*Quantum Electronics* **37** (12) 1097 – 1101 (2007) ©2007 Kvantovaya Elektronika and Turpion Ltd

PACS numbers: 04.80.Nn; 07.60.Ly
DOI: 10.1070/QE2007v037n12ABEH013675

# Parametric oscillatory instability in a signal recycled LIGO interferometer

S.P. Vyatchanin, S.E. Strigin

***Abstract*. The undesirable effect of parametric oscillatory instability in a LIGO (Laser Interferometer Gravitational-Wave Observatory) laser gravitational-wave antenna with a signal-recirculation mirror is analysed in detail. The instability is manifested in excitation of the Stokes optical mode and elastic mechanical mode of the mirror. It is shown that, if the eigenfrequencies of Fabry – Perot resonators in the interferometer arms are different, the parametric instability is quite small due to a small passband band width.**

***Keywords***: *quantum measurement theory, LIGO interferometer, gravitational waves.*

## 1. Introduction

Presently the sensitivity of a LIGO laser gravitational-wave antenna expressed in terms of the amplitude of the metric variation is approximately three times worse than its planned value $h \simeq 1 \times 10^{-21}$ in the 100-Hz frequency band [1, 2]. After the further improvement of a system of isolation from noises of different types in mirrors of 4-km optical Fabry – Perot resonators and increasing the optical power $W$ of a wave circulating in resonators up to $\sim 830$ kW, it is planned to achieve the sensitivity $h \simeq 1 \times 10^{-22}$.

In [3], the undesirable effect of parametric oscillatory instability in a Fabry – Perot resonator, which can considerably reduce the sensitivity of the antenna, was analysed. This effect appears if the optical power $W_c$ of a wave circulating in the fundamental optical mode exceeds a certain threshold value under the condition that the frequency difference $\omega_0 - \omega_1$ between the fundamental optical mode and the Stokes mode is close to the frequency of $\omega_m$ of the elastic mode of the resonator mirror. The interaction between these three modes appears due to the pondermotive pressure of light in optical and Stokes modes and the parametric action of mechanical vibrations of the mirror on optical modes. If the light power $W_c$ exceeds the critical value, the amplitude of mechanical vibrations, as the optical power in the Stokes mode, increases exponentially. Later, it was shown in [4] that, if the influence of the anti-Stokes mode at the frequency $\omega_{1a} = \omega_0 + \omega_m$ is taken into account, the effect of parametric oscillatory instability will be considerably weakened or even completely excluded.

In [5], a detailed analysis was performed for a power-recycling mirror LIGO interferometer and it was shown that the anti-Stokes mode cannot suppress completely the parametric oscillatory instability. This effect can be suppressed by varying the shape of mirrors or introducing the low-noise damping [6]. It was also proposed [7 – 9] to reduce the role of the parametric instability by heating probe masses to change the radius of curvature of interferometer mirrors and by controlling the values of detunings and also reducing the overlap factors for optical and elastic modes. Note that the parametric instability was recently observed experimentally [10].

The parametric instability in a signal-recirculation mirror LIGO interferometer with identical Fabry – Perot resonators in its arms was analysed in detail in [11]. However, in practice such resonators have in the general case different eigenfrequencies because the radii of curvature of the mirrors in different arms differ from each other by a few metres ($\sim$0.1 %). For example, for the Gauss – Hermite mode frequencies $q_{qmn}$ in a Fabry – Perot resonator with the radii of curvature $R_1$ and $R_2$ of mirrors separated by the distance $L$, we have [12]

$$\omega_{qmn} = \frac{\pi c}{L}\left[q + (m+n+1)\frac{\phi}{\pi}\right], \qquad (1)$$

$$\phi \equiv \arccos\left[\pm (g_1 g_2)^{1/2}\right], \qquad g_{1,2} = 1 - \frac{L}{R_{1,2}}. \qquad (2)$$

Here, $q = 0, 1, 2\ldots$ is the longitudinal coefficient and $m$ and $n$ are the transverse indices of optical modes; $g_1$ and $g_2$ are the $g$-factors; and $c$ is the speed of light. The sign ‘ + ’ in the expression for $\phi$ is used for $g_1 > 0$ and $g_2 > 0$, while the sign ‘$-$’ corresponds to $g_1 < 0$ and $g_2 < 0$; different signs (i,e. $g_1 g_2 < 0$) correspond to an unstable resonator.

