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Qutrits in multiparticle systems

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***Abstract*. The evolution of complex composite quantum systems which can be reduced to systems with a small dimensionality of the Hilbert space (of the qutrit, ququart type, etc.) is considered. In the case of the interaction of an ensemble of two-level atoms with light, the conditions are found under which a qutrit is produced from the light and atomic states. The properties and possible applications of a qutrit based of the Fock states of light in which two photons are distributed among three modes are discussed. It is shown that this state has the nonclassical photon statistics, is entangled and can be used as a quantum channel for the teleportation, dense coding, and key distribution.**

Keywords: *multiparticle systems, entangled states, quantum communications.*

1. Introduction

A qutrit as a system with three states composed from a few particles is well known in quantum optics. It can be formed, for example, by two biphotons composed from polarised photons. Such a state was observed experimentally [1] and is of interest for the key distribution problem in quantum cryptography [2]. The basic element in the quantum information theory is a qubit – a quantum system with two states. However, the use of the higher-dimensionality systems such as a qutrit, a ququart, etc. can extend the possibilities for solving the problems of quantum information transfer [3]. This determines interest in studying the methods of generation and properties of high-dimensional quantum-system states [4].

A state of a qutrit can be entangled, which is required for solving quantum communication problems such as teleportation, dense coding, and quantum key distribution. That is why the properties, the methods of generation, and applications of qutrits are of interest both from the theoretical and practical points of view and are being extensively studied. In particular, the characteristics or degrees of entanglement of three-level systems were discussed in [5], and the geometric measure for three-particle pure states was introduced in [6]. The generation and reconstruction (tomography) of the qutrit state based on the experimental characteristics of biphoton fields were performed in a number of papers [7]. The quantum protocol of the key distribution for three-level systems was studied in [8], and the entanglement exchange in the multiparticle state of multilevel systems was investigated in [9]. The application of the entangled state of two qutrits in the calculation Grover algorithm was proposed in [10]. The important specific features of the preservation of the state of a multiparticle qutrit system upon collective interaction were discussed in [11].

In this paper, we study some types of the interaction of two and three systems resulting in the qutrit formation. The reduction of a multicomponent system to a three-level system is based physically on the integral of motion describing the preservation of a total number of excitations. As a result, in the cases when the number of excitations is small, the excitation energy is transferred due to interaction from one system to another and a small number of states are involved in the evolution. We consider several physical models, in particular, the interaction of a mode with an ensemble of two-level atoms and the interaction of three modes in a nonlinear medium. In the latter case, a qutrit can appear which consists of the Fock light states in which two photons are distributed among three modes. This state has the sub-Poisson photon statistics, is squeezed and entangled. The protocols of teleportation, dense coding, and key distribution are presented in which the found qutrit can be used as a quantum channel.

2. Qutrit formed by two systems

Consider the formation of a qutrit (three-level system) during the interaction of systems A and B with a large number of degrees of freedom, which can have different physical nature. The reduction is possible due to the integral of motion preserving the total number e of excitations. In the case of $e = 2$, the two systems can form a qutrit because one excitation can be distributed between two systems, for example, in the following three ways: $\{y\} = |2,0\rangle,\ |1,1\rangle,\ |0,2\rangle$. In this case, the physical realisation of a qutrit depends on the nature of systems under study. We will consider below the two types of interaction: (i) the interaction of the two modes of an

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electromagnetic field and (ii) the interaction of one mode with an ensemble of two-level atoms.

Let the systems A and B be two modes of an electromagnetic field which are mixed on a nonabsorbing beamsplitter. This interaction is described by the Hamiltonian

$$H_1 = \mathrm{i}\hbar g(a^\dagger b - ab^\dagger), \quad (1)$$

where $\hbar$ is Planck's constant; g is the coupling constant; $a^\dagger, a$ and $b^\dagger, b$ are the creation and annihilation operators for the first and second modes. The problem has the integral corresponding to the preservation of a total number of photons in modes $a^\dagger a + b^\dagger b = \text{const}$, which gives simple solutions for operators: $a(t) = ca + sb$, $b(t) = -sa + cb$, where $c, s = \cos\theta,\ \sin\theta$; $\theta = gt$; and t is time. The quantities c and s can be considered as the transmission and reflection coefficients of the beamsplitter, respectively.

