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Complex periodic solutions of the nonlinear Schrödinger equation and nondegenerate multicomponent cnoidal waves in parametric frequency conversion

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***Abstract*. A new type of complex periodic solutions of the nonlinear Schrödinger equation is described which can be obtained in the collinear interaction of three plane monochromatic waves (modes) in a quadratic nonlinear medium. On passing to real variables (quadrature components), the solutions of this new type describe nondegenerate two-component cnoidal waves consisting of two 'incoherent' (noninterfering) components. The amplitudes of these components perform additional (with respect to the modulus oscillations) intricate nonlinear oscillations phase-shifted by $\pi/2$, which are consistent with oscillations of the solution modulus described by an elliptic function.**

Keywords: *nonlinear Schrödinger equation, nondegenerate multicomponent cnoidal wave, cascade cubic nonlinearity, parametric frequency conversion.*

1. Introduction

It was shown in [1] that in the interaction of three ($i = 1 - 3$) plane monochromatic waves (modes) propagating along the z axis in a medium with the quadratic nonlinearity $\chi^{(2)}$, the problem of frequency conversion, including second-harmonic generation (SHG) and parametric amplification involves the solution of three independent nonlinear Schrödinger equations (NSEs). Each of the equations determines the evolution of the complex amplitude $Y_i(z)$ of one ($i = 1 - 3$) of the interacting modes and is related to two other ($j = 1 - 3 \neq i$) NSEs only through boundary conditions ($z = 0$). The passage to real variables ($Y_i = Y_i' + \mathrm{i}Y_i''$) transforms each of the equations to a system of two coupled NSEs describing cnoidal waves consisting of two 'incoherent' (noninterfering) components $Y_i'(z)$ and $Y_i''(z)$. This approach is identical to the description of the result of competition between two processes ($\omega_1 + \omega_2 \to \omega_3$ and $\omega_3 \to \omega_1 + \omega_2$) simultaneously proceeding in a quadratic nonlinear medium with $\chi^{(2)}$ through the effective cascade cubic nonlinearity $\chi^{(3)}_{\mathrm{eff}}$ [2]. In this case, the schemes developed for solving such systems of equations allows one to obtain analytic solutions of the problem in the form of cnoidal waves by using standard algorithms.

An efficient algorithm of this type based on the fundamental solutions of the first- and second-order Lame equation [3] was proposed earlier in our paper [4]. However, because in the situation considered in [1] both equations of each of the systems ($i = 1 - 3$) describing two-component cnoidal waves $Y_i(z)$ are identical, the algorithm can be used only to obtain solutions of the so-called Manakov type, for which Y_i' and Y_i'' are proportional to the same function and the phase $\varphi_i = \arctan[Y_i'(z)/Y_i''(z)] = \mathrm{const}$ is independent of z [4]. This means a new (compared to that described in [4]) class of complex periodic solutions $Y_i(z)$ of the NSE should exist, which, on passing to real variables ($Y_i = Y_i' + \mathrm{i}Y_i''$) will describe nondegenerate two-component cnoidal waves in parametric frequency conversion. It is the solutions of this type that we will consider in our paper.

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2. Nonlinear Schrödinger equation for complex amplitudes

Let us write the NSE in a standard normalised form

$$\frac{\partial^2 Y}{\partial z^2} \pm 2(YY^* - \beta)Y = 0, \tag{1}$$

where the signs '$\pm$' correspond to nonlinearities of the so-called focusing and defocusing types, respectively, and $\beta = \mathrm{const}$. Let us assume that, unlike most often considered situations, Y in (1) is a complex function, which we represent in the form

$$Y(z) = X(z)\exp[\mathrm{i}\varphi(z)]. \tag{2}$$

By substituting (2) into (1) and separating the real and imaginary parts, we obtain the system of equations

$$\frac{\partial^2 X}{\partial z^2} - X\left(\frac{\partial\varphi}{\partial z}\right)^2 \pm 2(X^2 - \beta)X = 0, \tag{3a}$$

$$X\frac{\partial^2\varphi}{\partial z^2} + 2\frac{\partial X}{\partial z}\frac{\partial\varphi}{\partial z} = 0. \tag{3b}$$