For the improved LIGO scheme with

$$R_{1,2} = 2076 \pm 3 \text{ m}, \quad L = 4 \text{ km}, \qquad (3)$$

$$g_{1,2} = g \simeq -0.926 \pm 3 \times 10^{-3}, \quad \phi \simeq 0.385, \qquad (4)$$

the optical mode frequencies will differ from each other by the value

$$\Delta f_{qmn} = \pm\frac{\Delta\omega_{qmn}}{2\pi} \simeq \pm(m+n+1)100 \text{ Hz}. \qquad (5)$$

**S.P. Vyatchanin, S.E. Strigin** Department of Physics, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119992 Moscow, Russia; e-mail: svyatchanin@phys.msu.ru, strigin@phys.msu.ru

Received 5 July 2007; revision received 15 August 2007
*Kvantovaya Elektronika* **37** (12) 1097 – 1101 (2007)
Translated by M.N. Sapozhnikov

Taking into account that it is planned to obtain the frequency interval of the sensitivity of the improved LIGO interferometer scheme between 50 and 500 Hz, while the damping factor of optical modes in the interferometer is $\sim 2\ \mathrm{s}^{-1}$, this value is quite large. Note here that the fundamental modes of Fabry–Perot resonators in the two arms are tuned to the resonance by a feedback system. In this case, the frequencies of additional optical modes playing the role of Stokes modes differ slightly from each other and Fabry–Perot resonators in the interferometer arms are not optically identical, as was assumed in [13].

In this paper, we analysed the parametric instability in the improved LIGO interferometer scheme with optically different arms. It is shown that a small mismatch of the interferometer arms causes the shift of the normal modes of the complete interferometer, however, the probability of parametric instability only slightly differs from that in the case of optically identical arms. On the one hand, this results in the possibility of developing parametric instability in the interferometer at low optical power (a few watts) and a small mismatch $\Delta = \omega_0 - \omega_S - \omega_m$, and on the other, the probability of parametric oscillatory instability proves to be extremely small due to a small damping factor of optical modes (a few hertz).

## 2. Signal-recirculation mirror LIGO interferometer with optically different Fabry–Perot interferometers in arms

Figure 1 shows a LIGO interferometer with signal- and power-recycling mirrors (SR and PR mirrors, respectively).

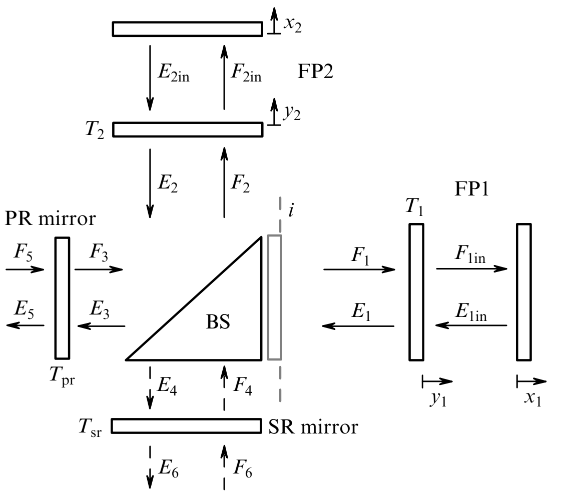


**Figure 1.** Improved scheme of a LIGO interferometer: FP1 and FP2: first and second Fabry–Perot resonators in the interferometer arms; BS: beamsplitter; $T_{pr}$, $T_{sr}$: transmission coefficients of PR and SR mirrors.

Let us neglect optical losses and noise in mirror suspensions and assume that the eigenfrequencies $\omega_1$ and $\omega_2$ of Stokes modes in the Fabry–Perot resonators in the interferometer arms slightly differ from the average frequency $\omega_S = (\omega_1 + \omega_2)/2$ by the mismatch $d \equiv (\omega_1 - \omega_2)/2$ of the interferometer arms, i.e. $\omega_{1,2} = \omega_S \pm d$. We assume also for simplicity that the damping factors and transmissions of the input mirrors are the same: $\gamma_1 = \gamma_2 = \gamma$ and $T_1 = T_2 = T$. The distances between the input mirrors of Fabry–Perot resonators and a beamsplitter and between the beamsplitter and SR and PR mirrors are sufficiently small (about several metres), so that we assume that the phase incursions of light waves circulating between these mirrors are constant and frequency-independent. The interferometer is pumped through the port $F_5$.