The evolution of a set of the Fock states $\{y\} = |1,1\rangle$, $|2,0\rangle$, $|0,2\rangle$ is described by the expressions

$$|1,1\rangle \to \sqrt{2}cs(|2,0\rangle - |0,2\rangle) + (c^2 - s^2)|1,1\rangle,$$

$$|0,2\rangle \to s^2|2,0\rangle + c^2|0,2\rangle + \sqrt{2}cs|1,1\rangle, \quad (2)$$

$$|2,0\rangle \to c^2|2,0\rangle + s^2|0,2\rangle - \sqrt{2}cs|1,1\rangle.$$

This means that after the propagation of modes through the beamsplitter, a qutrit appears which is indifferent to the interaction under study.

Consider the second example in which the system A is a mode of an electromagnetic field, while the system B is an ensemble of N two-level atoms. Their interaction is described by the Hamiltonian

$$H_2 = \mathrm{i}\hbar g(aS_{10} - a^\dagger S_{01}), \quad (3)$$

where $S_{10} = \sum_a s_{10}(a)$ is the collective atomic operator and $s_{xy} = |x\rangle_a\langle y|$ $(x, y = 0, 1)$ is the operator of an atom with the number a, the lower and upper states 0 and 1, respectively.

The integral of motion of the system describes the preservation of a total number of excitations $I = a^\dagger a - \Xi_z$, where $\Xi_z = (1/2)\sum_a (|0\rangle_a\langle 0| - |1\rangle_a\langle 1|)$. The interaction between atoms and the field leads to the qutrit formation. Consider, for example, atoms in the ground state assuming that the initial state of the field is the two-photon Fock state. Then, the initial excitation (the energy of two photons) will be periodically redistributed between the atomic ensemble and light, and three states will be involved in this process. In this case, the Dicke states $|h;N\rangle$ will appear in the atomic ensemble which describe an ensemble of N two-level atoms in which $h \leqslant N$ atoms are excited,

$$|h;N\rangle = \sum_z P_z|1_1, 1_2, ..., 1_h, 0_{h+1}, ..., 0\rangle = S_{10}^h|0, ..., 0\rangle/h!, \quad (4)$$

where P_z is one of the $C_h^N = N!/[h!(N-h)!]$ distinguishable permutations of particles. States (4) are normalised by the condition $\langle h;N|h;N\rangle = C_h^N$. In our case, $h = 2$, and the introduced states are a particular case of the Dicke state with $j = m = N/2$, where j and m are the eigenvalues corresponding to the two eigenvectors of collective operators $J^2 = J_x^2 + J_y^2 + J_z^2$ and J_z. Because $j = N/2$, these states are symmetric with respect to the permutation of particles. A particular case with $N = 3$ is known in the quantum information theory as the W state: $W = (1/\sqrt{3})\times(|100\rangle + |010\rangle + |001\rangle)$.

In the case considered here, a qutrit is formed by the set of states

$$|1,1\rangle = |1\rangle \otimes |1;N\rangle/\sqrt{N},$$

$$|2,0\rangle = |2\rangle \otimes |0;N\rangle,$$

$$|0,2\rangle = |0\rangle \otimes |2;N\rangle/\sqrt{C_2^N},$$

which are transformed to themselves due to the interaction:

$$|1,1\rangle \to \cos\theta|1,1\rangle + \theta\frac{1}{\sqrt{2N-1}}(\sqrt{N-1}\,|0,2\rangle - \sqrt{N}\,|2,0\rangle),$$

$$|2,0\rangle \to \frac{1}{2N-1}\{\sqrt{(2N-1)N}\sin\theta\,|1,1\rangle + \sqrt{N(N-1)}$$
$$\times(1-\cos\theta) + [N(1+\cos\theta) - 1]|2,0\rangle\}, \quad (5)$$
$$|0,2\rangle \to \frac{1}{2N-1}\{-\sqrt{(N-1)(2N-1)}\sin\theta|1,1\rangle$$
$$+ [N + (N-1)\cos\theta]|0,2\rangle + \sqrt{N(N-1)}$$
$$\times(1-\cos\theta)|2,0\rangle\}.$$

