
18. Ostrovskii L.A. *Pis'ma Zh. Eksp. Teor. Fiz.*, **5**, 331 (1967); Klyshko D.N., Polkovnikov B.F. *Kvantovaya Elektron.*, (4), 81 (1973) [*Sov. J. Quantum Electron.*, **3**, 324 (1973)].
19. Liu Xueming, Zhang Hanyi, Zhang Mingde. *Opt. Express*, **10**, 83 (2002).
20. Kuznetsov D.S. *Spetsial'nye funktsii* (Special Functions) (Moscow: Vysshaya shkola, 1965).
21. Landau L.D., Lifshits E.M. *Electrodynamics of Continuous Media* (Oxford: Pergamon Press, 1984; Moscow: Nauka, 1982).