Because solutions for which $X(z) \equiv 0$ are of no interest to us, we obtain from (3b) the integral

$$X^2\frac{\partial\varphi}{\partial z} = \left(X^2\frac{\partial\varphi}{\partial z}\right)\bigg|_{z=0} = \mathrm{const}, \tag{4}$$

which allows us to find cnoidal waves of new type for NSE (1). Note that, as far as we know, the integral of type (4) for the NSE was not considered earlier.

By introducing the notation $\partial\varphi/\partial z|_{z=0} = \varphi_0'$ and defining the intensity as $I(z) = X^2(z)$, which gives $I(0) = I_0 = X^2(0) = X_0^2$, we obtain

$$\frac{\partial\varphi}{\partial z} = \frac{\varphi_0' I_0}{X^2}. \tag{5}$$

By substituting now (5) into (3a), we find the ordinary differential equation

$$\frac{\partial^2 X}{\partial z^2} - \frac{(\varphi_0')^2 I_0^2}{X^3} \pm 2(X^2 - \beta)X = 0, \tag{6}$$

whose solution is the required function $X(z)$. By solving Eqn (6), thereby determining $X(z)$, we can also at once find $\varphi(z)$ by integrating (5):

$$\varphi(z) = \varphi_0 + \varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{X^2(z')}. \tag{7}$$

Here, $\varphi_0 = \varphi(0)$ is the initial ($z = 0$) phase of $Y(z)$. Note that, in the formulation of the problem described above, the known solutions of (1) in the form of cnoidal waves correspond to the limiting case for which

$$\frac{\partial\varphi}{\partial z} \equiv 0, \quad \varphi(z) \equiv \varphi_0 = \text{const}. \tag{8}$$

3. Complex periodic solutions of the NSE

We will seek the solution of (6) in the form

$$X(z) = \left[B + CF^2(z)\right]^{1/2}, \tag{9}$$

which is convenient for writing integrals (conservation of the total energy flux and the Manley–Row relation) of the initial system of equations [1, 5] describing parametric frequency conversion in terms of the intensities $I(z) = B + CF^2(z)$ of three interacting modes. Here, B and C are constants and $F = F(z)$ is a new periodic function of z. By substituting (9) into (6), we obtain

$$BC\left(\frac{\partial F}{\partial z}\right)^2 + CF(B + CF^2)\frac{\partial^2 F}{\partial z^2} - (\varphi_0')^2 I_0^2$$
$$\pm 2(B - \beta + CF^2)(B + CF^2)^2 = 0. \tag{10}$$

Let us assume now that

$$F = \mathrm{sn}(\gamma z), \tag{11}$$

which retains the functional form of the kernel of the auxiliary linear problem [4]. Note that the general solution corresponding to expressions (9) and (11) can be found by directly integrating (6) by using elliptic integrals [6]. Moreover, the substitutions $\mathrm{sn}(\gamma z) \to \mathrm{cn}(\gamma z)$ and $\mathrm{sn}(\gamma z) \to \mathrm{dn}(\gamma z)$ in (11), taking into account relations $\mathrm{cn}^2(\gamma z) = 1 - \mathrm{sn}^2(\gamma z)$ and $\mathrm{dn}^2(\gamma z) = 1 - k^2\mathrm{sn}^2(\gamma z)$, lead only to the renormalisation of constants B and C, not changing (8). Here, $0 \leqslant k \leqslant 1$ is the modulus of an elliptic function and γ is a constant. By differentiating (11) and substituting the obtained result into (10), expressing $\mathrm{cn}^2(\gamma z)$ and $\mathrm{dn}^2(\gamma z)$ in terms of $\mathrm{sn}^2(\gamma z)$ (see above) and equating coefficients at the different (from 0 to 3) powers of $\mathrm{sn}^2(\gamma z)$ to zero, taking into account that $B \equiv I_0$, we obtain that either $C = 0$ and