Here, $\theta = gt\sqrt{2(2N-1)}$. In the limit of a large number of atoms ($N \gg 1$), the structure of the states of the produced qutrit is determined by expression (2), where $\theta = gt \to gt\sqrt{N}$. Such analogy between the light and atomic states is caused by the fact that the commutation relations for atomic operators S_{01} and S_{10} in the limit $N \gg 1$ are reduced to boson relations and the Hamiltonian H_2 is reduced to H_1.

The squares of moduli of coefficients at the wave functions determine the probabilities of the corresponding states and depend on the duration of interaction, which can be interrupted at any moment, thereby producing one of the qutrit states. Consider, for example, the last equation in (5) in which the probabilities of the states in the superposition in the limit $N \gg 1$ are

$$\mathrm{Prob}(|0,2\rangle) = (1+\cos\theta)^2/4,$$

$$\mathrm{Prob}(|1,1\rangle) = \sin\theta^2/2,$$

$$\mathrm{Prob}(|2,0\rangle) = (1-\cos\theta)^2/4.$$

It follows from this that after switching on the interaction, the system passes to the superposition of three states with statistical weights periodically changing in time. This means that the three states $|1,1\rangle$, $|2,0\rangle$ and $|0,2\rangle$ will be simultaneously recorded and stored in the system. The two of them, $|0,2\rangle$ or $|2,0\rangle$, can be reproduced by switching off the interaction. Indeed, if $\theta=\pi$, then $\mathrm{Prob}(|2,0\rangle)=1$, and if $\theta=2\pi$, then $\mathrm{Prob}(|0,2\rangle)=1$. Other qutrit states can be formed in the same way. Thus, it follows from the first equation in (5) that a maximally entangled Einstein–Podolsky–Rozen (EPR) pair appears from the $|1,1\rangle$ state at $\theta=\pi/2$. This pair consists of two photons and an atomic ensemble with two excited atoms:

$$|\mathrm{EPR}\rangle=\left[\left(1/\sqrt{C_2^N}\right)|0\rangle\otimes|2,N\rangle-|2\rangle\otimes|0;N\rangle\right]/\sqrt{2}.$$

3. Qutrit formed by three systems

The three interacting systems A, B and C also can form a qutrit. In this case, many different possibilities appear. Consider, in particular, the case with the number of excitations $e=1$ and 2. For $e=2$, the states $\{y\}=|1,1,0\rangle$, $|1,0,1\rangle$, $|0,1,1\rangle$ are possible. We will discuss the realisation of such a qutrit based on three electromagnetic-field modes. This case is nontrivial because for $e=2$, a wider set of possible states $\{y\}=|1,1,0\rangle$, $|1,0,1\rangle$, $|0,1,1\rangle$, $|2,0,0\rangle$, $|0,2,0\rangle$, and $|0,0,2\rangle$ appears for photons.

In this example, the two modes a and b of the electromagnetic field play the role of systems A and B, while the system C is formed by an ensemble of N three-level atoms of the Λ-configuration with the $0\to2\to1$ transitions. We represent the interaction of modes and atoms in the form

$$H_3=\mathrm{i}\hbar g(aS_{20}-a^{\dagger}S_{02})+\mathrm{i}\hbar g(bS_{21}-b^{\dagger}S_{12}),\tag{6}$$

where S_{xy} $(x,y=0,1,2)$ are the collective atomic operators (see above) and the interaction constants g are assumed the same for simplicity. In this case, a qutrit is formed from the states with one excitation, when $\{y\}=|0,0,1\rangle$, $|1,0,0\rangle$, $|0,1,1\rangle$, where $|0,0,1\rangle=|0\rangle\otimes|0\rangle\otimes S_{20}|0;N\rangle/\sqrt{N}$; $|1,0,0\rangle=|1\rangle\otimes|0\rangle\otimes|0;N\rangle$; $|0,1,1\rangle=|0\rangle\otimes|1\rangle\otimes S_{10}|0;N\rangle/\sqrt{N}$. Thus, by considering the evolution of the initial state $|1,0,0\rangle$, we find