Let us introduce the average and small fluctuation amplitudes of light fields. For example, the amplitude in the first Fabry–Perot resonator is written in the form $F_{1in} = \mathscr{F}_0 + f_{1in}$, where the averaged field $\mathscr{F}_0$ corresponds to the fundamental mode at frequency $\omega_0$, and the fluctuation field $f_{1in}$ corresponds to the Stokes mode with the average frequency $\omega_S$. The detailed calculations of this configuration of the interferometer are presented in [13]:

$$f_{1in}(\Omega) = \mathscr{T}_1 f_1(\Omega) + N_1 \mathscr{F}_0 \frac{\mathscr{T}_1 2ikz_1^*(\Delta - \Omega)}{i\sqrt{T}}, \tag{6}$$

$$e_1(\Omega) = \mathscr{R}_1 f_1(\Omega) - N_1 \mathscr{F}_0 \mathscr{T}_1 2ikz_1^*(\Delta - \Omega), \tag{7}$$

$$\mathscr{T}_1 = \frac{2i\gamma}{\sqrt{T}[\gamma - i(\Omega - d)]}, \quad \mathscr{R}_1 = \frac{\gamma + i(\Omega - d)}{\gamma - i(\Omega - d)},$$
$$\Delta = \omega_0 - \omega_S - \omega_m, \quad \omega_S = \frac{\omega_1 + \omega_2}{2}, \quad d = \frac{\omega_1 - \omega_2}{2}, \tag{8}$$

$$f_{2in}(\Omega) = \mathscr{T}_2 f_2(\Omega) + N_2 \mathscr{F}_0 \frac{\mathscr{T}_2 2ikz_2^*(\Delta - \Omega)}{i\sqrt{T}}, \tag{9}$$

$$e_2(\Omega) = \mathscr{R}_2 f_2(\Omega) - N_2 \mathscr{F}_0 \mathscr{T}_2 2ikz_2^*(\Delta - \Omega), \tag{10}$$

$$\mathscr{T}_2 = \frac{2i\gamma}{\sqrt{T}[\gamma - i(\Omega + d)]}, \quad \mathscr{R}_2 = \frac{\gamma + i(\Omega + d)}{\gamma - i(\Omega + d)},$$

$$z_{1,2}(\Omega) \equiv x_{1,2}(\Omega) - y_{1,2}(\Omega), \quad \gamma = \frac{cT}{4L}, \tag{11}$$

$$E_1 = R_1 F_1, \quad E_2 = R_2 F_2, \quad R_{1,2} = \mathscr{R}_{1,2}(\Omega = 0).$$

Here, $f_{1in}$, $e_1$, $f_{2in}$, and $e_2$ are the fluctuation components of the fields $F_{1in}$, $E_1$, $F_{2in}$, and $E_2$; $N_{1,2}$ are the dimensionless overlap factors of optical and elastic modes; $k = \omega_S/c$ is the wave vector of the optical Stokes mode; $x_{1,2}$ and $y_{1,2}$ are mechanical displacements of the mirrors of Fabry–Perot resonators [13]; and $\Delta$ is the mismatch of modes in the case of parametric instability. Expressions (6)–(11) are written in the frequency representation for the case of the slowly varying amplitude of mechanical displacements $\bar{z}_1(t) = z_1(t)\exp(-i\omega_m t) + z_1^*(t)\exp(i\omega_m t)$.

Let us denote the amplitudes of light waves on the beamsplitter by $F_3$, $E_3$ and $F_4$, $E_4$ (Fig. 1). We assume that the beamsplitter transmission is $T_{bs} = 1/2$ and the phase appearing due the wave circulation between the second Fabry–Perot resonator and beamsplitter satisfies the relation $\exp(i\phi_2) = 1$, and between the first Fabry–Perot resonator and beamsplitter – $\exp(i\phi_1) = i$. For simplicity, we introduce the plane $i$ (Fig. 1) in which the phase incursion up to the beamsplitter is such that $\exp(i\phi_1) = i$ and the phase incursion up to the input mirror of the first Fabry–Perot resonator is $2\pi$. Then, the amplitudes $F_1$ and $E_1$ of fields on the input mirror of the first Fabry–Perot resonator (and in the plane *i*), the amplitudes $F_2$ and $E_2$ on the input mirror of the second Fabry–Perot resonator, and the amplitudes $F_3$, $E_3$, $F_4$, and $E_4$ on the beamsplitter will be related by the expressions

$$F_1 = \frac{-F_3 - \mathrm{i}F_4}{\sqrt{2}}, \quad F_2 = \frac{1}{\sqrt{2}}(-F_3 + \mathrm{i}F_4), \tag{12}$$

$$F_3 = \frac{-F_1 - F_2}{\sqrt{2}}, \quad F_4 = \frac{\mathrm{i}}{\sqrt{2}}(F_1 - F_2), \tag{13}$$

$$E_3 = \frac{-E_1 - E_2}{\sqrt{2}}, \quad E_4 = \frac{\mathrm{i}(-E_1 + E_2)}{\sqrt{2}}. \tag{14}$$