$$\left[(\varphi_0')^2 \mp 2(I_0 - \beta)\right] I_0^2 = 0, \tag{12}$$

or $C \neq 0$ and

$$\left[C\gamma^2 - (\varphi_0')^2 I_0 \pm 2(I_0 - \beta)I_0\right] I_0 = 0, \tag{13a}$$

$$[\gamma^2(1 + k^2) \mp (3I_0 - 2\beta)]I_0 = 0, \tag{13b}$$

$$(3I_0 - C)\gamma^2 k^2 - C\gamma^2 \pm 2(3I_0 - \beta)C = 0, \tag{13c}$$

$$k^2 = \mp C/\gamma^2. \tag{13d}$$

It follows from (13d) that the passage from the case of the focusing nonlinearity [sign '+' in (1)] to the defocusing one [sign '−' in (1)] changes the sign of C.

For $C \neq 0$, by substituting (13d) into (13b) and (13c), we find that either $I_0 = 0$ and

$$\gamma^2 = \pm C \mp 2\beta, \tag{14a}$$

$$k^2 = \frac{C}{2\beta - C}, \tag{14b}$$

or $I_0 \neq 0$ and

$$C\gamma^2 - (\varphi_0')^2 I_0 \pm 2(I_0 - \beta)I_0 = 0, \tag{15a}$$

$$\gamma^2 = \pm 3I_0 \pm C \mp 2\beta, \tag{15b}$$

$$k^2 = \frac{C}{2\beta - 3I_0 - C}, \tag{15c}$$

which determines the values of three of the four free parameters – k, γ, $B = I_0$, and C.

Let us assume that the initial intensity I_0 of the wave is a free parameter. Then, for $I_0 \neq 0$, by excluding γ^2 from (15a), we obtain the equation

$$C^2 - 2\left(\beta - \frac{3}{2}I_0\right)C + I_0\left[2(I_0 - \beta) \mp (\varphi_0')^2\right] = 0, \tag{16}$$

which gives for the required solution of NSE (1)

$$C_1 = \beta - \frac{3}{2}I_0 + \left[\left(\beta - \frac{1}{2}I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}, \tag{17a}$$

$$\gamma_1^2 = \pm\frac{3}{2}I_0 \mp \beta \pm \left[\left(\beta - \frac{1}{2}I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}, \tag{17b}$$

$$k_1^2 = \left\{\beta - \frac{3}{2}I_0 + \left[\left(\beta - \frac{1}{2}I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}\right\}$$
$$\times \left\{\beta - \frac{3}{2}I_0 - \left[\left(\beta - \frac{1}{2}I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}\right\}^{-1} \tag{17c}$$

or

$$C_2 = \beta - \frac{3}{2} I_0 - \left[\left(\beta - \frac{1}{2} I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}, \tag{18a}$$

$$\gamma_2^2 = \pm \frac{3}{2} I_0 \mp \beta \mp \left[\left(\beta - \frac{1}{2} I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}, \tag{18b}$$

$$k_2^2 = \left\{\beta - \frac{3}{2} I_0 - \left[\left(\beta - \frac{1}{2} I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}\right\}$$
$$\times \left\{\beta - \frac{3}{2} I_0 + \left[\left(\beta - \frac{1}{2} I_0\right)^2 \pm (\varphi_0')^2 I_0\right]^{1/2}\right\}^{-1}. \tag{18c}$$

As a result, a new complex periodic solution of (1)

$$Y_{1,2}(z) = \left[I_0 + C_{1,2}\mathrm{sn}^2(\gamma z)\right]^{1/2}$$
$$\times \exp\left[\mathrm{i}\varphi_0 + \mathrm{i}\varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{I_0 + C_{1,2}\,\mathrm{sn}^2(\gamma_{1,2} z')}\right] \tag{19}$$

taking (17) and (18) into account, proves to be completely specified by the boundary conditions (I_0, φ_0, φ_0'), and domains of its existence are determined by the requirements $\gamma^2 \geqslant 0$ and $1 \geqslant k^2 \geqslant 0$.