$$|1,0,0\rangle\to\frac{1}{N+1}[(1+N\cos\theta)|1,0,0\rangle$$

$$-\sqrt{N}(1-\cos\theta)|0,1,1\rangle+\sin\theta\sqrt{N(N+1)}\,|0,0,1\rangle],$$

where $\theta=\sqrt{N+1}gt$.

Consider the state of light in which two excitations are distributed among three modes a, b, and c in such a way that only one excitation corresponds to each of the modes:

$$\eta=A|110\rangle+B|101\rangle+C|011\rangle,\tag{7}$$

where $|A|^2+|B|^2+|C|^2=1$ and the states of light are the Fock states. For the atomic system consisting of three two-level atoms, such a state looks trivial and can appear upon absorption of two photons by atoms. This is not the case for the Fock states of light because one mode can contain two photons, for example, $|200\rangle$. This means that η cannot be obtained by the determinate method with the help of linear optical elements like beamsplitters. However, such a state can be obtained upon three-photon interaction in nonlinear media. Thus, by considering the simultaneous process of nondegenerate down conversion, in which three classical pump waves are transformed to photon pairs $a-b$, $a-c$ and $b-c$, we find the effective Hamiltonian

$$H_{\mathrm{eff}}=\mathrm{i}\hbar(k_1a^{\dagger}b^{\dagger}+k_2a^{\dagger}c^{\dagger}+k_3b^{\dagger}c^{\dagger}-k_1ab-k_2ac-k_3bc),\tag{8}$$

where a, b, and c are the annihilation operators for the corresponding modes and k_x $(x=1,2,3)$ are the interaction constants. Such a process was considered in [12] for the resonator scheme in which light fields were described by using continuous variables.

For the initial vacuum state of modes in the linear approximation in the interaction, we have

$$\eta'=\mu|\mathrm{vac}\rangle+\epsilon[A|110\rangle+B|101\rangle+C|011\rangle]_{abc},\tag{9}$$

where μ,ϵ are assumed real and it is also assumed that the normalisation condition $\mu^2+\epsilon^2=1$, and $\epsilon A=k_1t$, $\epsilon B=k_2t$, $\epsilon C=k_3t$. Unlike the state η, the state η' contains the vacuum state, which plays no role in a number of cases. Moreover, it can be excluded by postselection.

We will consider the physical properties of the qutrit formed by three modes taking into account the contribution of vacuum, by paying a special attention to the case $\epsilon\ll1$, which allows us to perform analysis taking into the possibility of the realisation of the qutrit. Without any calculations, we can point out that the state η' has two essentially quantum features. First, it is formed from the Fock states of modes, which can lead to the sub-Poisson statistics of photons. This statistics can be characterised by the Mandel parameter ξ describing the difference of the variance of the number of photons from the Poisson distribution with the help of the relation $\langle(\Delta n)^2=\langle n^2\rangle-\langle n\rangle^2=\langle n\rangle(1+\xi)$. Second, because the state of the entire field is pure and is described by the wave function, it has coherence.

State (9) has the following statistical properties:

(i) The photon statistics in each mode is sub-Poisson with the parameter $\xi_x=-\langle n_x\rangle$, where $x=a,b,c$. Thus, $\xi_a=-\langle n_a\rangle=-(|A|^2+|B|^2)\epsilon^2$, and for $\epsilon\ll1$, the deviation from the Poisson level is small.

(ii) The simultaneous photon counting rate and the random coincidence rate for each mode pair are equal to $\langle n_an_b\rangle=\epsilon^2|A|^2$ and $\langle n_a\rangle\langle n_b\rangle=\epsilon^4(|A^2+|BC|^2)$, respectively. This means that for $\epsilon\ll1$, the photon bunching appears because $\langle n_an_b\rangle>\langle n_a\rangle\langle n_b\rangle$. Due to the presence of only two photons in state (9), the combined counting rate for three photons is naturally zero and $\langle n_an_bn_c\rangle=0$.