For the PR mirror, we have

$$F_3 \exp(-\mathrm{i}\phi_{\mathrm{pr}}) = \mathrm{i}T_{\mathrm{pr}}^{1/2} F_5 - (1 - T_{\mathrm{pr}})^{1/2} E_3 \exp(\mathrm{i}\phi_{\mathrm{pr}}), \tag{15}$$

$$E_5 = \mathrm{i}T_{\mathrm{pr}}^{1/2} E_3 \exp(\mathrm{i}\phi_{\mathrm{pr}}) - (1 - T_{\mathrm{pr}})^{1/2} F_5, \tag{16}$$

$$\phi_{\mathrm{pr}} = (\omega_{\mathrm{S}} + \Delta_{\mathrm{pr}} + \Omega)\frac{l_{\mathrm{pr}}}{c}, \tag{17}$$

where $T_{\mathrm{pr}}$ is the transmission coefficient of the PR mirror and $\Delta_{\mathrm{pr}}$ is the detuning of a resonator formed by the beamsplitter and PR mirror. Let us also assume that this resonator is in the resonance and $\exp(\mathrm{i}\phi_{\mathrm{pr}}) = \mathrm{i}$, while $\phi_{\mathrm{pr}}$ is independent of the frequency $\Omega$ because of a small size of the resonator ($l_{\mathrm{pr}} \ll L$). Therefore,

$$F_3 = (1 - T_{\mathrm{pr}})^{1/2} E_3 - T_{\mathrm{pr}}^{1/2} F_5, \tag{18}$$

$$E_5 = -T_{\mathrm{pr}}^{1/2} E_3 - (1 - T_{\mathrm{pr}})^{1/2} F_5. \tag{19}$$

For the SR mirror, we write similarly

$$F_4 \exp(-\mathrm{i}\phi) = \mathrm{i}T_{\mathrm{sr}}^{1/2} F_6 - (1 - T_{\mathrm{sr}})^{1/2} E_4 \exp(\mathrm{i}\phi), \tag{20}$$

$$E_6 = \mathrm{i}T_{\mathrm{sr}}^{1/2} E_4 \exp(\mathrm{i}\phi) - (1 - T_{\mathrm{sr}})^{1/2} F_6, \tag{21}$$

where $T_{\mathrm{sr}}$ is transmission coefficient of the SR mirror. Let us assume that a resonator formed by the beamsplitter and SR mirror is not in the resonance and $\phi = (\omega_{\mathrm{S}} + \Omega) l_{\mathrm{sr}}/c$ is an arbitrary phase incursion, but $\phi$ is also independent of frequency $\Omega$ due to a small size of the resonator ($l_{\mathrm{sr}} \ll L$).

By substituting (13) and (14) into the system of equations (18), (20) and taking into account (6), (7), (9), (10), we obtain expressions for the light fields $f_{1\mathrm{in}}$ and $f_{2\mathrm{in}}$

$$f_{1\mathrm{in}}[\gamma_+ - \mathrm{i}(\Omega - d)] + f_{2\mathrm{in}}[\gamma_+ - \mathrm{i}(\Omega + d)] = \mathscr{L}_+,$$

$$f_{1\mathrm{in}}[\Gamma_- - \mathrm{i}(\Omega - d)] - f_{2\mathrm{in}}[\Gamma_- - \mathrm{i}(\Omega + d)] = \mathscr{L}_-,$$

$$\mathscr{L}_1 = \frac{\mathrm{i}cN_1\mathscr{F}_0 k z_1^*}{L}, \quad \mathscr{L}_2 = \frac{\mathrm{i}cN_1\mathscr{F}_0 k z_2^*}{L}, \tag{22}$$

$$\gamma_+ = \gamma\,\frac{1 - (1 - T_{\mathrm{pr}})^{1/2}}{1 + (1 - T_{\mathrm{pr}})^{1/2}}, \quad \mathscr{L}_\pm = \mathscr{L}_1 \pm \mathscr{L}_2,$$

$$\Gamma_- \equiv \gamma_- - \mathrm{i}\delta = \gamma\,\frac{1 - \exp(2\mathrm{i}\phi)(1 - T_{\mathrm{sr}})^{1/2}}{1 + \exp(2\mathrm{i}\phi)(1 - T_{\mathrm{sr}})^{1/2}}. \tag{23}$$