Note that a new class of solutions of NSE (1) considered here also includes two trivial situations in which

$$Y(z) = \sqrt{I_0}\exp[\mathrm{i}(\varphi_0 + \varphi_0' z)] \tag{20}$$

under the condition that

$$(\varphi_0')^2 = \pm 2(I_0 - \beta) \tag{21}$$

($C = 0$, parametric bleaching in terms of [1]) and

$$Y(z) = \sqrt{C}\,\mathrm{sn}(\gamma z)\exp(\mathrm{i}\varphi_0) \tag{22}$$

under the condition that

$$\gamma^2 = \pm C \mp 2\beta, \tag{23a}$$

$$k^2 = \frac{C}{2\beta - C} \tag{23b}$$

($I_0 = 0$, the Manakov type solution in terms of [1, 4] constructed by using the fundamental solution $\mathrm{sn}(\gamma z)$ of the first-order Lame equation). Moreover, it is easy to verify that the solutions of the class described above can be also aperiodic (i.e. solitary waves, $k = 1$), however, only in the case of defocusing nonlinearity. It follows from (17)–(19) in this case that the corresponding solitons differ from the known so-called dark solitons [7] by the unusual behaviour of the phase:

$$Y_{1,2}(z) = \left[I_0 + C_{1,2}\tanh^2(\gamma_{1,2} z)\right]^{1/2}$$
$$\times \exp\left[\mathrm{i}\varphi_0 + \mathrm{i}\varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{I_0 + C_{1,2}\tanh^2(\gamma_{1,2} z')}\right], \tag{24}$$

where

$$C_{1,2} = \beta_{1,2} - \frac{3}{2} I_0; \tag{25a}$$

$$\gamma_{1,2}^2 = -\frac{3}{2} I_0 + \beta_{1,2}; \tag{25b}$$

$$\beta_{1,2} = \frac{1}{2} I_0 \pm \sqrt{I_0}\,|\varphi_0'|. \tag{25c}$$

4. Specific features of a new class of periodic solutions of the NSE

A qualitatively new nature of the class of complex periodic solutions of the NSE described above becomes obvious if, by using from the very beginning the substitution

$$Y(z) = Y'(z) + \mathrm{i}\,Y''(z) \tag{26}$$

we separate the real and imaginary parts (quadrature components) in the required solution $Y(z)$. Then, by substituting (26) into (1) and separating the real and imaginary parts of the obtained expression, we obtain the classical system of two coupled NSEs for real variables $Y'(z)$ and $Y''(z)$ in the form

$$\frac{\partial^2 Y'}{\partial z^2} \pm 2\left[(Y')^2 + (Y'')^2 - \beta\right] Y' = 0, \tag{27a}$$

$$\frac{\partial^2 Y''}{\partial z^2} \pm 2\left[(Y')^2 + (Y'')^2 - \beta\right] Y'' = 0. \tag{27b}$$

By using now the scheme for constructing two-component cnoidal waves described in [4], taking into account that the values of β in (27a) and (27b) are the same, we find that the amplitudes of components Y' and Y'' are proportional to the same elliptic function $\theta(z)$ (i.e. to the same fundamental solution of the first- or second-order Lame equation):

$$Y'(z) = \theta(z)\sin\varphi, \tag{28a}$$

$$Y''(z) = \theta(z)\cos\varphi. \tag{28b}$$

In this case, we can consider formally that both components Y' and Y'' of this solution can be constructed simply by projecting the one-component solution $Y(z) = \theta(z)$ on the axes of a coordinate system turned through a fixed (independent of z) angle $\alpha = \varphi = \arctan(Y'/Y'')$ with respect to the initial system [4].