(iii) The difference intensity of the mode pair has the sub-Poisson statistics. By considering the operators of the difference and sum of the number of photons $n_-=n_a-n_b$ and $n_+=n_a+n_b$ we find the Mandel parameters for the variance n_- and n_+. For $\epsilon\ll1$ the corresponding expressions take the form

$$\xi_-\approx-\frac{2|A|^2}{2|A|^2+|B|^2+|C|^2},$$
$$\xi_+\approx\frac{2|A|^2}{2|A|^2+|B|^2+|C|^2}.\tag{10}$$

For $A=B=C$, the parameter $\xi_-=-1/2$ and variance are smaller by half than those for the Poisson distribution, which is caused by the quantum intermode pair photon correlation. Unlike the difference intensity, the sum-intensity statistics is super-Poisson.

The coherence of state (9) (which formally corresponds to the presence of nondiagonal elements of the density matrix) leads to the appearance of the phase-sensitive properties of this state, among which the squeezed states of light are of particular interest. The state η' has the following properties:

(i) Any pair of modes can be found in the squeezed state. Let modes a and b be mixed on a nonabsorbing beamsplitter which performs the transformation of type (1). Then, the variance $\langle(\Delta X)^2\rangle$ of the quadrature operator $X(\theta) = d^\dagger \exp(\mathrm{i}\theta) + \mathrm{H.c.}$ on one of the beamsplitter outputs, where $d = ca + sb$, will be equal to $1 + 2\epsilon^2(|A|^2 + |cB + sC|^2) + \mu\epsilon 2cs(A\mathrm{e}^{-2\mathrm{i}\theta} + \mathrm{c.c})$, where θ is the reference wave phase. For $c = s$, $B = C$, $\arg A - 2\theta = \pi$ and $\epsilon \ll \mu$ the squeezed state of light appears because $\langle(\Delta X)^2\rangle = 1 - 2\epsilon\mu|A|(1 - (\epsilon/\mu)|A|) < 1$. It is obvious that, if the mode is squeezed, it cannot have the sub-Poisson photon statistics due to the uncertainty relation. In conditions under study, the mode d (or the pair a, b) has the Mandel parameter $\xi_d = |A|^2 - \epsilon^2$, which cannot be negative in the case of squeezing, when $(\epsilon/\mu)|A| < 1$.

(ii) The state η' can be squeezed and can have the sub-Poisson photon statistics. Let the modes a and b be mixed on a beamsplitter and the modes d and r be obtained. Then, the state d can be squeezed and the statistics of the difference number of phonons in the r and c modes can be sub-Poisson.

The state η' has another important property. It is entangled because it cannot be represented as a product of the wave functions of individual modes. This property will be manifested, for example, depending on the result of the interference of modes a and b during the projection measurement of the third mode c. Let the modes a and b be mixed on a beamsplitter having a detector D_d located at one of its outputs, which detects the number of photons in the mode d. Let the mode c be detected with a detector D_c, which performs measurements in the $|0\rangle_c, |1\rangle_c$ basis. The results of these measurements appear with the probabilities $\mathrm{Prob}(0) = \mu^2 + \epsilon^2|A|^2$ and $\mathrm{Prob}(1) = \epsilon^2(|B|^2 + |C|^2)$. In this case, the wave function of the initial state η' is projected to the following states $\eta' \to (\mu|\mathrm{vac}\rangle + \epsilon A|110\rangle)/\sqrt{\mathrm{Prob}(0)}$ and $\eta' \to (B|101\rangle + C|011\rangle)/\sqrt{\mathrm{Prob}(1)}$. The results of interference of the modes a and b in these two cases are substantially different. If the detector D_c gave the result 0, the average number of photons, or the light intensity on the detector D_d, will be

$$\langle d^\dagger d\rangle_0 = \frac{\epsilon^2|A|^2}{\mu^2 + \epsilon^2|A|^2} \propto \epsilon^2. \tag{11}$$

If the result 1 is detected, then

$$\langle d^\dagger d\rangle_1 = \frac{|cB + sC|^2}{2(|B|^2 + |C|^2)} \leqslant 2. \tag{12}$$

One can easily see that for $\epsilon \ll 1$, the light intensity for the result 1 proves to be considerably higher.