Here, $\delta$ is the detuning of the asymmetric mode, which depends on the position of the SR mirror, and $\gamma_+$ and $\Gamma_-$ have a simple physical meaning. In the case of an interferometer with optically identical arms (for $d \to 0$), $\Gamma_- \to \gamma$ and $\gamma_+$ and $\gamma_-$ are the damping factors of the symmetric ($\sim F_1 + F_2$) and antisymmetric ($\sim F_1 - F_2$) modes [11]. By summing and subtracting these equations, we obtain

$$f_{1\mathrm{in}}[g_+ - \mathrm{i}(\Omega - d)] + f_{2\mathrm{in}} g_- = \mathscr{L}_1, \tag{24}$$

$$f_{1\mathrm{in}} g_- + f_{2\mathrm{in}}[g_+ - \mathrm{i}(\Omega + d)] = \mathscr{L}_2, \tag{25}$$

$$g_\pm = \frac{\gamma_+ \pm \Gamma_-}{2} = \frac{\gamma_+ \pm \gamma_- \mp \mathrm{i}\delta}{2}. \tag{26}$$

By adding now mechanical equations with the ponder-motive light pressure force, we find

$$\dot{\mathscr{Z}}_1 + \gamma_{\mathrm{m}}\mathscr{Z}_1 = 2\mathscr{Q} f_{1\mathrm{in}}(t)\exp(\mathrm{i}\Delta t), \tag{27}$$

$$\dot{\mathscr{Z}}_2 + \gamma_{\mathrm{m}}\mathscr{Z}_2 = 2\mathscr{Q} f_{2\mathrm{in}}(t)\exp(\mathrm{i}\Delta t), \tag{28}$$

$$2\mathscr{Q} = \frac{2\omega_{\mathrm{S}}|N_1|^2|\mathscr{F}_0|^2}{mc\mu\omega_{\mathrm{m}}L}, \quad \mu = \frac{1}{V}\int |\boldsymbol{u}(\boldsymbol{r})|^2 \mathrm{d}V, \tag{29}$$

where $\gamma_{\mathrm{m}}$ is the damping factor of an elastic mode of the mirror; $m$ is the mirror mass; $V$ is the mirror volume; and $\boldsymbol{u}(\boldsymbol{r})$ is the deformation vector of the elastic mode.

Because the system of equations (24), (25) describes coupled oscillators, it is reasonable to introduce normal modes with the complex amplitudes $\xi$, $\eta$ and eigenvalues $\lambda_1$, $\lambda_2$, respectively. By rewriting the system of equations (24), (25) and equations (27), (28) in the time representation [13]

$$(1 + \varkappa^2)(\partial_t - \lambda_1)\xi = z_\xi \exp(-\mathrm{i}\Delta t), \tag{30}$$

$$\dot{z}_\xi + \gamma_{\mathrm{m}} z_\xi = 2\mathscr{Q}(1 + \varkappa^2)\xi(t)\exp(\mathrm{i}\Delta t), \tag{31}$$

$$(1 + \varkappa^2)(\partial_t - \lambda_2)\eta = z_\eta \exp(-\mathrm{i}\Delta t), \tag{32}$$

$$\dot{z}_\eta + \gamma_{\mathrm{m}} z_\eta = 2\mathscr{Q}(1 + \varkappa^2)\eta(t)\exp(\mathrm{i}\Delta t), \tag{33}$$

$$\xi = \frac{f_{1\mathrm{in}} - \varkappa f_{2\mathrm{in}}}{1 + \varkappa^2}, \quad \eta = \frac{\varkappa f_{1\mathrm{in}} + f_{2\mathrm{in}}}{1 + \varkappa^2}, \tag{34}$$

$$z_\xi = \mathscr{Z}_1 - \varkappa\mathscr{Z}_2, \quad z_\eta = \varkappa\mathscr{Z}_1 + \mathscr{Z}_2, \tag{35}$$

$$\lambda_{1,2} = -g_+ \pm \left(g_-^2 - d^2\right)^{1/2}, \quad \varkappa = \frac{\left(g_-^2 - d^2\right)^{1/2} + \mathrm{i}d}{g_-}, \tag{36}$$