For the same substitution (26), the solution described in our paper will have the form

$$Y'(z) = \left[I_0 + C\,\mathrm{sn}^2(\gamma z)\right]^{1/2}$$
$$\times \sin\left[\varphi_0 + \varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{I_0 + C\,\mathrm{sn}^2(\gamma z')}\right], \tag{29a}$$

$$Y''(z) = \left[I_0 + C\,\mathrm{sn}^2(\gamma z)\right]^{1/2}$$
$$\times \cos\left[\varphi_0 + \varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{I_0 + C\,\mathrm{sn}^2(\gamma z')}\right]. \tag{29b}$$

It is easy to verify that, although the new solution is similar to the Manakov cnoidal wave with the modulus

$$|Y(z)| = [I_0 + C\,\mathrm{sn}^2(\gamma z)]^{1/2},$$

which also corresponds to the kernel of the Lame equation, the angle of rotation

$$\alpha(z) = \varphi_0 + \varphi_0' I_0 \int_0^z \frac{\mathrm{d}z'}{I_0 + C\,\mathrm{sn}^2(\gamma z')}$$

of the one-component solution upon a similar projection, providing the construction of the two quadrature components Y' and Y'' of this nondegenerate solution, performs now intricate nonlinear oscillations matched with the oscillations of $|Y(z)|$. Taking into account the transformation

$$\int_0^z \frac{\mathrm{d}z'}{I_0 + C\,\mathrm{sn}^2(\gamma z')} = \frac{1}{I_0}\int_0^{\mathrm{sn}(\gamma z)} \frac{\mathrm{d}x}{(1+\lambda x^2)\left[(1-x^2)(1-k^2x^2)\right]^{1/2}}, \tag{30}$$

where $\lambda = C/I_0$, these oscillations are described by the elliptic integral of the third kind [6]. Because some exact solutions of the problem of parametric frequency conversion (formulated, however, in the form of coupled nonlinear equations rather than in the NSE form) of type (19) were already presented earlier, the character of consistent nonlinear oscillations of the modulus and phase of solutions of this type is similar to that described in [8].

5. Conclusions

We have shown that the NSE has a new class of periodic solutions compared to that considered in [4]. These solutions are complex and, upon passing to real variables (quadrature components), describe nondegenerate cnoidal waves consisting of two 'incoherent' (noninterfering) components. The solutions of this new class are similar in form to degenerate cnoidal waves (so-called Manakov periodic solutions), however, the angle of projection [4], which provides the formation of the two components of the solution, performs intricate nonlinear oscillations consistent with the oscillations of their modulus.

Note that the solutions of this new class can be written in different forms. Thus, to change the boundary conditions, solution (19) presented above can be shifted along the z axis. For example, the shift of the argument $\xi \to \xi + K$ of the elliptic function by a quarter of a period is described by the well-known transformation $\mathrm{sn}\,\xi \to \mathrm{cn}\,\xi/\mathrm{dn}\,\xi$ [6]. Here K is the total elliptic integral of the first kind. There exist many other identical transformations and changes, which simultaneously change the argument ξ and modulus k of elliptic functions [6].

Because the NSE taking into account the lowest (cubic) order of inertialless nonlinearity has a rather universal character, one can expect that the new class of periodic solutions described above might be also useful for other fields of physics.

These solutions can be used in problems of the propagation of pulse trains in optical fibres [7, 9], the propagation of light beams with a special periodic transverse structure through photorefractive crystals [4, 10], and also in nonlinear hydrodynamics [11], plasma physics [12], the analytic description of coupled wave packets consisting of electron wave functions [13, 14], etc.

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