Being entangled, the state of the qutrit composed of three modes can be used as a quantum channel for various problems of the quantum information theory. Below, we will discuss protocols for the ideal case of the state η. This state can be used in these problems in accordance with two observations. Thus, the state η is related to the W state $A|001\rangle + B|010\rangle + C|001\rangle$ by the local unitary operation. This means that the properties of both states are identical from the information point of view, and both these states can be used to solve the same problems. A particular case is the asymmetric state $W^* = (1/\sqrt{2})|011\rangle + (1/2)(|110\rangle + |101\rangle)$, which can be related to the tree-particle Greenberg–Horne–Zeilinger (GHZ) state

$$\mathrm{GHZ} = 1/\sqrt{2}(|000\rangle + |111\rangle) \tag{13}$$

with the help of the two-particle unitary but nonlocal operation V

$$(1 \otimes V)|GHZ\rangle_{ABC} = |W^*\rangle, \tag{14}$$

where $V = |\Psi^+\rangle\langle 11| + |00\rangle\langle 10| + |\Psi^-\rangle\langle 01| + |11\rangle\langle 00|$; $\Psi^\pm = (|10\rangle \pm |01\rangle)/\sqrt{2}$. The GHZ state can be used as a quantum channel for the teleportation of one particle [13] and the unknown entangled state [14]. The quantum channel formed by the W states in these cases was studied in papers [15–17], respectively.

Consider the teleportation of the unknown purely entangles state of the form

$$|A\rangle = (\alpha|01\rangle + \beta|10\rangle)_{12}, \tag{15}$$

where $|\alpha|^2 + |\beta|^2 = 1$. This process involves five particles with the initial state $|A\rangle_{12} \otimes |\Omega\rangle_{ABC}$, the particles from the channel Ω being shared with the sender A and two receivers B and C, spatially separated. If the sender A wants to transmit state (15) to receivers B and C, it is necessary to perform the measurement on three particles 1, 2, and A in a certain basis $\{\Phi_x\}$. Thus, when the GHZ channel (13) is used, the measurement basis has the form

$$\{\Phi_x : \pi_1^\pm \otimes \Phi_{2A}^\pm, \pi_1^\pm \otimes \Psi_{2A}^\pm\}, \tag{16}$$

where $\Phi^\pm = (|00\rangle \pm |11\rangle)/\sqrt{2}$; $\pi^\pm = [|0\rangle \pm \exp(\mathrm{i}\theta)|1\rangle]/\sqrt{2}$. The performed measurement projects all the particles with the same probability $\mathrm{Prob}(\Phi_x) = 1/8$ to one of the eight states $|\Phi_x\rangle_{12A} \otimes |BC_x\rangle_{BC}$, where $|BC_x\rangle_{BC}$ is the state of particles of receivers. The probability of each measurement is independent of the properties of the initial state $|A\rangle_{12}$ determined by its coefficients. This means that the problem can be solved because there exists a set $\{U_x\}$ of recovering unitary operators, which act independently on particles B and C ($U_x = B_x \otimes C_x$) by reconstructing the unknown state. The equation describing teleportation by using such a channel has the form

$$|A\rangle_{12} \otimes |\mathrm{GHZ}\rangle_{ABC} = \sum_x |\Phi_x\rangle_{12A}\sqrt{\mathrm{Prob}(\Phi_x)}$$

$$\times (B_x \otimes C_x)|A\rangle BC, \tag{17}$$

where the operators B_x and C_x are determined by the known Pauli operators. For example, if the outcome corresponding to the basis function $\Phi_0 = \pi^+ \otimes \Phi^+$ is obtained, these operators are defined as $B_0 = \sigma_x$ and $C_0 = 1$. The change of the channel with the help of relation (14) leads to the change in the equation