we obtain, as in the case of two optically identical Fabry–Perot resonators in the interferometer arms, two independent pairs of equations. In the limit $d \to 0$, the mode $\xi$ is transformed to the antisymmetric mode [i.e. $\xi \to (f_{1\mathrm{in}} - f_{2\mathrm{in}})/2$, $z_\xi \to \mathscr{Z}_1 - \mathscr{Z}_2$], while the mode $\eta$ is transformed to the symmetric mode [i.e. $\eta \to (f_{1\mathrm{in}} + f_{2\mathrm{in}})/2$, $z_\eta \to \mathscr{Z}_1 + \mathscr{Z}_2$].

For the most probable case, we have the rigorous condition $|\gamma_+ - \gamma_-| \ll \delta$ (damping factors $\gamma_\pm$ lie in the interval between 1 and 10 $\mathrm{s}^{-1}$, while the detuning $\delta$, depending on the position of the SR mirror, is approximately equal to $10^3\ \mathrm{s}^{-1}$) and can use the following approximations for mode eigenvalues (36):

$$\lambda_{1,2} \simeq \left[-\frac{\gamma_+ + \gamma_-}{2} \pm \frac{\delta(\gamma_+ - \gamma_-)}{2D}\right] + \mathrm{i}\left(\frac{\delta}{2} \pm D\right), \qquad (37)$$

$$\varkappa \simeq \frac{2(d+D)}{\delta}, \quad D = \left[d^2 + \left(\frac{\delta}{2}\right)^2\right]^{1/2}. \qquad (38)$$

In the case of identical mirrors having the same mass and identical elastic eigenfrequencies, we can analyse independently two pairs of equations (30), (31) and (32), (33). For example, by using substitutions $\xi = \xi \exp(\lambda t - \mathrm{i}\Delta t)$ and $z_\xi = z_\xi \exp(\lambda t)$ in the system of equation (30) and (31), we obtain the characteristic equation

$$2\mathscr{Q} = (\lambda - \mathrm{i}\Delta - \lambda_1)(\lambda + \gamma_{\mathrm{m}}). \qquad (39)$$

Because it is known [13] that

$$\gamma_{\mathrm{m}} \ll \gamma_+, \gamma_-, \qquad (40)$$

we will seek the root of the characteristic equation close to the damping factor of the elastic mode: $|\lambda| \sim \gamma_{\mathrm{m}}$. Therefore, by using (40), we rewrite the characteristic equation (39) by assuming that $\lambda = 0$ in all parentheses containing $\gamma_+$, $\gamma_-$ (or $\lambda_1$):

$$\lambda \simeq -\gamma_{\mathrm{m}} - \frac{2\mathscr{Q}}{\mathrm{i}\Delta + \lambda_1}. \qquad (41)$$

The condition of parametric instability corresponds to the situation when $\mathrm{Re}\lambda < 0$, which gives

$$\frac{2\mathscr{Q}}{\gamma_{\mathrm{m}}} \mathrm{Re}\left(\frac{-1}{\lambda_1 + \mathrm{i}\Delta}\right) \geqslant 1. \qquad (42)$$

In the case of different mirrors, for which the eigenfrequencies of elastic modes do not coincide, we consider the displacement of only one mirror, for example, the displacement $x_1$ of the end mirror in the first Fabry–Perot resonator and introduce the quantity $\mathscr{X}_1 = \mathrm{i}cN_1\mathscr{F}_0 k x_1^*/L$. In this case, Eqns (27), (28), (30), and (32) can be written in the form

$$(1+\varkappa^2)(\partial_{\mathrm{t}} - \lambda_1)\xi = \mathscr{X}_1 \exp(-\mathrm{i}\Delta t), \qquad (43)$$

$$(1+\varkappa^2)(\partial_{\mathrm{t}} - \lambda_2)\eta = \varkappa\mathscr{X}_1 \exp(-\mathrm{i}\Delta t), \qquad (44)$$

$$\dot{\mathscr{X}}_1 + \gamma_{\mathrm{m}}\mathscr{X}_1 = \mathscr{Q}(\xi(t) + \varkappa\eta(t))\exp(\mathrm{i}\Delta t). \qquad (45)$$