$$|A\rangle_{12} \otimes (1 \otimes V)|\mathrm{GHZ}\rangle_{ABC}$$

$$= \sum_x |\Phi_x\rangle_{12A} \sqrt{\mathrm{Prob}(\Phi_x)}\, V(B_x \otimes C_x)|A\rangle_{BC}, \qquad (18)$$

which means the modification of the recover operators and their new set. Moreover, while upon using the GHZ-state quantum channel the recover operators were local and acted independently on their particles B and C, now the recover operators are nonlocal and cannot be represented as a direct product of operators acting on the corresponding subsystems. It is this circumstance that is the specific feature of the use of the W-state channel.

Another protocol of the quantum information theory, in which state (14) can be used, is the dense coding protocol. The corresponding scheme provides the increase in the classical capacity of the quantum channel. It contains a coder and a measurement scheme. The encoder performs coding of classical information by acting on a quantum state, which is described by a unitary operator. The measurement scheme is used to obtain information from the transmitted state. In the case of two independent qubits, each of them is measured in two independent bases, and as a result, the capacity of the channel is equal to unity. The property of the entanglement of a two-particle state of the EPR pair type makes it possible to considerably increase the capacity. Indeed, because particles in this case have the quantum correlation, the classical information is encoded by acting only on one of the particles in the pair. The information is obtained from the transmitted state by performing the measurement in the Bell basis. As a result, the capacity of the quantum channel achieves the value equal to two. If the three-particle GHZ channel or the W state is used, information is encoded by acting only on two subsystems of the channel, while the measurement is performed in the three-particle basis determined by using eight distinguishable states. As a result, the capacity of such a channel has the intermediate value equal to 3/2.

Consider now this protocol for the three-particle case in more detail. Let us assume that there exists the GHZ-state quantum channel of type (13). To transmit three bits of classical information 000, 001,111, a coder encodes them by using eight distinguishable states of the three-particle system. These states can be obtained by the action of a set of local unitary operators $U_x = B_x \otimes C_x$ on the particles B and C in channel (13)

$$|D_x\rangle_{ABC} = \mathbb{1} \otimes B_x \otimes C_x |\mathrm{GHZ}\rangle_{ABC}. \qquad (19)$$

As a result, a set of all functions is formed, each of them corresponding to a certain three-particle classical information bit. For example, the 000 bit is encoded by the state $|D_{000}\rangle_{ABC} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}|\mathrm{GHZ}\rangle_{ABC}$. The subsequent measurement of the channel state in the three-particle entangled basis allows one to identify and distinguish each of the transmitted states and, by comparing the result with one or another bit, to obtain information. The use of the W state of type (14) as a resource leads to a change in the form of the states encoding three bits of classical information. Because (13) and (14) are coupled by the nonlocal two-particle operator V, the new states can be obtained by the nonlocal transformation of generators U_x:

$$|D_x^{'}\rangle_{ABC} = (\mathbb{1} \otimes V)(\mathbb{1} \otimes B_x \otimes C_x)(\mathbb{1} \otimes V)^{\dagger}|W\rangle_{ABC}. \qquad (20)$$

Of course, to realise the protocol, it is also necessary, to change adequately the measurement basis.

Symmetric state (7) in which the probability amplitudes of the basis functions have the same value $A = B = C = 1/\sqrt{3}$, can be used to solve the problem of the quantum key distribution. Let this state be shared among the three parties A, B, and C. The observer A performs the measurement with his particle in the channel in the basis $|0\rangle_A, |1\rangle_A$. The outcomes obtained in such a measurement lead, according to the projection postulate, to two different states of particles of parties B and C. Let us assume that the outcome of measuring performed by A corresponds to the $|1\rangle_A$ state, then the state in the B and C hands will be the entangled $(1/\sqrt{2})(|10\rangle + |01\rangle)_{BC}$ state of the EPR pair. On the contrary, the measuring outcome $|0\rangle_A$ corresponds to the presence of the independent $|11\rangle_{BC}$ state in the B and C hands. The correlated state is very important in this protocol and means that some action can be performed only in the case if all the participants are interested in it. Note that A uses the measurement procedure corresponding to its basis states in a random way. In this case, the quantum-correlated state appears with the probability 2/3, while the independent state appears with the probability 1/3, which is determined by the form (7) of the used symmetric qutrit state.