By solving Eqns (27)–(45), we should take into account that the effective light pressure force should be half as much because it acts only on one mirror. By assuming that $\xi = \xi \exp(\lambda t - \mathrm{i}\Delta t)$, $\eta = \eta \exp(\lambda t - \mathrm{i}\Delta t)$, and $\mathscr{X}_1 = \mathscr{X}_1 \times \exp(\lambda t)$, we obtain the characteristic equation in the form similar to that for optically identical Fabry–Perot resonators ($d = 0$) [11]:

$$\lambda + \gamma_{\mathrm{m}} = \frac{\mathscr{Q}}{1+\varkappa^2}\left(\frac{1}{\lambda - \mathrm{i}\Delta - \lambda_1} + \frac{\varkappa^2}{\lambda - \mathrm{i}\Delta - \lambda_2}\right). \qquad (46)$$

By using (40), we find the condition of parametric instability in the form

$$\frac{\mathscr{Q}}{\gamma_{\mathrm{m}}} \mathrm{Re}\left[\frac{1}{1+\varkappa^2}\left(\frac{-1}{\mathrm{i}\Delta + \lambda_1} + \frac{-\varkappa^2}{\mathrm{i}\Delta + \lambda_2}\right)\right] \geqslant 1. \qquad (47)$$

Note that we used in all calculations the following values of damping factors: (see Appendix C in [13]): the damping factor for elastic modes $\gamma_{\mathrm{m}} \simeq 6 \times (10^{-4} - 10^{-2})\ \mathrm{s}^{-1}$; the damping coefficients for symmetric and antisymmetric modes $\gamma_+ \simeq 1.5\ \mathrm{s}^{-1}$ and $\gamma_- \geqslant 2\ \mathrm{s}^{-1}$, respectively; and the damping coefficient for a Fabry–Perot resonator $\gamma \simeq 100\ \mathrm{s}^{-1}$.

Note also that the conditions of parametric instability for the improved LIGO interferometer scheme with optically different arms exactly correspond to the instability conditions obtained earlier [11] for an interferometer with identical arms ($d = 0$). Indeed, conditions (42) and (47) will transform to Eqns (2.16) and (2.30) in [11] if we assume that

$$\lambda_1 \to -\gamma_- + \mathrm{i}\delta, \quad \lambda_2 \to -\gamma_+. \qquad (48)$$

The dependence of the eigenvalues $\lambda_{1,2}$ of normal modes on the arm mismatch $d$ gives complete information on the probability of appearance of parametric instability compared to the case of optically identical arms. Curves in Fig. 2 illustrate qualitatively this dependence. One can see that the real parts of $\lambda_{1,2}$ (with the opposite sign), which are damping factors for the normal resonator modes, weakly change over the entire range of the arm mismatch $d$. Therefore, a strong dependence of the parametric instability $\Delta$ on the mode mismatch is also preserved in the improved LIGO interferometer scheme with optically different arms (47), i.e. parametric instability can be observed if the value of $\Delta$ is relatively small: $|\Delta + \mathrm{Im}(\lambda_{1,2})| < \gamma_+, \gamma_- \simeq 2\ \mathrm{s}^{-1}$.

One the one hand, in the case of parametric resonance (the total mismatch is small), the parametric instability in a SR mirror interferometer appears for relatively low optical power. For example, if $|\Delta + \mathrm{Im}(\lambda_1)| \ll \gamma_+, \gamma_-$ and $|\Delta + \mathrm{Im}(\lambda_2)| \gg \gamma_+, \gamma_-$ [i.e. the second term in (47) is negligible small], the parametric oscillatory instability takes place even when the power of a wave circulating in the interfer-

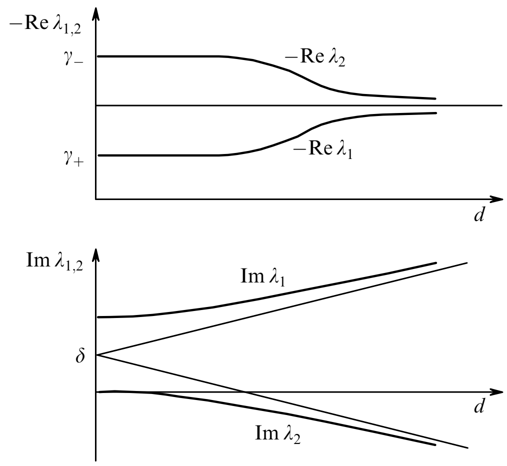


**Figure 2.** Qualitative dependences of the real and imaginary parts of the eigenvalues of normal modes on the mismatch $d$ of the LIGO interferometer arms.

ometer arms is low ($W_c \simeq 5$ W) (for $\omega_m = 10^5$ s$^{-1}$, $\gamma_m = 6 \times 10^{-4}$ s$^{-1}$, $|N_1|^2/\mu \simeq 1$) [13].

On the other hand, in the case of large mode mismatches $|\varDelta + \mathrm{Im}(\lambda_1)| > \gamma_+, \gamma_-$, the parametric instability can appear only at high optical powers [$W_c \sim |\varDelta + \mathrm{Im}(\lambda_1)|^2$)]. For example, if the mismatch is about 1 kHz [i.e. $|\varDelta + \mathrm{Im}(\lambda_1)| \simeq 6 \times 10^3$], by using parameters from [13], we obtain that the parametric instability occurs at $W_c \simeq 10^8$ W.

One can see from the lower curve in Fig. 2 that the mismatch $d$ of interferometer arms can strongly change the frequencies of the normal modes of the LIGO interferometer. At the same time, the mismatch does not change the density of optical modes in Fabry–Perot resonators and, hence, the probability of appearance of parametric instability.

## 3. Conclusions

We have analysed the parametric instability effect in the improved scheme of a signal-recirculation mirror LIGO interferometer with optically different arms. The neglect of this effect can lead to a considerably decrease in the sensitivity of a gravitational-wave antenna. The conditions of parametric instability have been obtained both in the case when all the mirrors have the same mass and identical elastic eigenfrequencies and in the case when these frequencies are different.

The calculations performed in the paper have shown that, although the mismatch $d$ of the LIGO interferometer arms changes the frequencies of the normal modes of the interferometer, the probability of the appearance of parametric instability is almost the same as that in the LIGO interferometer with optically identical arms. In addition, the parametric instability in this interferometer is relatively small because of a small width of the interferometer passband.

Note also that a direct experiment is the most efficient method for eliminating parametric instability. To observe the parametric instability in experiments with a SR mirror interferometer, we can change the mismatch $\delta$ of a SR mirror resonator and the mismatch $d$ of interferometer arms. The methods for changing the mismatch $\delta$ by varying the position of the SR mirror and the arm mismatch $d$ by varying the radius of curvature of mirrors due to nonuniform heating were proposed in papers [7–9, 11]. These methods allow one to study in detail a relatively broad frequency interval for finding parametric instabilities. Together with calculations of the elastic eigenmodes of mirrors, these methods give very important information on the possibility of excluding parametric instability in the LIGO interferometer.

***Acknowledgements.*** The authors thank V.B. Braginsky, A. Melissinos, and W. Kells for their valuable advices and remarks. This work was supported by Grant No. MK-195.2007.2 of the President of the Russian Federation for the State Support of Young Russian Scientists, State Contract Nos 5178.2006.2 and 02.445.11.7423 of the Ministry of Industry and Science, and NSF Grant PHY-0353775.

## References

1. Abramovici A. et al. *Science*, **256**, 325 (1992).
2. Abramovici A. et al. *Phys. Lett. A*, **218**, 157 (1996).
3. Braginsky V.B., Strigin S.E., Vyatchanin S.P. *Phys. Lett. A*, **287**, 331 (2001).
4. D'Ambrosio E., Kells W. *Phys. Lett. A*, **299**, 326 (2002).
5. Braginsky V.B., Strigin S.E., Vyatchanin S.P. *Phys. Lett. A*, **305**, 111 (2002).
6. Braginsky V.B., Vyatchanin S.P. *Phys. Lett. A*, **293**, 228 (2002).
7. Zhao C., Ju L., Degallaix J., Gras S., Blair D.G. *Phys. Rev. Lett.*, **94**, 121102 (2005).
8. Ju L., Gras S., Zhao C., Degallaix J., Blair D.J. *Phys. Lett. A*, **354**, 360 (2006).
9. Ju L., Zhao C., Gras S., Degallaix J., Blair D.J., Munch J., Reitze D.H. *Phys. Rev. Lett.*, **96**, 231101 (2006).
10. Corbitt T., Ottaway D., Innerhofer E., Pelc J., Mavalvala N. *Phys. Rev. A*, **74**, 021802 (2006).
11. Gurkovsky A.G., Strigin S.E., Vyatchanin S.P. *Phys. Lett. A*, **362**, 91 (2007).
12. Siegman A.E. *Lasers* (London: University Science Book, 1986).
13. Strigin S.E., Vyatchanin S.P. *Phys. Lett. A*, **365**, 10 (2007).