4. Conclusions

We have considered one of the features of the behaviour of a complex physical system when only a small part of the quantum states from its Hilbert space is involved in the evolution process. In this case, the behaviour of the system can be described with the help of a simple system with a small number of levels of the qutrit type. We have shown that such behaviour is determined by the integrals of motion describing the preservation of the total number of excitations in the system. As examples we considered a qutrit formed by mixing two light modes in the Fock state on a beamsplitter and a qutrit formed due to the interaction of ensembles of two-level atoms and the electromagnetic field in the Fock state. These two examples correspond to the case when one excitation is distributed among the three degrees of freedom. Based on the three-photon parametric interaction in a transparent nonlinear medium, we considered another case – the appearance of the light state upon the distribution of two excitations among three modes. From the point of view of quantum statistics, such a state of bosons looks nontrivial because no more than one excitation corresponds to each mode in it. This stage of light proves to be entangled, which makes possible to use it as a quantum channel for the teleportation, dense coding, and quantum key distribution.

References

1. Krivitsky L.A., Kulik S.P., Penin A.N., Chekhova M.V. *Zh. Eksp. Teor. Fiz.*, **124**, 943 (2003).
2. Bechmann-Pasquinucci H., Peres A. *Phys. Rev. Lett.*, **85**, 3313 (2000).
3. Bechmann-Pasquinucci H., Tittel W. http://trefoil.math.ucdavis.edu/9910.4095.
4. Maslennikov G., Moreva E., Spraupe S., Kulik S.P. *Phys. Rev. Lett.*, **97**, 023602 (2006).

5. Caves C.M., Milburn G.J. *Opt. Commun.*, **179**, 439 (2000).
6. Kazakov A.Ya. *Int. J. Quantum Inform.*, **4**, 907 (2006).
7. Burlakov A.V., Krivitsky L.A., Kulik S.P., Maslennikov G.A., Chekhova M.V. *Opt. Spektrosk.*, **94**, 743 (2003); Zhukov A.A., Maslennikov G.A., Chekhova M.V. *Pis'ma Zh. Eksp. Teor. Fiz.*, **76**, 696 (2002); Bogdanov Yu., Chekhova M., Krivitsky L., Kulik S.P., Kwekh C., Tey M.K., Oh C., Zhukov A. *Phys. Rev. A*, **70**, 042303 (2004).
8. Bruss D., Macchiavello C. *Phys. Rev. Lett.*, **88**, 127901 (2002).
9. Bouda J., Buzek V. *J. Phys. A Math. Gen.*, **34**, 4301 (2001).
10. Chamoli A., Bhandari C.M. http://front.math.ucdavis.edu/0609.4010.
11. Mandilara A., Akulin V.M. http://front.math.ucdavis.edu/0609.4121v1.
12. Bradley A.S., Olsen M.K., Pfister O., Pooser R.C. *Phys. Rev. A*, **72**, 053805, (2005).
13. Karlsson A., Bourennane M. *Phys. Rev. A*, **58**, 4394 (1998).
14. Gorbachev V.N., Trubilko A.I. *Zh. Eksp. Teor. Fiz.*, **118**, 1036 (2000); Marianatto L., Weber T. *Found. Phys. Lett.*, **13**, 119 (2000).
15. Agrawal P., Pati A. *Phys. Rev. A*, **74**, 0602320 (2006).
16. Gorbachev V.N., Trubilko A.I., Rodichkina A.A., Zhiliba A.I. *J. Quantum Inform. Comput.*, **2**, 367 (2002).
17. Gorbachev V.N., Trubilko A.I., Rodichkina A.A. *Phys. Lett. A*, **314**, 267 (2003).